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Numerical Real Inversion Formulas of the Laplace Transform by using a Fredholm integral equation of the second kind

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1 Introduction

We shall give a very natural and numerical real inversion formula of the Laplace transform

$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt}F(t)dt, \quad p > 0 \tag{1.1}$$

for functions F of some natural function space. This integral transform is, of course, very fundamental in mathematical science. The inversion of the Laplace transform is, in general, given by a complex form, however, we are interested in and are requested to obtain its real inversion in many practical problems. However, the real inversion will be very involved and one might think that its real inversion will be essentially involved, because we must catch "analyticity" from the real or discrete data. Note that the image functions of the Laplace transform are analytic on some half complex plane. For complexity of the real inversion formula of the Laplace transform, we recall, for example, the following formulas:

$$\lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} f^{(n)}\left(\frac{n}{t}\right) = F(t)$$

(Post [15] and Widder [25,26]), and

$$\lim_{n\to\infty} \Pi_{k=1}^{n} \left(1 + \frac{t}{k}\frac{d}{dt}\right) \left[\frac{n}{t} f\left(\frac{n}{t}\right)\right] = F(t),$$

([25,26]). Furthermore, see [1-7,10,11,17,18,21] and the recent related articles [10] and 11]. See also the great references [27,28]. The problem may be related to analytic extension problems, see [10] and [11].

In this paper, we shall give new type and very natural real inversion formulas from the viewpoints of best approximations, generalized inverses and the Tikhonov regularization by combining these fundamental ideas and methods by means of the theory of reproducing kernels. However, in this paper we shall propose a new method for the real inversion formulas of the Laplace transform based essentially on a Fredholm integral equation of the second kind. We may think that these real inversion formulas are practical and natural. We can give good error estimates in our inversion formulas. Furthermore, we shall illustrate examples, by using computers.

2 Background General Theorems

Let E be an arbitrary set, and let H_K be a reproducing kernel Hilbert space (RKHS) admitting the reproducing kernel $K(p,q)$ on E. For any Hilbert space $\mathcal{H}$ we consider a bounded linear operator L from H_K into $\mathcal{H}$. We are generally interested in the best approximate problem

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} \tag{2.2}$$

for a vector $\mathbf{d}$ in $\mathcal{H}$. However, when there exits, this extremal problem is involved in the both senses of the existence of the extremal functions in (2.2) and their representations. See [16] for the details. So, we shall consider its Tikhonov regularization.

We set, for any fixed positive $\alpha > 0$

$$K_L(\cdot, p; \alpha) = \frac{1}{L^*L + \alpha I} K(\cdot, p),$$

where L^* denotes the adjoint operator of L. Then, by introducing the inner product

$$(f, g)_{H_K(L;\alpha)} = \alpha (f, g)_{H_K} + (Lf, Lg)_{\mathcal{H}}, \tag{2.3}$$

we shall construct the Hilbert space $H_K(L; \alpha)$ comprising functions of H_K. This space, of course, admits a reproducing kernel. Furthermore, we obtain, directly

Proposition 2.1 *([18]) The extremal function $f_{\mathbf{d},\alpha}(p)$ in the Tikhonov regularization*

$$\inf_{f \in H_K} \{\alpha \|f\|_{H_K}^2 + \|\mathbf{d} - Lf\|_{\mathcal{H}}^2\} \tag{2.4}$$

exists uniquely and it is represented in terms of the kernel $K_L(p, q; \alpha)$ as follows:

$$f_{\mathbf{d},\alpha}(p) = (\mathbf{d}, LK_L(\cdot, p; \alpha))_{\mathcal{H}} \tag{2.5}$$

where the kernel $K_L(p, q; \alpha)$ is the reproducing kernel for the Hilbert space $H_K(L; \alpha)$ and it is determined as the unique solution $\tilde{K}(p, q; \alpha)$ of the equation:

$$\tilde{K}(p, q; \alpha) + \frac{1}{\alpha}(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\alpha} K(p, q) \tag{2.6}$$

with

$$\tilde{K}_q = \tilde{K}(\cdot, q; \alpha) \in H_K \quad \textit{for} \quad q \in E, \tag{2.7}$$

and

$$K_p = K(\cdot, p) \in H_K \quad \textit{for} \quad p \in E.$$

In (2.5), when $\mathbf{d}$ contains errors or noises, we need its error estimate. For this, we can obtain the general result:

Proposition 2.2 *([14]). In* (2.5), *we have the estimate*

$$|f_{\mathbf{d},\alpha}(p)| \leq \frac{1}{\sqrt{\alpha}}\sqrt{K(p,p)}\|\mathbf{d}\|_{\mathcal{H}}.$$

For the convergence rate or the results for noisy data, see, ([9]).

3 A Natural Situation for Real Inversion Formulas

In order to apply the general theory in Section 2 to the real inversion formula of the Lapace transform, we shall recall the "natural situation" based on [17,13].

We shall introduce the simple reproducing kernel Hilbert space (RKHS) H_K comprised of absolutely continuous functions F on the positive real line $\mathbf{R}^+$ with finite norms

$$\left\{\int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt\right\}^{1/2}$$

and satisfying $F(0) = 0$. This Hilbert space admits the reproducing kernel $K(t,t')$

$$K(t,t') = \int_0^{\min(t,t')} \xi e^{-\xi} d\xi \tag{3.8}$$

(see [9], pages 55-56). Then we see that

$$\int_0^\infty |(\mathcal{L}F)(p)p|^2 dp \leq \frac{1}{2}\|F\|^2_{H_K}; \tag{3.9}$$

that is, the linear operator on H_K

$$(\mathcal{L}F)(p)p$$

into $L_2(\mathbf{R}^+, dp) = L_2(\mathbf{R}^+)$ is bounded ([17]). For the reproducing kernel Hilbert spaces H_K satisfying (3.9), we can find some general spaces ([17]). Therefore, from the general theory in Section 2, we obtain

Proposition 3.1 *([17]). For any $g \in L_2(\mathbf{R}^+)$ and for any $\alpha > 0$, the best approximation $F^*_{\alpha,g}$ in the sense*

$$\inf_{F\in H_K}\left\{\alpha\int_0^\infty |F'(t)|^2\frac{1}{t}e^t dt + \|(\mathcal{L}F)(p)p - g\|^2_{L_2(\mathbf{R}^+)}\right\}$$

$$= \alpha\int_0^\infty |F^{*\prime}_{\alpha,g}(t)|^2\frac{1}{t}e^t dt + \|(\mathcal{L}F^*_{\alpha,g})(p)p - g\|^2_{L_2(\mathbf{R}^+)} \tag{3.10}$$

exists uniquely and we obtain the representation

$$F^*_{\alpha,g}(t) = \int_0^\infty g(\xi)\left(\mathcal{L}K_\alpha(\cdot,t)\right)(\xi)\xi d\xi. \tag{3.11}$$

Here, $K_\alpha(\cdot,t)$ is determined by the functional equation

$$K_\alpha(t,t') = \frac{1}{\alpha}K(t,t') - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}K_t)(p)p)_{L_2(\mathbf{R}^+)} \tag{3.12}$$

for

$$K_{\alpha,t'} = K_\alpha(\cdot,t')$$

and

$$K_t = K(\cdot,t)$$

4 New Algorithm

We shall look for the approximate inversion $F^*_{\alpha,g}(t)$ by using (3.11). For this purpose, we take the Laplace transform of (3.12) in t and change the variables t and t' as in

$$(\mathcal{L}K_\alpha(\cdot,t))(\xi)$$
$$= \frac{1}{\alpha}(\mathcal{L}K(\cdot,t'))(\xi) - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}(\mathcal{L}K_t)(p)p))(\xi))_{L_2(\mathbf{R}^+)}. \tag{4.13}$$

Note that

$$K(t,t') = \begin{cases} -te^{-t} - e^{-t} + 1 & \text{for} \quad t \le t' \\ -t'e^{-t'} - e^{-t'} + 1 & \text{for} \quad t \ge t'. \end{cases}$$

$$
\begin{aligned}
&(\mathcal{L}K(\cdot,t'))(p)\\
&= e^{-t'p}e^{-t'}\left[\frac{-t'}{p(p+1)}+\frac{-1}{p(p+1)^2}\right]+\frac{1}{p(p+1)^2}.
\end{aligned} \tag{4.14}
$$

$$
\int_0^\infty e^{-qt'}(\mathcal{L}K(\cdot,t'))(p)dt' = \frac{1}{pq(p+q+1)^2}. \tag{4.15}
$$

Therefore, by setting

$$
(\mathcal{L}K_\alpha(\cdot,t))(\xi)\xi = H_\alpha(\xi,t),
$$

which is needed in (3.11), we obtain the Fredholm integral equation of the second type

$$
\begin{aligned}
&\alpha H_\alpha(\xi,t) + \int_0^\infty H_\alpha(p,t)\frac{1}{(p+\xi+1)^2}dp\\
&= -\frac{e^{-t\xi}e^{-t}}{\xi+1}\left(t+\frac{1}{\xi+1}\right)+\frac{1}{(\xi+1)^2}.
\end{aligned} \tag{4.16}
$$

5 Numerical Experiments

We shall give a numerical experiment for the typical example

$$
F_0(t) = \begin{cases} -te^{-t} - e^{-t} + 1 & \text{for} \quad 0 \le t \le 1 \\ 1 - 2e^{-1} & \text{for} \quad 1 \le t, \end{cases}
$$

whose Laplace transform is

$$
(\mathcal{L}F_0)(p) = \frac{1}{p(p+1)^2}\left[1-(p+2)e^{-(p+1)}\right]. \tag{5.17}
$$

We set

$$
g(\xi) = (\mathcal{L}F_0)(\xi)\xi
$$

in (3.11) with $(\mathcal{L}K_\alpha(\cdot,t))(\xi)\xi = H_\alpha(\xi,t)$, then

$$
F^*_{\alpha,g}(t) \sim F_0(t)
$$

for a small α ?

For fixed t, we calculate the integral (4.16) over $[0, 50]$ with span 0.01 by the trapezoidal rule. Here we solve the linear simultaneous linear equations of 5000 by using Matlab. For t, we take the values over $[0, 5]$ with span 0.01. By (3.11), we caluculate the inversion by the trapeziodal rule over $[0, 50]$ with the span 0.01.

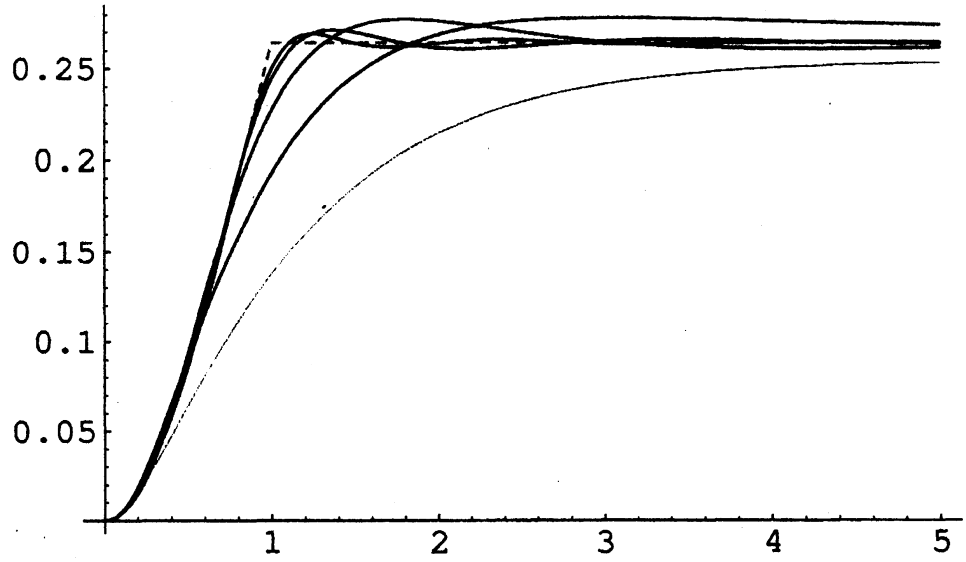

Figure 1: For $F_0(t)$ and for $\alpha = 10^{-1}, 10^{-2}, 10^{-4}, 10^{-8}, 10^{-10}$.

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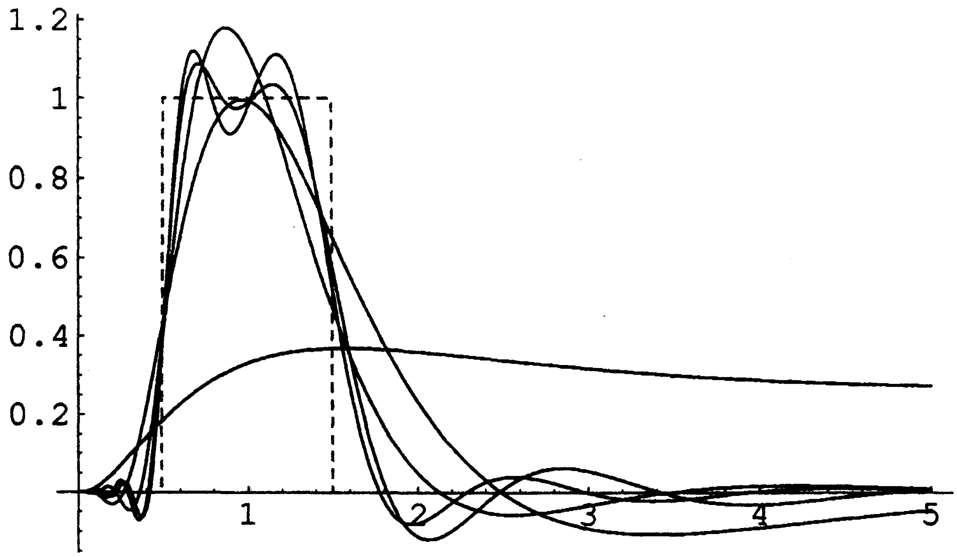

Figure 2: For $F(t) = \chi(t, [1/2, 3/2])$, the characteristic function and for $\alpha = 10^{-1}, 10^{-4}, 10^{-8}, 10^{-12}, 10^{-16}$. $(\mathcal{L}F)(p) = \frac{1}{p}(\exp(-\frac{1}{2}p) - \exp(-\frac{3}{2}p))$.

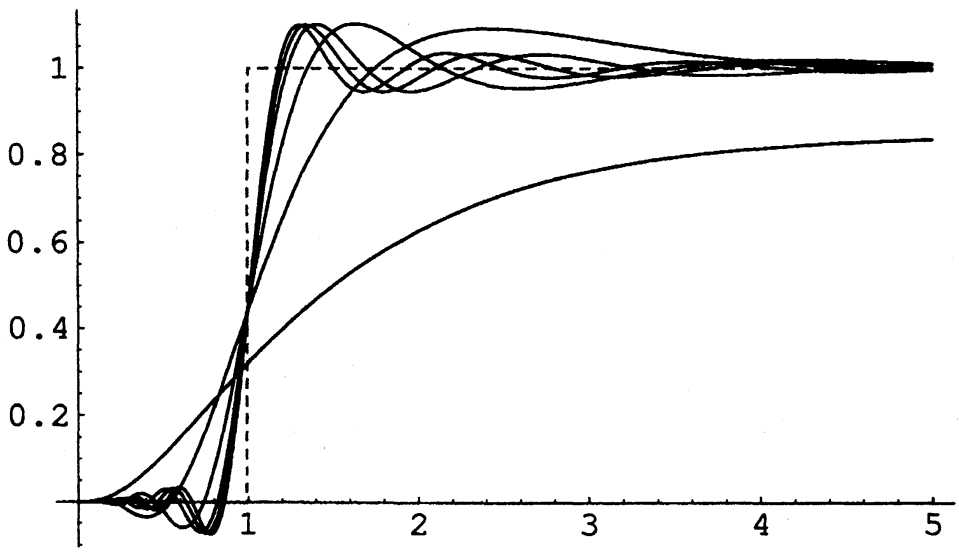

Figure 3: For $U(t, [1, \infty])$, the step function and for $\alpha = 10^{-1}, 10^{-4}, 10^{-8}, 10^{-12}, 10^{-16}$. $(\mathcal{L}U)(p) = \frac{1}{p}e^{-p}$.

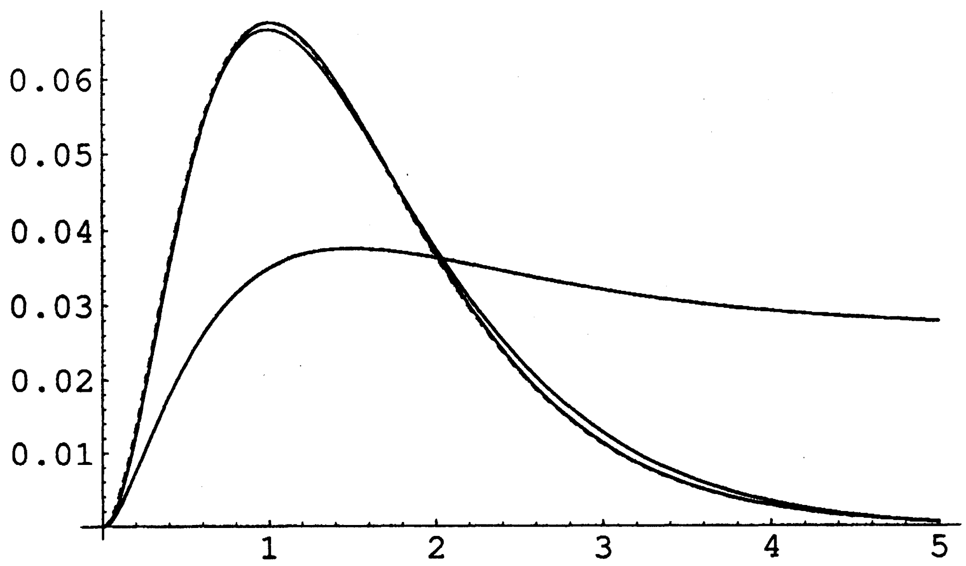

Figure 4: For $F(t) = 1/2t^2 \exp(-2t)$ and for $\alpha = 10^{-1}, 10^{-4}, 10^{-8}$. $(\mathcal{L}F)(p) = \frac{1}{(p+2)^3}$.

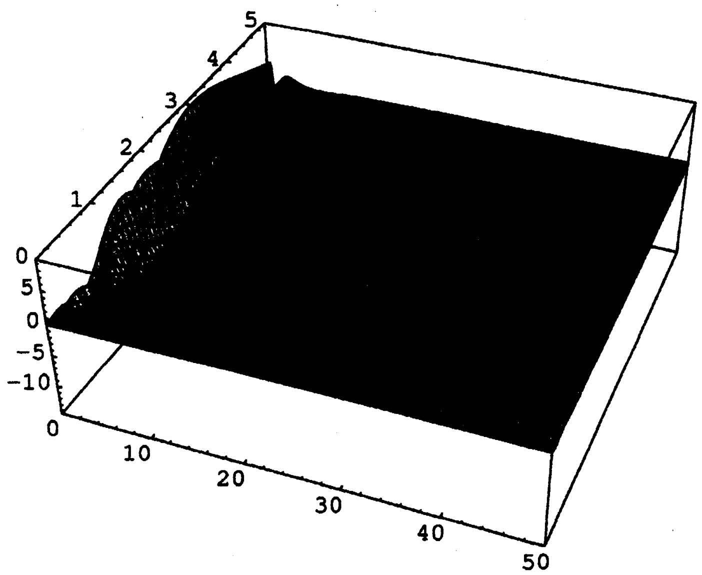

Figure 5: For $H_\alpha(\xi, t)$ and for $\alpha = 10^{-4}$.

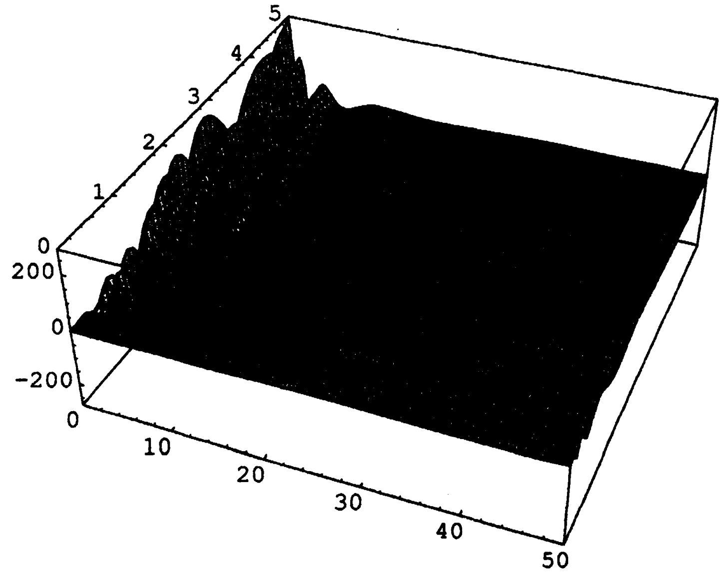

Figure 6: For $H_\alpha(\xi, t)$ and for $\alpha = 10^{-8}$.

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