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1次元周期的2次元Stokes流に対する基本解法
Fundamental Solutionxs Method for Two-Dimensional Stokes Flow Problems with One-Periodic Periodicity

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Abstract

In this paper, we propose a fundamental solution method for the problems of two-dimensional Stokes flow past a one-dimensional periodic array of cylinders. In the presented method, the solution is approximated by a superposition of the periodic Stokeslets, the periodic fundamental solutions of the Stokes flow equation. The numerical example included in this paper show the effectiveness of the presented method.

1 Introduction

Flow problems with spatial periodicity are attracting subjects from the viewpoint of theoretical fluid mechanics and important in applications to science and engineering. The aim of this paper is to present a fundamental solution method for the problems of two-dimensional Stokes flows past a one-dimensional periodic array of cylinders as shown in Figure 1.

There are many works on spatially periodic flows as follows. As a work on periodic potential flows, Ogata et al. presented a charge simulation method (fundamental solution method) for numerical conformal mappings of two-dimensional Euclidean domains, which is identified with one-dimensional complex domain, with one-dimensional periodic periodicity [19] and applied their method to the analysis of two-dimensional potential flow past a one-dimensional array of cylinders. As a work on periodic

Oseen flow, Tamada and Fujikawa studied the problem of steady two-dimensional Oseen flow past an infinite row of circular cylinders [22]. As works on periodic Stokes flow, the problems with which we are concerned in this paper, Hasimoto presented the periodic fundamental solution method of the two of three-dimensional Stokes flow equation and applied it to the analysis of Stokes flow past a periodic array of spheres [7], which was improved by Sangani and Acrivos [20]. Ishii presented the fundamental solution of three-dimensional Stokes flow with planar periodicity and applied it to the study of three-dimensional Stokes flow problem with planar arrays of small spheres [8]. In addition, as works on application of periodic Stokes flow studies, Liron presented studies of Stokes flow due to an infinit array of Stokeslets, which are applied to the analysis of ciliary transport [12, 13, 14].

The fundamental solution method is a numerical solver of partial differential equation problems and is widely used in science and engineering, especially, in potential problems, where the method is usually called the "charge simulation method" [15, 21], for the reasons that it is easy to program, (ii) its computational cost is low and (iii) it achieves high accuracy under some conditions. In this method, the solution is approximated by a linear combination of the fundamental solutions of the partial differentiation operator with singularities outside the problem domain. In terms of physics, the potential which is the solution of a potential problem is approximated by a superposition of the Coulomb potentials due to the charges positioned outside the problem domain. Katsurada and Okamoto showed the solvability and the high accuracy of the fundamental solution method from theoretical viewpoints [9, 10, 11]. As works on applications of the fundamental solution method, Amano et al. presented numerical conformal mappings by the fundamental solution method [1, 2, 3]. Related to this paper, Chuwang and Wu presented a fundamental solution method for Stokes flow problems, whose approximation is based on the Stokeslet [4].

It is, however, difficult to apply the fundamental solution method by Chuwang and Wu to our problem of periodic Stokes flow because the approximation of Chuwang and Wu's method may not be able to approximate accurately the solution of our periodic problem which may include periodic functions. In the method presented in this paper, we modify the Stokeslet so that it illustrates the flow due to an infinite periodic array of concentrated forces of equal magnitude and construct an approximate solution by a linear combination of the above *periodic Stokeslet*. It is expected that this method inherits the advantages of the ordinary fundamental solution method and can approximate the solution including periodic functions with high accuracy. We here remark that, as work related to our method, Ogata et al. presented fundamental solution methods for two-dimensional potential problems with one-dimensional periodicity [19], two-dimensional Stokes flow problems with a two-dimensional periodic array of cylinders [18], three-dimensional Stokes flow problems with a two or three-dimensional periodic array of obstacles [16, 17]. Greengard and Kropinski presented an integral equation method for two-dimensional Stokes flow problemsin double-periodic domains, which is based on elliptic function theory and incorporated into the fast multipole method [6], and Zick and Homsy also presented an integral equation method for Stokes flow problems with a periodic array of spheres, which is based on the periodic fundamental solution of the Stokes flow equation [23].

The contents of this paper are as follows. In Section 2, we formulate mathematically our problem and prepare some notations. In Section 3, we present a fundamental solution method for our problems. In Section 4, we show a typical numerical example of our method. In Section 5, we give concluding remarks.

2 Formulation of Problems

We first formulate our problem mathematically and give some notations. Throughout this paper, we denote by $\mathbb{Z}$ the set of all the integers and by $\mathbb{Z}$ the set of all the complex numbers. We denote the Cartesian coordinates of the two-dimensional Euclidean plane $\mathbb{R}^2$ by (x_1, x_2) and identify a point $(x_1, x_2) \in \mathbb{R}^2$ with the complex number $z = x_1 + \mathrm{i}x_2 \in \mathbb{C}$.

We here consider the problem of a stationary two-dimensional Stokes flow past a one-dimensional periodic array of cylinders as shown in Figure 1). In the figure, D_n, $n \in \mathbb{Z}$ are the cylinders of the same shape which are arranged in a one-periodic array of period $\mathrm{i}a$ $(a > 0)$. In terms of mathematics, D_n, $n \in \mathbb{Z}$ are simply-connected domain in the complex plane $\mathbb{C}$ and mutually related by the equality

$$D_n = D_0 + ina = \{\, z + ina \mid z \in D_0 \,\}, \quad \forall n \in \mathbb{Z} \tag{1}$$

$\mathscr{D}$ is the region of the Stokes flow past an array of cylinders D_n

$$\mathscr{D} = \{\, z \in \mathbb{C} \mid |\operatorname{Re} z| < d, \quad z \notin \overline{D_n}, \ \forall n \in \mathbb{Z} \,\} \tag{2}$$

with a positive constant d, where the flow is assumed to be uniform sufficiently far from the obstacles (the cylinders D_n) and we will pose below the boundary condition that the flow is uniform at $\operatorname{Re} z = \pm d$.

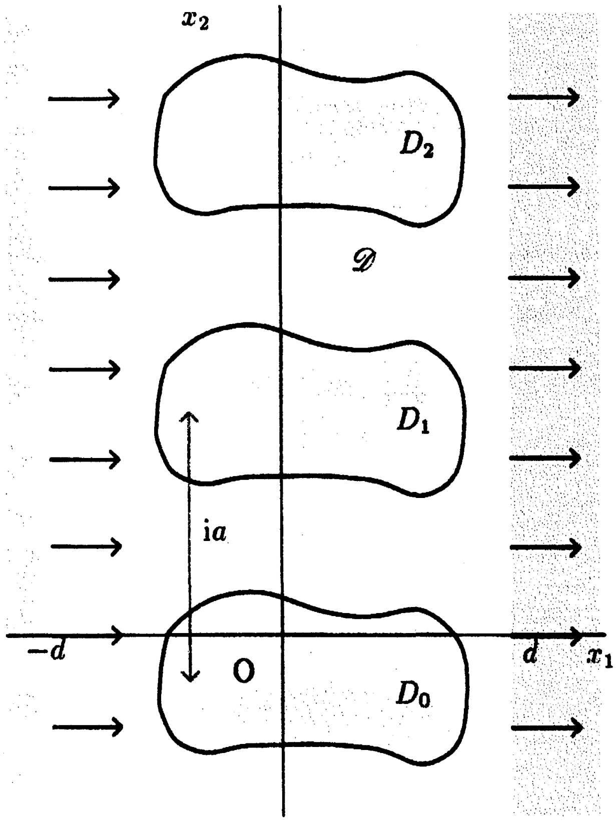

Figure 1: Two-dimensional Stokes flow past a one-dimensional periodic array of cylinders.

3 Fundamental Solution Method

Our problem of the periodic Stokes flow is given in terms of mathematics by the boundary value problem of the Stokes flow equation and the continuity equation

$$
\begin{array}{rlll}
\text{Stokes flow equation} & \mu\Delta \boldsymbol{v} - \nabla p = 0 & \text{in } \mathscr{D} & (3)\\
\text{continuity equation} & \nabla \cdot \boldsymbol{v} = 0 & \text{in } \mathscr{D} & (4)\\
\text{boundary conditions} & \boldsymbol{v} = \boldsymbol{0} & \text{on } \partial D_n \ (\, n \in \mathbb{Z}\,) & (5)\\
 & \boldsymbol{v} = (U, 0) & \text{on } x_1 = \pm d. & (6)
\end{array}
$$

From (4), there exists a stream function $\Psi(x_1, x_2)$ such that it gives the velocity $\boldsymbol{v} = (v_1, v_2)$ by

$$v_1 = \frac{\partial \Psi}{\partial x_2}, \quad v_2 = -\frac{\partial \Psi}{\partial x_1}. \tag{7}$$

From (3) and (7), we can easily find that

$$\Delta^2 Psi = 0, \tag{8}$$

that is, the stream function Ψ is a biharmonic function. Therefore, the stream function can be written as

$$\Psi(z) = \mathrm{Im}\left\{\overline{z}\varphi(z) + \int^z \psi(z')\mathrm{d}z'\right\} \quad (\, z = x_1 + \mathrm{i}x_2\,) \tag{9}$$

with analytic functions $\varphi(z)$, $\psi(z)$, which is called "Goursat's representation" [5]. Based on (9), the complex velocity is written as

$$W \equiv v_1 - \mathrm{i}v_2 = 2\mathrm{i}\frac{\partial \Psi}{\partial z} = \overline{z}\varphi'(z) - \overline{\varphi}(\overline{z}) + \psi(z). \tag{10}$$

As a fundamental solution of the Stokes flow equation (3) and the continuity equation (4), we know the "Stokeslet", the flow such that the analytic function $\varphi(z), \psi(z)$ in (9) is given by

$$\varphi(z) = -\, Q_0 \log(z - \zeta_0), \tag{11}$$

$$\psi(z) = \overline{Q_0}(z - \zeta_0)\log(z - \zeta_0) - Q_0\frac{z - 2\,\mathrm{Re}\,\zeta_0}{z - \zeta_0} \tag{12}$$

where Q_0 is a complex constant and ζ_0 is a fixed point in the complex plane, and the complex velocity is given by

$$W = 2\overline{Q_0}\log|z - \zeta_0| - 2Q_0\frac{\mathrm{Re}(z - \zeta_0)}{z - \zeta_0}. \tag{13}$$

In terms of physics, the Stokeslet is a Stokes flow due to a concentrated force $-8\pi\mu\boldsymbol{Q}_0 = -8\pi\mu(\operatorname{Re} Q_0, \operatorname{Im} Q_0)$ on the point ζ_0 in the complex plane. Therefore, in the fundamental solution method for ordinary Stokes flow problem in a domain $\mathscr{D}$, the analytic function $\varphi(z)$, $\psi(z)$ in (9) are approximated by

$$\varphi(z) \simeq -\sum_{j=1}^{N} Q_j \log(z-\zeta_j), \tag{14}$$

$$\psi(z) \simeq \sum_{j=1}^{N} \overline{Q_j}(z-\zeta_j)\left\{\log(z-\zeta_j)-1\right\}, \tag{15}$$

and, then, the complex velocity is approximated by

$$W \simeq 2\sum_{j=1}^{N} \overline{Q_j} \log|z-\zeta_j| - 2\sum_{j=1}^{N} Q_j \frac{\operatorname{Re}(z-\zeta_j)}{z-\zeta_j}. \tag{16}$$

In (14–16), ζ_j, $j=1,2,\ldots,N$ are the singularity points given in the exterior of $\mathscr{D}$ and $Q_1, Q_2, \ldots, Q_N$ are the complex coefficients to be determined so that the flow satisfies the boundary conditions in a sufficient accuracy. In terms of physics, the above approximation (14–16) illustrates the superposition of the Stokes flows due to the concentrated forces $-8\pi\mu\boldsymbol{Q}_j = -8\pi\mu(\operatorname{Re} Q_j, \operatorname{Im} Q_j)$, $j=1,2,\ldots,N$ on the point ζ_j, $j=1,2,\ldots,N$.

It is, however, difficult to approximate the solution of our problem by the ordinary fundamental solution method (14–16) because these approximation are not periodic functions. Therefore, we have to modify the above fundamental solution method so that it can approximate the periodic solutions of our problems. A primitive modification may be arranging the Stokeslets in a periodic array, that is, approximate the analytic functions $\varphi(z)$, $\psi(z)$ by

$$\varphi(z) \simeq -\sum_{n\in\mathbb{Z}}\sum_{j=1}^{N} Q_j \log(z-(\zeta_j+ina)), \tag{17}$$

$$\psi(z) \simeq \sum_{n\in\mathbb{Z}}\sum_{j=1}^{N}\left\{\overline{Q_j}\log(z-(\zeta_j+ina)) - Q_j\frac{z-2\operatorname{Re}(\zeta_j+ina)}{z-(\zeta_j+ina)}\right\}, \tag{18}$$

but the infinite sums in the above approximation are generally divergent as they are. Then, we modify the infinite sums so that they are convergent, namely,

$$\sum_{n\in\mathbb{Z}}\sum_{j=1}^{N} Q_j \log(z-(\zeta_j+ina)) \to \sum_{j=1}^{N} Q_j \left\{ \log(z-\zeta_j) + \sum_{n\neq 0}\left[\log\left(1-\frac{z-\zeta_j}{ina}\right)+\frac{z-\zeta_j}{ina}\right]\right.$$
$$= \sum_{j=1}^{N} Q_j \log\sinh\left[\frac{\pi}{a}(z-\zeta_j)\right], \qquad (19)$$

$$\sum_{n\in\mathbb{Z}}\sum_{j=1}^{N} Q_j \frac{z-2\,\mathrm{Re}(\zeta_j+ina)}{z-(\zeta_j+ina)} = \sum_{j=1}^{N} Q_j(z-2\,\mathrm{Re}\,\zeta_j)\sum_{n\in\mathbb{Z}}\left(\frac{1}{z-\zeta_j-ina}+\frac{1}{ina}\right)$$
$$\to \sum_{j=1}^{N} Q_j(z-2\,\mathrm{Re}\,\zeta_j)\left\{\frac{1}{z}+\sum_{n\neq 0}\left(\frac{1}{z-\zeta_j-ina}+\frac{1}{ina}\right)\right\}$$
$$= \frac{\pi}{a}\sum_{j=1}^{N} Q_j(z-2\,\mathrm{Re}\,\zeta_j)\coth\left[\frac{\pi}{a}(z-\zeta_j)\right], \qquad (20)$$

and we have

$$\varphi(z) \simeq \varphi_N(z) \equiv -\sum_{j=1}^{N} Q_j \log\sinh\left[\frac{\pi}{a}(z-\zeta_j)\right], \qquad (21)$$

$$\psi(z) \simeq \psi_N(z) \equiv \sum_{j=1}^{N} \overline{Q_j}\log\sinh\left[\frac{\pi}{a}(z-\zeta_j)\right] - \frac{\pi}{a}\sum_{j=1}^{N} Q_j(z-2\,\mathrm{Re}\,\zeta_j)\coth\left[\frac{\pi}{a}(z-\zeta_j)\right]. \qquad (22)$$

These give an approximate complex velocity by

$$W \simeq W_N \equiv u_1^{(N)} - iu_2^{(N)}$$
$$\equiv 2\sum_{j=1}^{N} \overline{Q_j}\log\left|\sinh\left[\frac{\pi}{a}(z-\zeta_j)\right]\right| - \frac{2\pi}{a}\sum_{j=1}^{N} Q_j\,\mathrm{Re}(z-\zeta_j)\coth\left[\frac{\pi}{a}(z-\zeta_j)\right]. \qquad (23)$$

The approximation (21–23) is suitable for our problem of periodic Stokes flow because it is expressed by periodic functions of period ia, and it is expected to inherit the advantages of the ordinary fundamental solution method that it is easy to compute and it gives approximation with high accuracy. This approximation is given by a linear combination of the 'Mperiodic Stokeslet", the periodic fundamental solutions of the Stokes flow equation and the continuity equation and, in terms of physics, it illustrates the superpositions of the flows due to an infinite periodic array of concentrated forces $-8\pi\mu\boldsymbol{Q}_j = -8\pi\mu(\operatorname{Re} Q_j, \operatorname{Im} Q_j)$, $j = 1, 2, \ldots, N$ on the points $\zeta_j + ina$, $n \in \mathbb{Z}$.

The complex coefficients Q_j are determined by the collocation condition, the condition that the approximate velocity (23) satisfies the boundary conditions (6) only at a finite number of boundary points. Namely, we choose boundary points

$$
\begin{gathered}
z_1^{(0)}, z_2^{(0)}, \ldots, z_{N_0}^{(0)} \in \partial D_0, \\
z_1^{(+)}, z_2^{(+)}, \ldots, z_{N_+}^{(+)} \in \{\, z \in \mathbb{C} \mid \operatorname{Re} z = d \,\}, \quad z_1^{(-)}, z_2^{(-)}, \ldots, z_{N_-}^{(-)} \in \{\, z \in \mathbb{C} \mid \operatorname{Re} z = -d \,\} \\
(\, N_0 + N_+ + N_- = N \,)
\end{gathered}
\tag{24}
$$

and pose the boundary contitions (6) on W_N at the above points

$$W_N(z_i^{(0)}) = 0 \quad i = 1, 2, \ldots, N_0, \tag{25}$$

$$W_N(z_i^{(+)}) = U \quad i = 1, 2, \ldots, N_+, \tag{26}$$

$$W_N(z_i^{(-)}) = U \quad i = 1, 2, \ldots, N_-. \tag{27}$$

The equations (25–27) form a system of linear equations with respect to the coefficients $Q_1, Q_2, \ldots, Q_N$. We determine the coefficients Q_j by solving the above system of linear equations and obtain the approximate velocity W_N.

4 Numerical Example

We here show a numerical example of the presented method. All the computations were carried out using programs coded in C++ with double precision working.

The example is the problem of Stokes flow past a periodic array of circular cylinders, that is, the Stokes flow problem in the domain

$$\mathscr{D} = \{\, z \in \mathbb{C} \mid |\operatorname{Re} z| < d, \quad z \notin D_n, \ \forall n \in \mathbb{Z} \,\} \tag{28}$$

with

$$D_n = \{ z \in \mathbb{C} \mid |z - na| < r \}, \quad n \in \mathbb{Z}. \tag{29}$$

The collocation points and the singularity points are taken as

$$z_i^{(0)} = r \exp\left(\mathrm{i}\frac{2\pi(i-1)}{N_0}\right), \; \zeta_i^{(0)} = 0.5r \exp\left(\mathrm{i}\frac{2(i-1)}{N_0}\right), \quad i = 1, 2, \ldots, N_0, \tag{30}$$

$$z_i^{(\pm)} = \pm d + \mathrm{i}\left(-\frac{a}{2} + \frac{a(i-1/2)}{N_0}\right), \; \zeta_i^{(\pm)} = \pm\frac{3}{2}d + \mathrm{i}\left(-\frac{a}{2} + \frac{a(i-1/2)}{N_0}\right), \quad i = 1, 2, \ldots, I \tag{31}$$

Figure 2 illustrates the velocity field of the flow obtained by the presented fundamental solution method.

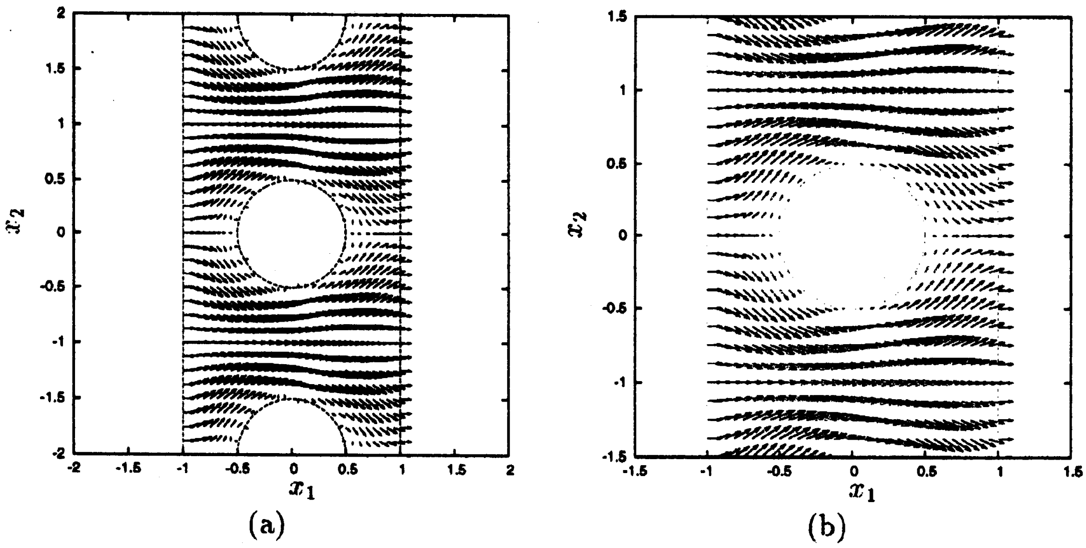

Figure 2: The velocity field of Stokes flow past a periodic array of circular cylinders computed by the presented method. The figure (b) is a magnification of the figure (a).

In order to estimate the accuracy of the presented method, we computed the error on the boundaries

$$\epsilon(\text{circle}) = \sup_{z \in \partial D_0} \frac{|W_N(z)|}{U}, \quad \epsilon(\text{left}) = \sup_{\mathrm{Re}\, z = -d} \frac{|W_N(z) - U|}{U}, \quad \epsilon(\text{right}) = \sup_{\mathrm{Re}\, z = d} \frac{|W_N(z) - U}{U} \tag{32}$$

where the suprima are actually computed as the maxima on 1000 boundary points distributed by using the uniform random numbers. Figure 3 shows the error estimates (32) as the total numbers of the nodes $N = 3N_0$, though we cannot distinguish between the error on the left boundary ϵ(left) and the error on the right boundary ϵ(right). From this figure, we find that the errors on the left and right boundaries ϵ(left), ϵ(right) are of the order of the square of the error on the surfaces of the cylinders ϵ(circle) with the same number of the nodes, and the total error decays exponentially as the number of nodes $N = 3N_0$ increases.

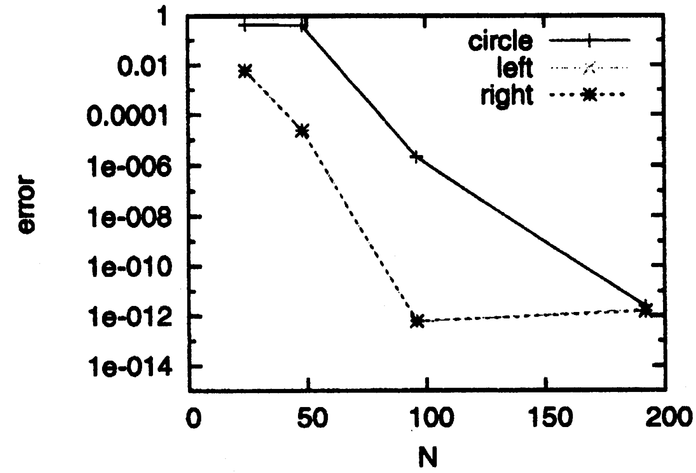

Figure 3: The error estimates of the presented method as functions of the total number of the nodes $N = 3N_0$.

5 Concluding Remarks

In this paper, we presented the fundamental solution method for the problems of two-dimensional Stokes flow past a one-dimensional periodic array of cylinders. We obtained our method by using the biharmonic function theory based on analytic functions and by modifying the Stokeslet included in the approximation so that the method gives a good approximation of the solution including periodic functions. The numerical examples for a problem with circular cylinders shows an exponential convergence of our method.

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