Construction of smooth actions on spheres for Smith equivalent representations

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1. PROBLEMS AND RESULTS

Throughout this paper, let $G$ be a finite group. A real $G$-representation of finite dimension is meant by a real $G$-module, a smooth manifold is meant by a manifold, and a smooth $G$-manifold is meant by a $G$-manifold. For a $G$-manifold $X$, let $\mathcal{T}\mathcal{R}(X)$ denote the set of all isomorphism classes (as real $G$-modules) of tangential representations $T_x(X)$, where $x$ runs over the $G$-fixed point set $X^G$. We are interested in $\mathcal{T}\mathcal{R}(X)$ for manifolds $X$ such that $X^G$ consists of exactly two points. In particular, the case where $X$ are homotopy spheres has been studied as Smith Problem.

Smith Problem. Let $\Sigma$ be a homotopy sphere with $G$-action such that the $G$-fixed point set consists of exactly two points $a, b$. Are the tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ isomorphic to each other (namely $|\mathcal{T}\mathcal{R}(\Sigma)| = 1$)?

We have affirmative answers (e.g. Atiyah-Bott, Milnor, Sanchez) as well as negative answers (e.g. Petrie, Cappell-Shaneson, Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawalowski, Pawalowski-Solomon), to Smith Problem under various hypotheses. There are surveys relevant to studies on Smith Problem in [24], [18] and [6].

To study the problem, we define the following relations $\sim_{\Delta}$, $\sim_{\emptyset}$ and $\sim_{\Delta \emptyset}$. In the definition below, $V$ and $W$ are real $G$-modules.

1. $V \sim_{\Delta} W$ if there exists a disk $D$ with $G$-action such that $D^G = \{a, b\}$ and $[[V], [W]] = \mathcal{T}\mathcal{R}(D)$.
2. $V \sim_{\emptyset} W$ if there exists a homotopy sphere $\Sigma$ with $G$-action such that $\Sigma^G = \{a, b\}$ and $[[V], [W]] = \mathcal{T}\mathcal{R}(\Sigma)$.
3. $V \sim_{\Delta \emptyset} W$ if $V \sim_{\Delta} W$ and $V \sim_{\emptyset} W$.

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Here $\sim_\mathfrak{D}$ and $\sim_\mathfrak{S}$ may not be equivalence relations, although they stably yield equivalence relations. We have been interested in the relation $\sim_\mathfrak{S}$ (namely the Smith equivalence), but in the present paper we will mainly pay our attention to the relation $\sim_\mathfrak{D}\mathfrak{S}$.

Let $RO(G)$ denote the real representation ring. We define the subsets $\mathfrak{D}(G)$, $\mathfrak{S}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ of $RO(G)$ by

$$\mathfrak{D}(G) = \{ V - W \in RO(G) \mid V \sim_\mathfrak{D} W \}$$

$$\mathfrak{S}(G) = \{ V - W \in RO(G) \mid V \sim_\mathfrak{S} W \}$$

$$\mathfrak{D}\mathfrak{S}(G) = \mathfrak{D}(G) \cap \mathfrak{S}(G)$$

The set $\mathfrak{S}(G)$ was usually denoted by $Sm(G)$. By R. Oliver [16], there exists a disk with $G$-action with $|D^G| = 2$ if and only if $G$ is an Oliver group (namely, $G$ is not a mod $\mathcal{P}$ hyperelementary group). Thus it is worthwhile to study $\mathfrak{D}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ only for Oliver groups $G$.

If $M$ is a subset of $RO(G)$ then for families $\mathcal{A}$, $\mathcal{B}$ consisting of subgroups of $G$ we define

$$M^\mathcal{A} \overset{\text{def}}{=} \{ x \in M \mid \text{res}_H^G x = 0 \ \forall \ H \in \mathcal{A} \}$$

$$M^\mathcal{B} \overset{\text{def}}{=} \{ x = V - W \in M \mid V^K = 0 = W^K \ \forall \ K \in \mathcal{B} \}$$

$$M^\mathcal{B}_\mathcal{A} \overset{\text{def}}{=} \{ x = V - W \in M^\mathcal{A} \mid V^K = 0 = W^K \ \forall \ K \in \mathcal{B} \}.$$ 

Using the notation with the families

$$\mathcal{P} = \mathcal{P}(G) \overset{\text{def}}{=} \{ P \leq G \mid |P| = p^a \ (p \text{ a prime}) \}$$

$$\mathcal{N}_2 = \mathcal{N}_2(G) \overset{\text{def}}{=} \{ N \leq G \mid |G/N| = 1, 2 \}$$

$$\mathcal{N} = \mathcal{N}(G) \overset{\text{def}}{=} \{ N \leq G \mid |G/N| = 1 \text{ or a prime} \}$$

$$\mathcal{L} = \mathcal{L}(G) \overset{\text{def}}{=} \{ L \leq G \mid L \supseteq G^{\{p\}} \text{ for some prime } p \},$$

we study the subsets $\mathfrak{D}(G)$, $\mathfrak{S}(G)$ and $\mathfrak{D}\mathfrak{S}(G)$ of $RO(G)$. Here the group $G^{\{p\}}$ is the smallest normal subgroup of $G$ with prime power index, namely

$$G^{\{p\}} = \bigcap_{H \leq G : |G/H| = p^a \text{ for some } a} H.$$

An element in $\mathcal{L}$ defined above is called a large subgroup of $G$.

Many authors (e.g. Petrie-Randall, Petrie-Dovermann, Dovermann-Washington, Dovermann-Suh, Laitinen-Pawalowski, Pawalowski-Solomon) found various pairs $(V, W)$ of nonisomorphic $\mathfrak{D}\mathfrak{S}$-related real $G$-modules $V$, $W$. But their $(V, W)$ with $V \sim_\mathfrak{D}\mathfrak{S} W$ satisfy $V^N = 0 = W^N$ for all $N \triangleleft G$ with prime index. In other words, they showed

$$\mathfrak{D}\mathfrak{S}(G)^\mathcal{N} \neq 0.$$
for various $G$. Now we recall the next proposition.

**Proposition 1** ([12], [13]). The implications $\mathfrak{G}(G) \subseteq \text{RO}(G)^{N_2}_Q$ and $\mathfrak{D}\mathfrak{G}(G) \subseteq \text{RO}(G)^{N_2}_P$ hold.

These facts motivate us to study the following problem.

**Problem A.** Does there exist a finite group $G$ satisfying $\mathfrak{D}\mathfrak{G}(G) \neq \mathfrak{D}\mathfrak{G}(G)^N$?

The notion *gap module* is convenient to study this problem as well as Smith Problem. A real $G$-module $V$ is called a *gap module* if it satisfies the following conditions.

1. $V^L = 0$ for all $L \in \mathcal{L}(G)$.
2. $\dim V^P > \dim V^H$ for all pairs $(P, H)$ of subgroups of $G$ such that $P \in \mathcal{P}(G)$ and $H > P$.

A finite group $G$ is called a *gap group* if $G$ admits a gap real $G$-module. Pawalowski-Solomon showed in [18] that for an arbitrary nonsolvable gap group $G$ with $a_G \geq 2$ and $G \not\cong P\Sigma L(2,27)$,

$$\mathfrak{D}\mathfrak{G}(G) \supseteq \text{RO}(G)^{\mathcal{L}}_P \neq 0.$$

Since the appearance of this result, the next problem has been asked.

**Problem B.** Are the sets $\mathfrak{G}(G)$ and $\mathfrak{D}\mathfrak{G}(G)$ nontrivial in the case $G = P\Sigma L(2,27)$?

The purpose of the present paper is to answer to Problems A and B, and we obtained the following results.

**Theorem 2.** For each odd prime $p$, there exist a gap Oliver group $G$ and real $G$-modules $V$ and $W$ such that $V \sim \mathfrak{D}\mathfrak{G} W$, $\dim V^N > 0$ and $\dim W^N = 0$ for some $N \triangleleft G$ with $|G/N| = p$, hence $\mathfrak{D}\mathfrak{G}(G) \neq \mathfrak{D}\mathfrak{G}(G)^N$.

Let $SG(m, n)$ denote the small group of order $m$ and type $n$ appearing in the computer software GAP [5].

**Theorem 3.** Let $G = P\Sigma L(2,27)$, $SG(864, 2666)$, or $SG(864, 4666)$. Then $\text{RO}(G)^G = 0$ but

$$\mathfrak{G}(G) = \mathfrak{D}(G) = \mathfrak{D}\mathfrak{G}(G) = \text{RO}(G)^{\mathcal{P}}_G \cong \mathbb{Z}.$$

2. **Additional Information**

For $g \in G$, let $(g)$ denote the conjugacy class of $g$ in $G$. The *real conjugacy class* $(g)^{\pm}$ of $g$ is the union of $(g)$ and $(g^{-1})$. Let $a_G$ denote the number of all real conjugacy classes...
of elements $g$ of $G$ such that $g$ does not have prime power order. By the representation theory, we have
\[ a_G = \text{rank } RO(G)_P. \]
Let $\delta$ denote the homomorphism from $RO(G)_P$ to $\mathbb{Z}$ given by
\[ \delta([V] - [W]) = \dim V^G - \dim W^G. \]
Then by definition,
\[ RO(G)^{(G)}_P = \text{Ker}[\delta : RO(G)_P \to \mathbb{Z}]. \]
B. Oliver [17] showed that if $a_G \geq 1$ then
\[ \text{Image}[\delta : RO(G)_P \to \mathbb{Z}] \supseteq 2\mathbb{Z}. \]
Thus the next proposition immediately follows.

**Proposition** (Laitinen-Pawalowski [8]). If $a_G \geq 1$ then $\text{rank } RO(G)^{(G)}_P = a_G - 1$.

In addition, B. Oliver [17] implies the next result.

**Theorem** (Oliver). If $G$ is an Oliver group then $\mathfrak{D}(G) = RO(G)^{(G)}_P$.

Viewing these facts, E. Laitinen conjectured the next.

**Laitinen’s Conjecture.** If $G$ is an Oliver group with $a_G \geq 2$ then $\mathfrak{D}^*(G) \neq 0$.

This conjecture had been positively expected until 2006. We, however, have a negative example.

**Theorem 4** ([12], [13]). Let $G = \text{Aut}(A_6)$. Then Laitinen’s Conjecture fails, in fact $a_G = 2$ and $\mathfrak{S}(G) = 0 = \mathfrak{D}^*(G)$.

Most finite Oliver groups are gap groups, but neither $S_5$ nor $\text{Aut}(A_6)$ is a gap group, where $S_5$ is the symmetric group on five letters and $A_6$ is the alternating group on six letters. Pawalowski-Solomon [18] showed the next theorem using a deleting-inserting theorem of $G$-fixed point sets to disks ([10], [15, Appendix]).

**Theorem** (Pawalowski-Solomon [18]). If $G$ is a gap Oliver group then
\[ \text{RO}(G)^{(G)}_P \subseteq \mathfrak{D}^*(G). \]
On the other hand, they also showed the next result using the finite group theory.

**Theorem** (Pawalowski-Solomon [18]). Let $G$ be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong \text{PSL}(2,27)$ then
\[ RO(G)^{(G)}_P \neq 0. \]
Putting these results together, we obtain a corollary.

**Corollary** (Pawalowski-Solomon [18]). Let $G$ be a nonsolvable gap group with $a_G \geq 2$. If $G \not\cong P\Sigma L(2,27)$ then $\mathcal{D}\mathcal{S}(G) \neq 0$.

Since $S_5 \times C_2$, where $C_2$ is the cyclic group of order 2, is not a gap group, the next result is also interesting.

**Theorem** (X.M. Ju [6]). In the case $G = S_5 \times C_2$, the equalities
\[ \mathcal{S}(G) = \mathcal{D}\mathcal{S}(G) = \text{RO}(G)^{\mathcal{P}} \cong \mathbb{Z} \]
hold.

We obtained a deleting-inserting theorem [14] of new kind by employing an equivariant interpretation of Cappell-Shaneson's surgery obstruction theory for getting homology (possibly, not homotopy) equivalences as well as employing the induction theory of Wall's surgery obstruction groups. We state here the theorem in a simplified form.

**Theorem 5.** Let $G$ be an Oliver group and $Y$ a disk with $G$-action. Suppose the following conditions are satisfied.

1. $Y^G = \{y_1, \ldots, y_m\}$, where $m \geq 1$.
2. $\partial Y^L = \emptyset$ for all $L \in \mathcal{L}(G)$.
3. $\dim Y^H \geq 5$ for all mod $\mathcal{P}$ cyclic subgroups $H$, i.e. \( 1 \triangleleft P \triangleleft H \text{ cyclic} \).
4. $\dim Y^P > 2(\dim Y^H + 1)$ for all $P \in \mathcal{P}(G)$ and $H > P$.
5. $|\pi_1(Y^P)| < \infty$ and $(|\pi_1(Y^P)|, |P|) = 1$ for all $P \in \mathcal{P}(G)$.
6. The inclusion induced maps $\pi_1(\partial Y^P) \rightarrow \pi_1(Y^P)$ are isomorphisms for all $P \in \mathcal{P}(G)$.

Then there exists a disk $X$ with $G$-action such that $\partial X = \partial Y$ and $X^G = \emptyset$.

Remark that the union $\Sigma = X \cup_{\partial} Y$ identified along the boundaries of $X$ and $Y$ in the theorem above is a homotopy sphere such that $\mathcal{T}\mathcal{R}(\Sigma) = \mathcal{T}\mathcal{R}(Y)$. Since various $G$-actions on disks $Y$ are constructed by Oliver's theory [17], we would obtain $G$-actions on homotopy spheres $\Sigma$ from those on disks. In fact, the next result is an outcome of Theorem 5.

**Theorem 6.** Let $p$ be an odd prime. Let $G$ be an Oliver group such that $G = G^{(q)}$ for all primes $q \neq p$ and $|G/G^{(p)}| = p$. If $G$ has a dihedral subquotient $D_{2qr}$ (order $2qr$) with distinct primes $q$ and $r$ and further that $G$ contains distinct real $G$-conjugacy classes
of elements $x$, $y$ not of prime power order, then $\mathfrak{D}\mathfrak{S}(G)$ contains a direct summand of $\text{RO}(G)$ of rank 1.

Theorems 2 and 3 follow from Theorem 6. In addition, we conclude the next.

Theorem 7. Laitinen's Conjecture is affirmative for any finite nonsolvable gap group.

REFERENCES