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Borsuk-Ulam Theorems for Set-valued Mappings

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1 Introduction

S.Eilenberg and D. Mongomery [2] gave the fixed point formula of acyclic mappings which is a generalization of Lefschetz's fixed point theorem. L. Górniewicz [6] has studied set-valued mappings and fixed point theorems for acyclic mappings. In this paper, the author shall give a proof of a coincidence theorem for a Vietoris mapping and a compact mapping and prove Borsuk-Ulam type theorems for a class of set-valued mappings.

When a closed subset $\varphi(x)$ in Y is assigned for a point x in X, we say that the correspondence is a set-valued mapping and write $\varphi : X \to Y$ by the Greek alphabet. For single-valued mapping, we write $f : X \to Y$ etc. by the Roman alphabet. A set-valued mapping is studied particularly in Chapter 2 in [6]. We assume that any set-valued mapping is upper semi-continuous.

The following theorem is our main theorem (cf. Theorem 2.7). From the theorem we obtain the fixed point theorem for admissible mapping.

Main Theorem 1. *Let X be an ANR space and Y a paracompact Hausdorff space. Let $p : Y \to X$ be a Vietoris mapping and $q : Y \to X$ be a compact mapping. Then $(p^*)^{-1}q^*$ is a Leray endomorphism. If the Lefschetz number $L((p^*)^{-1}q^*)$ is not zero, there exists a coincidence point $z \in Y$, that is, $p(z) = q(z)$.*

Borsuk-Ulam type theorems are proved in the following theorems which are the generalizations of Theorem 43.10 in L.Górniewicz [6]. (cf. Theorem 3.5, Theorem 3.9). The author shall give the related results and the detail proofs in [13].

Main Theorem 2. *Let N be a paracompact Hausdorff space with a free involution T and M an m-dimensional closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is $*$-admissible and satisfies $\varphi^* = 0$ for positive dimension and $c(N,T)^m \neq 0$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover if N is an n-dimensional closed topological manifold, it holds $\dim A(\varphi) \geqq n-m$ where $A(\varphi) = \{x \in N \mid \varphi(x) \cap \varphi(T(x)) \neq \emptyset\}$.*

Main Theorem 3. *Let N be a closed topological manifold with a free involution T which has the homology group of the n-dimensional sphere and M be a closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is admissible and $\varphi(N) \neq M$ and $n \geqq m$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover it holds $\dim A(\varphi) \geqq n-m$ and $\mathrm{Ind}A(\varphi) \geqq n-m$.*

2 Coincidence Theorem

We give some remarks about several cohomology theories. Alexander-Spanier cohomology theory $\bar{H}^*(-)$ is isomorphic to the singular cohomology theory $H^*(-)$ (cf. Theorem 6.9.1 in [14]), that is,

$$\bar{H}^*(X) \cong H^*(X)$$

if the singular cohomology theory satisfies the continuity: $\lim_{\overrightarrow{\{U\}}} H^*(U) = H^*(x)$ where $\{U\}$ is a system of neighborhood of x.

For a paracompact Hausdorff space X, it holds also the isomorphism between Čech cohomology theory $\check{H}^*(-)$ with a constant sheaf and Alexander cohomology theory $\bar{H}^*(-)$ (cf. Theorem 6.8.8 in [14])

$$\check{H}^*(X) \cong \bar{H}^*(X).$$

For a locally compact subset A of Euclidean neighborhood retract X (cf. Chapter 4 in [1]), it holds also the isomorphism between Čech cohomology theory $\check{H}^*(-)$ and the singular cohomology theory $H^*(-)$

$$\check{H}^*(A) = \lim_{\overrightarrow{\{U\}}} H^*(U)$$

where U is a neighborhood of A in X. For Euclidean neighborhood retract X, it holds also the isomorphism $\check{H}^*(X) \cong H^*(X)$. Hereafter we use Alexander-Spanier (co)homology theory with a field as the coefficient and use the notation $H^*(X)$ instead of $\bar{H}^*(X)$. When we have to distinguish them, we use the corresponding notation.

For a covering $\mathcal{U}$ of X, the simplicial complex $K(\mathcal{U})$ called the nerve of $\mathcal{U}$ is defined in §1.6 of Chapter 3 in [14] and the simplicial complex $X(\mathcal{U})$ called the Vietoris simplicial complex of $\mathcal{U}$ is defined in §5 of Chapter 6 in [14]. They are chain equivalent each other (cf. Exercises D of Chapter 6 in [14]). Clearly by the definition of Alexander cohomology theory, we have the isomorphism:

$$\lim_{\overrightarrow{\{\mathcal{U}\}}} H^*(C^*(X(\mathcal{U})) \cong \bar{H}(X).$$

We have the cross products $\bar{\mu} : \bar{H}^*(X,A) \otimes \bar{H}^*(Y,B) \to \bar{H}^*((X,A) \times (Y,B))$ and $\mu : H^*(X,A) \otimes H^*(Y,B) \to H^*((X,A) \times (Y,B))$ and the natural transformation $\tau : \bar{H}(-) \to H^*(-)$ which satisfy the commutativity $\mu(\tau \otimes \tau) = \tau\bar{\mu}$.

In this paper, we shall work in the category of paracompact Hausdorff spaces and continuous mappings. We give some definitions and notation. Let $w_K^U \in H_n(U, U-K)$ be the cycle such that $(i_x)_*(w_K^U) = w_x \in H_n(\mathbf{R}^n, \mathbf{R}^n - x)$ where $i_x : (U, U-K) \to (\mathbf{R}^n, \mathbf{R}^n - x)$. Define $\gamma_0 \in H^n(\mathbf{R}^n, \mathbf{R}^n - 0)$ the dual cocycle of w_0.

Definition 1. *Define a class $\gamma_K^U \in H^n((U, U-K) \times K)$ by $\gamma_K^U = d^*(\gamma_0)$ where $d : (U, U-K) \times K \to (\mathbf{R}^n, \mathbf{R}^n - 0)$ defined by $d(x,y) = x - y$.*

Definition 2. *A mapping $f : X \to Y$ is called a Vietoris mapping, if it satisfies the following conditions:*

1. *f is proper and onto continuous mapping.*
2. *$f^{-1}(y)$ is an acyclic space for any $y \in Y$, that is, $\tilde{H}^*(f^{-1}(y) : G) = 0$.*

When f is closed and onto continuous mapping and satisfies the condition (2), *we call it weak Vietoris mapping (abbrev. w-Vietoris mapping).*

Note that a proper mapping is closed mapping. We need Alexander-Spanier cohomology for the proof of the Vietoris theorem (cf. Theorem 6.9.15 in [14]).

Theorem 2.1 (Vietoris). *Let $f : X \to Y$ be a w-Vietoris mapping between paracompact Hausdorff spaces X and Y. Then,*

$$f^* : H^m(Y : G) \to H^m(X : G)$$

is an isomorphism for all $m \geqq 0$.

A mapping $f : X \to Y$ is called a compact mapping, if $f(X)$ is contained in a compact set of Y, or equivalently its closure $\overline{f(Y)}$ is compact.

Definition 3. *Let U an open set of the n-dimensional Euclidean space $\mathbf{R}^n$ and Y be a paracompact Hausdorff space. For a w-Vietoris mapping $p : Y \to U$ and a compact mapping $q : Y \to U$, the coincidence index $I(p,q)$ of p and q is defined by*

$$I(p,q)w_0 = \bar{q}_*(\bar{p})_*^{-1}(w_K^U)$$

where K is a compact set satisfying $q(Y) \subset K \subset U$, and $\bar{p} : (Y, Y - p^{-1}(K)) \to (U, U - K)$ and $\bar{q} : (Y, Y - p^{-1}(K)) \to (\mathbf{R}^n, \mathbf{R}^n - 0)$ are defined by $\bar{p}(y) = p(y)$ and $\bar{q}(y) = p(y) - q(y)$ respectievly.

Lemma 2.2. *It holds a formula:*

$$d_*(1 \times q_*(p_*)^{-1})\Delta_*(w_K^U) = I(p,q)w_0$$

where $\Delta(x) = (x,x)$, $d(x,y) = x - y$.

In this section, we give a proof of the coincidence theorem which is different from L.Górniewicz [5, 6] and depends on the line of M. Nakaoka [8]. The following theorem is easily verified.

Theorem 2.3. *Let U be an open set of the n-dimensional Euclidean space $\mathbf{R}^n$ and Y a paracompact Hausdorff space. For $p : Y \to U$ a w-Vietoris mapping and $q : Y \to U$ a compact mapping, if the index $I(p,q)$ is not zero, there exists a coincidence point $z \in Y$, that is, $p(z) = q(z)$.*

Let $\mathbf{V}$ be a vector space and $f : \mathbf{V} \to \mathbf{V}$ a linear mapping. Let f^k be the k time iterated composition of f. Set $N(f) = \cup_{k \geq 0} \ker f^k$ a subspace of $\mathbf{V}$ and $\tilde{\mathbf{V}} = \mathbf{V}/N(f)$. Then f induces the linear mapping $\tilde{f} : \tilde{\mathbf{V}} \to \tilde{\mathbf{V}}$ which is a monomorphism. When $\dim \tilde{\mathbf{V}} < \infty$, we define $\mathrm{Tr}(f)$ by $\mathrm{Tr}(\tilde{f})$. In the case of $\dim \mathbf{V} < \infty$, it coincides with the classical one $\mathrm{Tr}(f)$.

Definition 4. *Let $\{\mathrm{V}_k\}_k$ be a graded vector space and $f = \{f_k : \mathrm{V}_k \to \mathrm{V}_k\}_k$ graded linear mapping. Define the generalized Lefschetz number for the case of $\sum_{k\geqq 0} \dim \tilde{\mathrm{V}}_k < \infty$:*

$$L(f) = \sum_{k\geqq 0}(-1)^k \mathrm{Tr}(f_k)$$

In this case, $f = \{f_k\}_k$ is called a Leray endomorphism.

Lemma 2.4. *In the following commutative diagram of graded vector spaces:*

$$\begin{tikzcd} \mathbf{V}_k \arrow[r, "\phi_k"] \arrow[d, "f_k"'] & \mathbf{W}_k \arrow[d, "g_k"] \arrow[dl, "\psi_k"'] \\ \mathbf{V}_k \arrow[r, "\phi_k"'] & \mathbf{W}_k \end{tikzcd}$$

If one of $f = \{f_k\}_k$ and $g = \{g_k\}_k$ is a Leray endomorphism, the other is also a Leray endomorphism, and $L(f) = L(g)$ holds.

The following theorem is a new proof of a coincidence theorem which is based on M.Nakaoka [8].

Theorem 2.5. *Let U be an open set in the n-dimensional Euclidean space R^n and Y a paracompact Hausdorff space. Let $p : Y \to U$ be a w-Vietoris mapping and $q : Y \to U$ be a compact mapping. Then $(p^*)^{-1}q^* : H^*(U) \to H^*(U)$ is a Leray endomorphism and we have the following formula:*

$$L((p^*)^{-1}q^*) = I(p, q)$$

Especially, if the Lefschetz number $L((p^*)^{-1}q^*)$ is not zero, there exists a coincidence point $z \in Y$ such that $p(z) = q(z)$.

Proof. At first we remark that there exists a finite complex K in U such that $q(Y) \subset K \subset U$. Here we subdivide U into small boxes whose faces are parallel to axes and construct the complex K by collecting small boxes which intersect with $f(Y)$. Consider the following diagram:

$$\begin{tikzcd} H^*(U) \arrow[r, "i^*"] \arrow[d, "q^*"'] & H^*(K) \arrow[d, "q'^*"] \arrow[dl, "q''^*"'] \\ H^*(Y) \arrow[r, "j^*"] \arrow[d, "(p^*)^{-1}"'] & H^*(p^{-1}(K)) \arrow[d, "(p'^*)^{-1}"] \\ H^*(U) \arrow[r, "i^*"] & H^*(K) \end{tikzcd}$$

where p', q' are restriction mappings of p, q to the subspace $p^{-1}(K)$ respectively and $q'' : Y \to K$ is defined by $q' = q''j$ and $q = iq''$. Since $(p'^*)^{-1}q'^*$:

$H^*(K) \to H^*(K)$ is a Leray endomorphism, $(p^*)^{-1}q^* : H^*(U) \to H^*(U)$ is also a Leray endomorphism by Lemma 2.4. Then, we have

$$L((p'^*)^{-1}q'^*) = L((p^*)^{-1}q^*).$$

Consider the following diagram:

$$\begin{array}{ccc}
H^*(K) & \xrightarrow{=} & H^*(K) \\
\big\downarrow{\scriptstyle (p^*)^{-1}q''^*} & & \big\downarrow{\scriptstyle (p'^*)^{-1}q'^*} \\
H^*(U) & \xrightarrow{i^*} & H^*(K) \\
\big\downarrow{\scriptstyle (-)\cap w_K^U} & & \big\uparrow{\scriptstyle (-1)^q\gamma_K^U/(-)} \\
H_*(U, U-K) & \xrightarrow{=} & H_*(U, U-K)
\end{array}$$

Clearly the upper square is commutative. The commutativity of lower square is proved by Lemma 3 in [8] for the singular (co)homology theory, that is, $i^*(x) = (-1)^q\gamma_K^U/(x \cap w_K^U)$ for $x \in H^q(U)$. Here since K is a finite complex, $i^* : H^*(U) \to H^*(K)$ of Alexander-Spanier cohomology coincides with the one of the singular cohomology. We use i^* of the singular cohomology to calculate i^* of Alexander-Spanier cohomology. Note that Alexander-Spanier cohomology groups $H^*(U)$, $H^*(U, U-K)$, $H^*((U, U-K) \times K)$, $H^*(K)$ are coincide with ones of the singular cohomology.

Let $\{\alpha_\lambda\}, \{\beta_\mu\}, \{\gamma_\nu\}$ be basis of $H^*(U)$, $H^*(U, U-K)$, $H^*(K)$ respectively. We represent $\gamma_K^U \in H^*((U, U-K) \times K)$ as follows:

$$\gamma_K^U = \sum_{\mu,\nu} c_{\mu\nu}\beta_\mu \times \gamma_\nu$$

Since p^* is isomorphic, we set

$$(p^*)^{-1}q''^*(\gamma_\xi) = \sum_\lambda m_{\lambda\xi}\alpha_\lambda$$

We calculate the Lefschetz number $L((p'^*)^{-1}q'^*)$:

$$\begin{aligned}
(-1)^q(p'^*)^{-1}q'^*(\gamma_\xi) &= (-1)^q i^*(p^*)^{-1}q''^*(\gamma_\xi) \\
&= \gamma_K^U/((p^*)^{-1}q''^*(\gamma_\xi) \cap w_K^U) \\
&= \sum_{\mu,\nu} c_{\mu\nu}(\beta_\mu \times \gamma_\nu)/((p^*)^{-1}q''^*(\gamma_\xi) \cap w_K^U) \\
&= \sum_{\mu,\nu} c_{\mu\nu} < \beta_\mu, (p^*)^{-1}q''^*(\gamma_\xi) \cap w_K^U > \gamma_\nu \\
&= \sum_{\mu,\nu} c_{\mu\nu} < \beta_\mu, (\sum_\lambda m_{\lambda\xi}\alpha_\lambda) \cap w_K^U > \gamma_\nu \\
&= \sum_{\lambda,\mu,\nu} c_{\mu\nu}m_{\lambda\xi} < \beta_\mu \cup \alpha_\lambda, w_K^U > \gamma_\nu
\end{aligned}$$

Hence we obtain a result :

$$L((p'^*)^{-1}q'^*) = \sum_{\lambda,\mu,\xi} c_{\mu\xi} m_{\lambda\xi} < \beta_\mu \cup \alpha_\lambda, w_K^U >$$

Next we calculate the incidence index $I(p,q)$:

$$\begin{aligned} I(p,q) &= < \Delta^*(1 \times (p^*)^{-1}q''^*)(\gamma_K^U), w_K^U > \\ &= \sum_{\mu,\nu} c_{\mu\nu} < \Delta^*(\beta_\mu \times (p^*)^{-1}q''^*)(\gamma_\nu), w_K^U > \\ &= \sum_{\mu,\nu} c_{\mu\nu} < \Delta^*(\beta_\mu \times (\sum_\lambda m_{\lambda\nu}\alpha_\lambda), w_K^U > \\ &= \sum_{\lambda,\mu,\nu} c_{\mu\nu} m_{\lambda\nu} < \beta_\mu \cup \alpha_\lambda, w_K^U > \end{aligned}$$

From these results, we have $L((p'^*)^{-1}q'^*) = I(p,q)$. Since $L((p'^*)^{-1}q'^*)$ is equal to $L((p^*)^{-1}q^*)$, we obtain the result $L((p^*)^{-1}q^*) = I(p,q)$.

We obtain the second statement by the above result and Theorem 2.3. Q.E.D.

We can generalize the result above to the case of ANR spaces through the line of L. Górniewicz [5, 6] by using the approximation theorem of Schauder.

Theorem 2.6. *Let U be an open set in a norm space* E *and Y a paracompact Hausdorff space. Let $p : Y \to U$ a w-Vietoris mapping and $q : Y \to U$ be a compact mapping. Then $(p^*)^{-1}q^*$ is a Leray endomorphism. We assume that the graph of qp^{-1} is closed. If the Lefschetz number $L((p^*)^{-1}q^*)$ is not zero, there exists a coincidence point $z \in Y$, that is, $p(z) = q(z)$.*

Theorem 2.7. *Let X be an ANR space and Y a paracompact Hausdorff space. Let $p : Y \to X$ be a Vietoris mapping and $q : Y \to X$ be a compact mapping. Then $(p^*)^{-1}q^*$ is a Leray endomorphism. If the Lefschetz number $L((p^*)^{-1}q^*)$ is not zero, there exists a coincidence point $z \in Y$, that is, $p(z) = q(z)$.*

3 Borsuk-Ulam Type Theorem

When M has an involution T, the equivariant diagonal $\Delta : M \to M \times M$ is given by $\Delta(x) = (x, T(x))$. If T is trivail, Δ is the ordinary diagonal. The involution T on M^2 is given by $T(x,x') = (x',x)$. Hence Δ is an equivariant mapping. Hereafter, we use the same notation for involutions, if there is not confusion. M.Nakaoka defined the equivariant Thom class in Lemma 2.2 of [12] (cf. §1 in [10]):

$$\hat{U}_M \in H^m(S^\infty \times_\pi (M^2, M^2 - \Delta M))$$

where the involution $\tilde{T}$ on $S^\infty \times_\pi M^2$ is given by $\tilde{T}(x,y,y') = (Tx, y', y)$.

For a paracompact Hausdorff space N with a free involution T, there exists an equivariant mapping $h : N \to S^\infty$. We also define the element:

$$\hat{U}_{N,M} \in H^m(N \times_\pi (M^2, M^2 - \Delta M)$$

by $\hat{U}_{N,M} = (h \times_\pi id_{M^2})^*(\hat{U}_M)$ for $h \times_\pi id_{M^2} : N \times_\pi (M^2, M^2 - \Delta M) \to S^\infty \times_\pi (M^2, M^2 - \Delta M)$. Set

$$\Delta_N = j^*(\hat{U}_{N,M}) \in H^m(N \times_\pi M^2)$$

where $j : N \times_\pi M^2 \to N \times_\pi (M^2, M^2 - \Delta(M))$. In the case of $N = S^\infty$ and the trivial involution T on M, M.Nakaoka determined θ_∞ by Proposition 3.4 in [11].

A mapping $\hat{f}_\pi : N_\pi \to N \times_\pi M^2$ is defined by $\hat{f}_\pi(x) = (x, f(x), f(Tx))$. Since we use Alexander-Spanier cohomology theory in this paper, we must treat carefully the results of M.Nakaoka. The following theorem is given in Theorem 3.5 in [11].

Theorem 3.1 (Nakaoka). *Let N be a paracompact Hausdorff space with a free involution T, and M be an m-dimensional closed topological manifold. Let $\{\alpha_1, \dots, \alpha_s\}$ be a basis for $H^*(M)$, and set*

$$d_*([M]) = \sum_{j,k} \eta_{jk} a_j \times a_k \quad (\eta_{jk} \in Z/2)$$

where $a_i = \alpha_i \cap [M]$. Then, for any continuous mapping $f : N \to M$, it holds

$$\hat{f}_\pi^*(\theta_N) = \sum_{i \geqq 0} c^{m-2i} Q(f^* v_i) + \sum_{j<k} (\eta_{jk} + \eta_{jj}\eta_{kk}) \phi^*(f^*(\alpha_j) \cup T^* f^*(\alpha_k)) \qquad (1)$$

where $c = c(N,T)$ and $v_i = v_i(M)$ Wu class of M and $\phi^* : H^*(N) \to H^*(N_\pi)$ is the transfer homomorphism.

The next theorem is proved in Proposition 1.3 in [10].

Theorem 3.2. *Let N be a paracompact Hausdorff space with a free involution T and M a closed topological manifold. If a continuous mapping $f : N \to M$ satisfies $\hat{f}_\pi^*(\theta_N) \neq 0$, the set $A(f) = \{y \in N \mid f(y) = f(Ty)\}$ is not empty set.*

Definition 5. *A set-valued mapping $\varphi : X \to Y$ is called admissible, if there exists a paracompact Hausdorff space Γ satisfying the following conditions:*

1. *there exist a Vietoris mapping $p : \Gamma \to X$ and a continuous mapping $q : \Gamma \to Y$.*

2. *$\varphi(x) \supset q(p^{-1}(x))$ for each $x \in X$.*

$\varphi : X \to Y$ is called w-admissible, if it satisfies the condition (2) and p is a w-Vietoris mapping.

A pair (p,q) of mappings p, q is called a selected pair of φ. If $\varphi : X \to Y$ satisfies the first condition and $\varphi(x) = q(p^{-1}(x))$ for each $x \in X$, it is called s-admissible mapping.

Definition 6. *A set-valued mapping $\varphi : X \to Y$ is called $*$-admissible mapping, if it is admissible and satisfies $p_\varphi : \Gamma_\varphi \to X$ induces an isomorphism $p_\varphi^* : H^*(X) \to H^*(\Gamma_\varphi)$.*

Theorem 3.3. *Let X be an ANR space and $\varphi : X \to X$ compact admissible mapping. If $L(\varphi^*)$ contains non-zero element, there exists a fixed point $x_0 \in X$, that is, $x_0 \in \varphi(x_0)$.*

Proof. We can choose a selected pair (p,q) where a Vietoris mapping $p : \Gamma \to X$ and a compact mapping $q : \Gamma \to X$. We may assume $L((p^*)^{-1}q^*) \neq 0$. By Theorem 2.7, there exists a coincidence point $z \in \Gamma$ such that $p(z) = q(z)$. we obtain the result. **Q.E.D.**

Let N be a paracompact Hausdorff space with a free involution T and M a closed topological manifold without involution. For a set-valued mapping $\varphi : N \to M$, $\tilde{N}$ is defined by

$$\tilde{N} = \{(x,y,y') \in N \times M^2 \mid x \in N,\ y \in \varphi(x),\ y' \in \varphi(T(x))\ \}$$

A free involution $\tilde{T}$ on $\tilde{N}$ is given by $\tilde{T}(x,y,y') = (Tx, y', y)$. $\tilde{p} : \tilde{N} \to N$ is the projection. The following Lemma is a key result.

Lemma 3.4. *Let $\varphi : N \to M$ be an admissible mapping with a selected pair $p : \Gamma \to N$ and $q : \Gamma \to M$. Then $H^*(\tilde{N})$ and $H^*(\tilde{N}_\pi)$ have direct summands $H^*(N)$ and $H^*(N_\pi)$ respectively. Moreover if N is a metric space and A is a π-invariant closed or open subspace of N, then $H^*(\tilde{N} - \tilde{p}^{-1}(A))$ and $H^*(\tilde{N}_\pi - \tilde{p}_\pi^{-1}(A_\pi))$ have direct summands $H^*(N - A)$ and $H^*(N_\pi - A_\pi)$ respectively.*

Theorem 3.5. *Let N be a paracompact Hausdorff space with a free involution T and M an m-dimensional closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is $*$-admissible and satisfies $\varphi^* = 0$ for positive dimension and $c(N,T)^m \neq 0$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover if N is an n-dimensional closed topological manifold, it holds $\dim A(\varphi) \geqq n - m$ where $A(\varphi) = \{x \in N \mid \varphi(x) \cap \varphi(T(x)) \neq \emptyset\}$.*

Proof. We can define a free involution $\tilde{T}$ on $\tilde{N}$ by $\tilde{T}(x,y,y') = (T(x), y', y)$ and a mapping $\tilde{\varphi} : \tilde{N} \to M$ by $\tilde{\varphi}(x,y,y') = y$. We note:

$$A(\tilde{\varphi}) = \{(x,y,y) \in \tilde{N} \mid y \in \tilde{\varphi}(x),\ y \in \tilde{\varphi}(\tilde{T}(x))\}$$

Now consider the following diagram:

$$\begin{tikzcd}
\tilde{N} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{p}"'] \arrow[dr, "\tilde{p}'"] & M \\
N & \Gamma_\varphi \arrow[l, "p_\varphi"] \arrow[u, "q_\varphi"']
\end{tikzcd}$$

where $\tilde{p}(x,y,y') = x$, $\tilde{p}'(x,y,y') = (x,y)$ and $p_\varphi(x,y) = x$, $q_\varphi(x,y) = y$.

We see $\tilde{\varphi}^* = 0$ from $\varphi^* = 0$. The mapping $\tilde{p} : \tilde{N} \to N$ is π-equivarint, that is $\tilde{p}(\tilde{T}(x,y,y')) = T(\tilde{p}(x,y,y'))$. Since $\tilde{p}_\pi^*$ is injective by Lemma 3.4. We have $\tilde{c}^m = c(\tilde{N},\tilde{T})^m = \tilde{p}_\pi^*(c^m) \neq 0$ because of π-equivariant mapping $\tilde{p} : \tilde{N} \to N$.

Now we calculate $\hat{\tilde{\varphi}}^*(\theta_{\tilde{N}})$. Since we have $\phi^*(\tilde{\varphi}^*(\alpha_j) \cup T^*\tilde{\varphi}^*(\alpha_k)) = 0$ and $\tilde{c}^{m-2i}Q(\tilde{\varphi}^*(v_i)) = 0$ for $i > 0$ from our condition and $\tilde{c}^m Q(\tilde{\varphi}^*(v_0)) = \tilde{c}^m \neq 0$, we obtained $\hat{\tilde{\varphi}}^*(\theta_{\tilde{N}}) = \tilde{c}^m \neq 0$ from the formula (1) in Theorem 3.1. We conclude $A(\tilde{\varphi}) \neq \emptyset$ from Theorem 3.2. Hence we obtain the former result.

Since $\tilde{N} - A(\tilde{\varphi})$, $\tilde{N} - \tilde{p}^{-1}A(\varphi)$, $N - A(\varphi)$ have natural involutions induced by $\tilde{T}$, T, we obtained $\tilde{N}_\pi - A(\tilde{\varphi})_\pi$, $\tilde{N}_\pi - \tilde{p}^{-1}A(\varphi)_\pi$, $N_\pi - A(\varphi)_\pi$. For the latter proof, we consider the following diagram:

$$
\begin{array}{ccccc}
H^*(\tilde{N}_\pi, \tilde{N}_\pi - A(\tilde{\varphi})_\pi) & \xrightarrow{j_1^*} & H^*(\tilde{N}_\pi) & \xrightarrow{i_1^*} & H^*(\tilde{N}_\pi - A(\tilde{\varphi})_\pi) \\
\downarrow{\scriptstyle k_1^*} & & \downarrow{\scriptstyle id^*} & & \downarrow{\scriptstyle k_2^*} \\
H^*(\tilde{N}_\pi, \tilde{N}_\pi - \tilde{p}^{-1}A(\varphi)_\pi) & \xrightarrow{j_2^*} & H^*(\tilde{N}_\pi) & \xrightarrow{i_2^*} & H^*(\tilde{N}_\pi - \tilde{p}^{-1}A(\tilde{\varphi})_\pi) \\
\uparrow{\scriptstyle \tilde{p}_{1\pi}^*} & & \uparrow{\scriptstyle \tilde{p}_\pi^*} & & \uparrow{\scriptstyle \tilde{p}_{2\pi}^*} \\
H^*(N_\pi, N_\pi - A(\varphi)_\pi) & \xrightarrow{j_3^*} & H^*(N_\pi) & \xrightarrow{i_3^*} & H^*(N_\pi - A(\varphi)_\pi)
\end{array}
$$

where k_1, k_2 are induced by natural inclusions and $\tilde{p}_1$, $\tilde{p}_2$ are induced by $\tilde{p}$. Here we note $\bar{H}^*(-) \cong H^*(-)$ for manifolds. Since $A(\varphi)$ is a π-invariant closed subset of N, we have an into-isomorphism $(\tilde{p}_2)_\pi^* : H^*(N_\pi - A(\varphi)_\pi) \to H^*(\tilde{N}_\pi - \tilde{p}_\pi^{-1}(A(\tilde{\varphi})_\pi))$ by Lemma 3.4. We note that $\hat{\tilde{\varphi}}_\pi^*(\theta_{\tilde{N}}) = \tilde{c}^m \neq 0$ is an image of $c^m \in H^*(N_\pi)$, that is, $(\tilde{p}_\pi)^*(c^m) = \tilde{c}^m$. Since $\tilde{c}^m$ is an image of $\hat{\tilde{\varphi}}_\pi^*(U_{\tilde{N},M})$ under j_1^*, it holds $i_2^*(\tilde{c}^m) = 0$. From this, we see $(\tilde{p}_2)_\pi^* i_3^*(c^m) = i_2^*\tilde{p}_\pi^*(c^m) = (i_2)^*(\tilde{c}^m) = 0$ in the above diagram and hence $(i_3)^*(c^m) = 0$ because of the injectivity of $(\tilde{p}_2)_\pi^*$. If $H^m(N_\pi, N_\pi - A(\varphi)_\pi) = 0$, we easily see $c^m = 0$ which contradicts $c^m \neq 0$. Hence we obtain $H^m(N_\pi, N_\pi - A(\varphi)_\pi) \neq 0$.

Since N and $N - A(\varphi)$ are manifolds, the singular homology group $H_m(N_\pi, N_\pi - A(\varphi)_\pi) \neq 0$ by the universal coefficient theorem. We obtain that the Čech cohomology group $\check{H}^{n-m}(A(\varphi)_\pi) \neq 0$ by Poincaré duality. In this case $\check{H}^{n-m}(A(\varphi)_\pi)$ is equal to Alexander-Spanier cohomology group $H^{n-m}(A(\varphi)_\pi)$. We see $\dim A(\varphi)_\pi \geqq n - m$ and hence $\dim A(\varphi) \geqq n - m$. Q.E.D.

Cororally 3.6. *Let N be a paracompact Hausdorff space with a free involution T which has a homology group of n-dimensional sphere and M be an m-dimensional closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is $*$-admissible and satisfies $\varphi^* = 0$ and $n \geqq m$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover if N is an n-dimensional closed topological manifold, it holds* $\dim A(\varphi) \geqq n - m$.

Let X be a space with a free involution T and S^k the k-dimensional sphere with the antipodal involution. Define

$$
\begin{aligned}
\gamma(X) &= \inf \{k \mid f : X \to S^k \text{ equivariant mapping}\} \\
\mathrm{Ind}(X) &= \sup \{k \mid c^k \neq 0\}
\end{aligned}
$$

where $c \in H^1(X_\pi)$ is the class $c = f_\pi^*(\omega)$ for an equivariant mapping $f : X \to S^\infty$. If X is a compact space with a free involution, it holds the following formula (cf. §3 in [3]):

$$\mathrm{Ind}(X) \leqq \gamma(X) \leqq \dim X.$$

K. Gęba and L. Górniewicz determined $\mathrm{Ind}A(\varphi)$ of an admissible mapping $\varphi : S^{n+k} \to \mathbf{R}^n$ in [3]. We generalize their result.

Cororally 3.7. *Let N be a closed topological manifold with a free involution T which has a homology group of n-dimensional sphere and M be an m-dimensional closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is $*$-admissible and $\varphi^* = 0$ and $n \geqq m$, it holds* $\mathrm{Ind}A(\varphi) \geqq n - m$.

Proof. At first, we remark commutativity of the following diagram for n-dimensional closed topological manifold X and a closed subset Y of X:

$$\begin{array}{ccc} H_k(X) & \xrightarrow{j_*} & H_k(X, X-Y) \\ \big\downarrow{\scriptstyle -\backslash U_0} & & \big\downarrow{\scriptstyle -\backslash U_1} \\ H^{n-k}(X) & \xrightarrow{i^*} & H^{n-k}(Y) \end{array}$$

where U_0, U_1 are restrictions of $U \in H^n(X^2, X^2 - d(X))$ for $k : (X^2, \emptyset) \to (X^2, X^2 - d(X))$, $l : (X, X - Y) \times Y \to (X^2, X^2 - d(X))$ respectively. Here the vertical arrows are Poincaré isomorphisms.

We apply the above diagram for the case $X = N_\pi$, $Y = A(\varphi)$. In the proof of the Theorem 3.5, we find a class $\alpha \in H^m(N_\pi, N_\pi - A(\varphi)_\pi)$ such that $j^*(\alpha) = c^m$. Let $b \in H_m(N_\pi)$ be the dual element of $c^m \in H^m(N_\pi)$ and $a \in H_m(N_\pi, N_\pi - A(\varphi)_\pi)$ be the dual class of α. Then we obtain $j_*(b) = a \neq 0$. Since the Poincaré dual of b is c^{n-m}, we obtain $i^*(c)^{n-m} = i^*(c^{n-m}) \neq 0$ by the above diagram. Hence we obtain the result. Q.E.D.

Theorem 3.8. *Let N be a paracompact Hausdorff space with a free involution T and M be an m-dimensional closed topological manifold which has a homology group of m-dimensional sphere. If a set-valued mapping $\varphi : N \to M$ is admissible and satisfies $c(N,T)^m \neq 0$ and $\varphi(N) \neq M$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ Moreover if N is an n-dimensional closed topological manifold, it holds* $\dim A(\varphi) \geqq n - m$.

Proof. We use the notation and method in the proof of Theorem 3.5. A homology group of $M' = M - \{a\}$ is trivial for positive dimensions by a homology group of M. From the fact and $\varphi(N) \neq M$, we have $\tilde{\varphi}^* = 0$ for positive dimensions. We see that $\tilde{c}^m = c(\tilde{N}, \tilde{T})^m \neq 0$ by our assumption. By the similar method of Theorem 3.5, we see

$$\hat{\tilde{\varphi}}^*(\theta_{\tilde{N}}) = \tilde{c}^m \neq 0$$

by $\tilde{\varphi}^* = 0$ for positive dimension and $c(N,T)^m \neq 0$. Hence there exists a point $z_0 \in \tilde{N}$ such that $\tilde{\varphi}(z_0) = \tilde{\varphi}(\tilde{T}(z_0))$. We obtain $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ for $x_0 \in N$.

We can prove the last statement as in the proof of Theorem 3.5. We omit the proof. Q.E.D.

Theorem 3.9. *Let N be a closed topological manifold with a free involution T which has the homology group of the n-dimensional sphere and M be a closed topological manifold. If a set-valued mapping $\varphi : N \to M$ is admissible and $\varphi(N) \neq M$ and $n \geqq m$, then there exists a point $x_0 \in N$ such that $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$. Moreover it holds $\dim A(\varphi) \geqq n - m$ and $\mathrm{Ind} A(\varphi) \geqq n - m$.*

Proof. We use the notation and method in the proof of Theorem 3.5. We remark $v_i(M) = 0$ for $i > \frac{m}{2}$ by the definition of Wu class. Therefore we see $\tilde{\varphi}(v_i(M))) = 0$ for $i > 0$ because of $H^*(N) = H^*(S^n)$. We see also $\phi^*(\tilde{\varphi}^*(\alpha_i) \cup \tilde{T}^*\tilde{\varphi}^*(\alpha_j)) = 0$ by $H^*(N) = H^*(S^n)$ and $\deg \alpha_i + \deg \alpha_j = m$ and $\tilde{\varphi}^*(\alpha_0) = 0$ for the class α_0 such that $\deg \alpha_0 = m$. Note $\tilde{c}^m = c(\tilde{N}, \tilde{T})^m \neq 0$ by our assumption. From this remark we see

$$\hat{\tilde{\varphi}}^*(\theta_{\tilde{N}}) = \tilde{c}^m = c(\tilde{N}, \tilde{T})^m \neq 0.$$

Therefore there exists a point $z_0 \in \tilde{N}$ such that $\tilde{\varphi}(z_0) = \tilde{\varphi}(\tilde{T}(z_0))$. We obtain $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$ for $x_0 \in N$. We can prove the last statement as in the proof of Theorem 3.5 and Corollary 3.7. We omit the proof. Q.E.D.

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