Assembly in Surgery

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1. Introduction

In [Y], I discussed glueing and splitting operations of geometric quadratic Poincaré complexes, and studied the \( L^{-\infty} \)-theory assembly map

\[
A : H_*(X; L^{-\infty}(p : E \rightarrow X)) \rightarrow L^{-\infty}(\pi_1 E)
\]

for certain polyhedral stratified systems of fibrations \( p : E \rightarrow X \), following the general description of assembly maps by Quinn [Q, §8]. This assembly map was constructed in two steps; first we used the gluing operation to construct a map

\[
\alpha : H_*(X; L^{-\infty}(p : E \rightarrow X)) \rightarrow L_*^{-\infty}(p)
\]

from the homology to the controlled \( L \)-group, and then composed it with the forget-control map

\[
F : L_*^{-\infty}(p : E \rightarrow X) \rightarrow L_*^{-\infty}(E \rightarrow \{\ast\}) = L_*^{-\infty}(\pi_1 E).
\]

The following was claimed in (3.9) of [Y].

**Theorem.** If \( p : E \rightarrow X \) is a polyhedral stratified system of fibrations on a finite polyhedron \( X \), then the map \( \alpha \) is an isomorphism.

The map \( \alpha \) was constructed in the following way: an element of \( H_k(K; L^{-\infty}(p : E \rightarrow X)) \) can be thought of as a \( PL \)-triangulation \( V \) of the product \( S^N \times \Delta^k \) of a sphere \( S^N \) (\( N \) large) and the \( k \)-simplex \( \Delta^k \) together with

1. a simplicial map \( \phi : V \rightarrow X \), and
2. a compatible family \( \{\rho(\Delta) \mid \Delta \in V \} \), where \( \rho(\Delta) \) is a quadratic Poincaré \((\dim \Delta + 2)\)-ad on the pullback \( q \) of \( \bar{\rho} : \mathbb{R}^l \times E \rightarrow E \rightarrow X \) via the map \( \Delta \rightarrow V \rightarrow X \), and \( \rho(\Delta) \) is 0 if \( \Delta \) is a simplex in the boundary.

I claimed that these ads \( \rho(\Delta) \)'s can be glued together to give a geometric quadratic Poincaré complex on \( q \):
**Theorem** (Glueing over a manifold) [Y, 2.10] Let $L$ be the barycentric subdivision of a PL-triangulation $K$ of a compact $n$-dimensional manifold $M$ possibly with a non-empty boundary $\partial M$ and $p : E \rightarrow M$ be a map. And suppose each $n$-simplex $\Delta \in L$ is given an $m$-dimensional geometric quadratic Poincaré $(n+2)$-ad on $(p^{-1}(\Delta), p^{-1}(\partial \Delta))$ which are compatible on common faces. Then one can glue them together to get an $m$-dimensional geometric quadratic Poincaré pair on $(E, p^{-1}(\partial M))$.

If this is possible, then its functorial image on $\overline{p}$ gives a geometric quadratic complex on $\overline{p}$. By the 'barycentric subdivision argument' [Y, p.589], this assembled complex is equivalent to arbitrarily small complex and defines an element of $L_{-\infty}^\ast(p)$.

Unfortunately the argument given in [Y] is insufficient to prove this. The aim of this short note is to describe how to remedy this.

2. Glueing over a manifold

In [Y], I described the glueing operation of two quadratic Poincaré pairs along a common codimension 0 subcomplex of the boundaries. If there is an order of the $n$-simplices $\Delta_1, \ldots, \Delta_r$ of $L$ so that $(\Delta_1 \cup \ldots \cup \Delta_i) \cap \Delta_{i+1}$ is the union of $(n - 1)$-simplices for each $i$, then we can successively glue the pieces in this linear order. But this seems very difficult to achieve. The strategy used in [Y] is the following:

For each vertex $v$ of $K$, consider its star $S(v)$ in $L$, i.e. the dual cone of $v$. Two such dual cones are either disjoint or meet along codimension 1 cells. The glueing problem over $S(v)$ can be solved by looking at the link $L(v)$ of $v$ in $L$. Note that $L(v)$ is an $(n - 1)$-dimensional sphere or disk and the triangulation is the first barycentric subdivision of another. Thus we can keep on reducing the dimension until the link becomes a circle or an arc, and in this case there is an obvious order of 2-simplices and glueing can be done.

The fact is that the induction fails, since any two $n$-simplices of $S(v)$ have the vertex $v$ in common and are never disjoint.
There are two possible remedies for this. The first one is to use a different definition for the homology groups. This was actually done in [R].

Here I propose another remedy. Let us look at the dual cone at the vertex $v$. Let $c$ denote the quadratic Poincaré complex lying over $v$. Split each of the pieces of the dual cone so that the pieces near $v$ are of the form $c \otimes (a$ small simplex):

Here we do not need stabilization to split. We would like to glue the pieces away from $v$ first, and then fill in the hole with a piece of the form $c \otimes (a$ small copy of the dual cone):
To carry out the induction steps, we need to deal with holes of more complicated forms, and I have not worked out the details yet.

Remarks. (1) The control map should be a polyhedral stratified system of fibrations.
(2) The picture above may be misleading. The 'hole' itself lies over the vertex $v$, because $c \otimes$ (a small copy of the dual cone) can only live over $v$.
(3) Splitting needs a similar treatment.

References