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On 8-manifolds with $SU(3)$-actions

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ABSTRACT. In this article, we study about compact manifolds which have $SU(3)$-actions with codimension one orbits. We get more precise classification than the paper of Gambioli [G].

1. Why do we consider 8 dimensional manifolds?

The purpose of this paper gives the outline to classify compact manifolds which have $SU(3)$-actions with codimension one orbits in some case. Obviously codimension one orbits are principal orbits in this case, where principal orbits mean orbits which have the largest dimension. We also remark $\dim SU(3) = 8$. So the dimension of the manifold M which have $SU(3)$-actions with codimension one orbits must be less than or equal to 9, that is, $\dim M \leq 9$. In this section we mention why 8 dimensional manifolds.

Through all of this paper, we will use the classical Lie theory, the transformation group theory and the Lie group representation theory. The referenaces of the classical Lie theory (in particular the classification result of compact Lie groups) are [**MT91**], of the transformation group theory is [**B72**] and [**Ka91**] and of the Lie group representation theory is [**Y73**]. Sometimes we will use the classification result about transitive actions on sphere in [**HH65**].

1.1. The cases whose dimension is less than or equal to 4.

First we consider the cases $\dim M \leq 4$. From the following proposition, there is no non-trivial $SU(3)$-action on such M.

PROPOSITION 1.1. If a compact manifold M such that $0 \leq \dim M \leq 4$, then there is no $SU(3)$-actions on M with codimension one principal orbits.

PROOF. If $\dim M = 0$, the $SU(3)$-action is trivial action. Hence the case $\dim M = 0$ does not occur.

If $\dim M = 1$, M is a 1-dimensional circle S^1 because M is compact. Since $SU(3)$ can not act on S^1 non-trivially (by [**HH65**]), the $SU(3)$-action is also trivial. Hence the case $\dim M = 1$ does not occur.

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If $\dim M = 2$, a principal orbit is 1-dimensional compact manifold. So a principal orbit must be S^1. However $SU(3)$ does not act on S^1 (by **[HH65]**). Hence the case $\dim M = 2$ does not occur.

If $\dim M = 3$ then a principal orbit is 2-dimensional compact manifold G/K, where $G = SU(3)$ and K is its compact subgroup. Since $\dim G/K = \dim G - \dim K$, $\dim G = \dim SU(3) = 8$, and the dimension of the maximal torus of $SU(3)$ is 2 (rank $SU(3) = 2$),

$$\dim K = 6, \quad \text{rank } K \leq 2.$$

Therefore the universal covering of K^o is $SU(2) \times SU(2)$ (by the classification result in **[MT91]**). Hence rank G = rank K^o. Since $H^{odd}(G/K^o) = 0$ (iff rank G = rank K^o, see **[U77]** or **[Ku]**) and G/K^o is orientable and compact 2-dimensional manifold, G/K^o is the 2-dimensional sphere S^2. This gives a contradiction, because $SU(3)$ can not act on S^2 non-trivially (by **[HH65]**). Hence the case $\dim M = 3$ does not occur.

If $\dim M = 4$, a principal orbit is 3-dimensional compact manifold G/K. Then the dimension of K is 5. However there are not 5-dimensional Lie group K which satisfies rank $K \leq 2$ (see **[MT91]**). Hence the case $\dim M = 4$ does not occur.

Therefore we conclude the statement of this proposition. □

From the proof of Proposition 1.1, we also have the following corollary.

COROLLARY 1.2. If $0 \leq \dim M \leq 3$, there is no non-trivial $SU(3)$-action on M.

1.2. The case whose dimension is 5. Next we consider the case $\dim M = 5$. Sometimes we denote such manifold by M^5.

First we prove the following lemma.

LEMMA 1.3. *Orbits of an $SU(3)$-action on M^5 with codimension one principal orbits $SU(3)/H$ are not singular orbits, that is, all dimension of orbits are* 4.

PROOF. If there is an singular orbit (whose dmension is less than 4), then the sngular orbit must be one point because there is no K such that $1 \leq \dim SU(3)/K \leq 3$ by the proof of Proposition 1.1. Therefore its isotorpy subgroup $SU(3)$ has representation to $O(5)$ and $SU(3)$ acts transitively on S^4 through this action because of the differentiable slice theorem (see e.g. **[B72]** or **[Ka91]**). However there is no such action by **[HH65]**. Therefore there is no singular orbits. □

Since H satisfies $\dim H = 4$ and rank $H \leq 2$, it is isomorphic to $U(2)$ (see **[MT91]**). Hence $S(U(1) \times U(2)) \simeq H \subset SU(3)$. Because H is a maximal subgroup, that is, $H \subset \hat{H}$ then $\hat{H} = H$ or $SU(3)$, we see that there is no exceptional orbits. Therefore $M^5/SU(3) \simeq S^1$. Moreover there is no fixed points in this case because there is no transitive $SU(3)$-action on $S^4 \subset \mathbb{R}^5$ by **[HH65]**. Hence we have the follwing proposition.

PROPOSITION 1.4. A compact $SU(3)$-manifold M^5 with codimension one orbits is equivariant diffeomorphic to

$$M_l^5 = SU(3) \times_{S(U(1)\times U(2))} S^1,$$

where $S(U(1)\times U(2))$ acts on S^1 through the following representation $\rho : S(U(1)\times U(2)) \to U(1)$:

$$\rho\begin{pmatrix} t & 0 \\ 0 & A \end{pmatrix} = t^l,$$

where $t \in U(1)$ and $A \in U(2)$ such that $t = \det A^{-1}$ and $l \in \mathbb{Z}$.

REMARK 1.5. M_l^5 is the restricted circle bundle of the complex line bundle over $\mathbb{C}P(2)$ such that its first chern class is l.

Finally in this subsection, we remark the following corollaries.

Because of the proofs of Proposition 1.1 and Lemma 1.3, the following corollaries can be shown.

COROLLARY 1.6. If M^4 has a non-trivial $SU(3)$-action, then this action is transitively and $M^4 = \mathbb{C}P(2) \simeq SU(3)/S(U(1)\times U(2))$.

COROLLARY 1.7. There is no $SU(3)$-action on M^5 with codimension less than or equal to two orbits.

1.3. The case whose dimension is 6. Next we consider the case M^6, that is, $\dim M = 6$. Let $K \subset SU(3)$ be a subgroup such that $\dim K = 3$. Then $K^o \simeq SU(2)$ or $SO(3)$ by [**MT91**]. So we have the following lemma.

LEMMA 1.8. *Let $K \subset SU(3)$ and $\dim SU(3)/K = 5$. Then $K^o \simeq SU(2)$ or $SO(3)$.*

First we consider the case $K^o = SO(3)$. Let $SU(3)/SO(3) = \mathbb{L}$ (the notation of Gambioli in [**G**]). Now $N(SO(3)) \simeq \mathbb{Z}_3 \times SO(3)$, where $N(SO(3))$ is a normal subgroup of $SU(3)$ and $\mathbb{Z}_3 \subset U(1)$ is a center of $SU(3)$. Hence in this case there is no singular orbits because if H is a singular isotropy subgroup then $K \subset H \subset SU(3)$ and $H/K \simeq S^m$ $(1 \le m \le 6)$. Therefore we have the following proposition.

PROPOSITION 1.9. If an $SU(3)$-manifold M^6 has codimension one orbits with $SO(3)$ as their connected components, then all orbits are principal orbits and M^6 is equivariant diffeomorphic to one of the following manifolds:

$$\mathbb{L}\times S^1, \quad SU(3)/N(SO(3))\times S^1, \quad SU(3)\times_{N(SO(3))} S^1,$$

where in the last case $N(SO(3))$ acts on S^1 by the following representation:

$$N(SO(3)) \simeq \mathbb{Z}_3 \times SO(3) \to \mathbb{Z}_3 \to U(1)$$

by the natural inclusion $\mathbb{Z}_3 \subset U(1)$.

Next we consider the case $K^o = SU(2)$. Then $N(SU(2)) \simeq S(U(1)\times U(2))$. Therefore all subgroups $K \subset SU(3)$ such that $\dim K = 5$ are isomorphic to

$$K \simeq S(\mathbb{Z}_l \times U(2)) = \left\{ \begin{pmatrix} t & 0 \\ 0 & A \end{pmatrix} \in S(U(1)\times U(2)) \;\middle|\; t \in \mathbb{Z}_l \subset U(1);\ t^l = 1 \right\},$$

where $l \in \mathbb{N}$ (if $l = 1$ then $\mathbb{Z}_1 = \{1\}$, that is, $S(\mathbb{Z}_1 \times U(2)) = SU(2)$). If there is no singular orbits and exceptional orbits, by the similar argument of Proposition 1.4 we have the following proposition.

PROPOSITION 1.10. If an $SU(3)$-manifold M^6 has codimension one orbits with $SU(2)$ as their connected component and all orbits are principal orbits, then M^6 is equivariant diffeomorphic to one of the following manifolds:

$$M_l^6 = SU(3) \times_{S(\mathbb{Z}_l \times U(2))} S^1, \quad SU(3)/S(\mathbb{Z}_l \times U(2)) \times S^1 \simeq S^5/\mathbb{Z}_l \times S^1,$$

where $l \in \mathbb{N}$ ($\mathbb{Z}_1 = \{1\}$) and in the left case $S(\mathbb{Z}_l \times U(2))$ acts S^1 through the following representation:

$$S(\mathbb{Z}_l \times U(2)) \to \mathbb{Z}_l \to U(1)$$

by the natural projection $S(\mathbb{Z}_l \times U(2)) \to \mathbb{Z}_l$ and the natural inclusion $\mathbb{Z}_l \to U(1)$.

Remark that $M_1^6 = S^5 \times S^1$.

Next we assume there is a singular orbit G/K_1. Since $S(\mathbb{Z}_l \times U(2)) \subset K_1$ and $\dim S(\mathbb{Z}_l \times U(2)) < \dim K_1$, we see that $K_1 \simeq S(U(1) \times U(2))$ or $SU(3)$. Moreover because of Theorem 8.2 in [**B72**] and the differentiable slice theorem, we see that there are two singular orbits G/K_1, $G/K_2 \simeq SU(3)/S(U(1) \times U(2))$ or $\{*\}$ and there are two type slice representations $\rho_i : K_i \simeq S(U(1) \times U(2)) \xrightarrow{\sigma_i} U(1) \simeq SO(2)$ such that

$$\sigma_i \begin{pmatrix} t & 0 \\ 0 & A \end{pmatrix} = t^l,$$

where $t = \det A^{-1}$, $l \geq 1$ for $i = 1,\ 2$, or the natural inclusion $\iota_i : K_i \simeq SU(3) \to SU(3) \subset SO(6)$. Therefore we see that the tubular neighborhood X_i of G/K_i is unique and there are three cases:

(1) $X_1 = X_2 = D^6 \subset \mathbb{C}^3$,
(2) $X_1 = X_2 = SU(3) \times_{S(U(1)\times U(2))} D^2$,
(3) $X_1 = D^6 \subset \mathbb{C}^3$ and $X_2 = SU(3) \times_{S(U(1)\times U(2))} D^2$

where the slice representation ρ_i of X_i in the second case and the last case ($i = 2$) is defined by $l = 1$. By the Uchida's criterion (see [**G**]) and the connectedness of $N(S(\mathbb{Z}_l \times U(2))) = S(U(1) \times U(2))$, we have that the attaching map $\partial X_1 \to \partial X_2$ is also unique. Therefore we have the following proposition.

PROPOSITION 1.11. If M^6 has an $SU(3)$-action with codimension one orbits and singular orbits, then M^6 is equivariant diffeomorphic to one of the following manifolds:

$$S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}, \quad SU(3) \times_{S(U(1)\times U(2))} S^2(\mathbb{C}_l \oplus \mathbb{R}), \quad \mathbb{C}P(3)$$

where $S^2(\mathbb{C}_l \oplus \mathbb{R})$ is a 2-dimensional sphere and has $S(U(1) \times U(2))$-action through σ_i.

REMARK 1.12. $SU(3) \times_{S(U(1)\times U(2))} S^2(\mathbb{C}_l \oplus \mathbb{R})$ is the projectification of the complex line bundle over $\mathbb{C}P(2)$ such that its first chern class is l.

We omit the case which has exceptional orbits (we can easily see that such case satisfies $M^6/SU(3) \simeq S^1$ and there exists infinitely many cases).

Finally in this subsection, we remark the following corollaries.

Because of Lemma 1.8 and the above arguments, we have the following corollary.

COROLLARY 1.13. If M^5 has a transitive $SU(3)$-action, then M^5 is equivariant diffeomorphic to one of the followings:

$$SU(3)/SO(3), \quad SU(3)/N(SO(3)), \quad SU(3)/S(\mathbb{Z}_l \times U(2)).$$

Because of the proofs of Proposition 1.1 and Lemma 1.3, the following corollaries can be shown .

COROLLARY 1.14. If M^6 has an $SU(3)$-action with codimension two principal orbits, then all orbits are principal orbits $\mathbb{C}P(2)$ and and there is a fibration $\mathbb{C}P(2) \to M^6 \xrightarrow{\pi} \Sigma^2$ where π is a projection to the orbit space and the orbit space Σ^2 is a 2-dimensional manifols.

COROLLARY 1.15. There is no $SU(3)$-action on M^6 with codimension less than or equal to three orbits.

1.4. The case whose dimension is 7. Next we consider the case M^7, that is, $\dim M = 7$. If $H \subset SU(3)$ such that $\dim H = 2$ then $H^o \simeq T^2$ (maximal torus in $SU(3)$) by [**MT91**]. So we have the following lemma.

LEMMA 1.16. *Let $K \subset SU(3)$ and $\dim SU(3)/K = 6$. Then $K^o \simeq T^2$.*

Therefore we have the following proposition.

PROPOSITION 1.17. Let M^6 be a transitive $SU(3)$-manifold. Then M^6 is equivariant diffeomorphic to one of the following manifolds:

$$SU(3)/T^2, \quad SU(3)/(\mathbb{Z}_2 \times T^2), \quad SU(3)/(\mathbb{Z}_3 \times T^2), \quad SU(3)/N(T^2),$$

where $N(T^2)$ is a normal subgroup of T^2 in $SU(3)$ and $\mathbb{Z}_2 \times T^2$ and $\mathbb{Z}_3 \times T^2$ are subgroups of $N(T^2)$ with the same connected component T^2.

Therefore candidates of principal orbits are the above 4 manifolds.

We omit the cases which satisfy all orbits are principal orbits and there exist an exceptional orbit.

Assume there is a singular orbit,

Because rank $SO(3) = 1 =$ rank $SU(2)$ and the connected component of the principal isotorpy subgroup need to include T^2, the singular isotropy subgroups are isomorphic to $S(U(1) \times U(2))$ by Corollaries 1.6 and 1.13. Hence the following lemma holds because $\mathbb{Z}_3 \times T^2 \not\subset S(U(1) \times U(2))$ (also see Section IV Theorem 8.2 in [**B72**]).

LEMMA 1.18. *Assume M^7 has an $SU(3)$-action with codimension one orbits and singular orbits. Then there is just two singular orbits*

$$\{*\} \quad \text{or} \quad \mathbb{C}P(2) \simeq SU(3)/S(U(1) \times U(2)),$$

there is no exceptional orbits and the principal orbit is

$$SU(3)/T^2 \quad \text{or} \quad SU(3)/(\mathbb{Z}_2 \times T^2).$$

Let us consider the slice representation of singular orbits. Assume $K_i \simeq S(U(1) \times U(2))$. Because K_i acts on the normal sphere S^2 transitively through the slice representation, so the slice representation $\rho_i : K_i \to SO(3)$ need to be surjective. Now we can consider

$$S(U(1) \times U(2)) = \left\{ \begin{pmatrix} t^{-2} & 0 \\ 0 & tA \end{pmatrix} \;\middle|\; t \in U(1),\ A \in SU(2) \right\}.$$

Then the slice representation $\rho_i : K_i \simeq S(U(1) \times U(2)) \xrightarrow{\sigma_i} SO(3)$ is unique up to equivalence by [**Y73**], as follows:

$$\sigma_i \begin{pmatrix} t^{-2} & 0 \\ 0 & tA \end{pmatrix} = \tau(A) \in SO(3),$$

where $\tau : SU(2) \to SO(3)$ is the double covering. Assume $K_i = SU(3)$. In this case K_i acts on the normal sphere S^6 transitively through the slice representation. However $SU(3)$ does not act on S^6 transitively. Therefore all principal orbits are $SU(3)/T^2$ and two tubular neighborhoods $X_1 \simeq X_2 \simeq SU(3) \times_{S(U(1)\times U(2))} D^2$ of $G/K_1 \simeq G/K_2 \simeq SU(3)/S(U(1) \times U(2))$ are unique. Hence we only need to study about attaching maps.

Consider the attaching maps. Because we can take an attaching map $f : \partial X_1 \simeq G/K \to G/K \simeq \partial X_2$ form $N(K)/K$ and $K = T^2$, $N(T^2)/T^2 \simeq S_3$, so we see that there are at most 6 attaching maps. Since we can consider $T^2 \subset SU(3)$ is a diagonal subgroup, $N(T^2)/T^2$ is as follows:

$$\left\{ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, x = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, x^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

Let $M(f) = X_1 \cup_f X_2$ where $f \in N(T^2)/T^2$. By the Uchida's criterion (see [**G**]) and $xx^{-1} = I$, we see that $M(x) \simeq M(x^{-1})$. Fix $K_1 = S(U(1) \times U(2)) \subset SU(3)$. Because $\alpha \in S(U(1) \times U(2)) = K_1$, we can easily have $M(\alpha) \simeq M(I)$ (see [**U77**] or [**Ku**]). Since $\beta x = \alpha = \gamma x^{-1}$, we also have $M(\beta) \simeq M(x) \simeq M(x^{-1}) \simeq M(\gamma)$. Therefore there are two cases $M(I)$ and $M(\beta)$.

PROPOSITION 1.19. If M^7 has an $SU(3)$-action with codimension one orbits and singular orbits, then M^7 is equivariant diffeomorphic to the following manifolds:

$$S^7, \quad SU(3) \times_{S(U(1)\times U(2))} S^3(\mathbb{R}^3 \oplus \mathbb{R}),$$

where in the left case $SU(3)$ acts on $S^7 \subset \mathfrak{su}(3) \simeq \mathbb{R}^8$ (Lie algebra of $SU(3)$) by the adjoint $SU(3)$-action on $\mathfrak{su}(3)$ and in the right case $S(U(1) \times U(2))$ acts on $S^3(\mathbb{R}^3 \oplus \mathbb{R}) \simeq S^3$ by the representation $\sigma_i : S(U(1) \times U(2)) \to SO(3)$.

REMARK 1.20. $SU(3) \times_{S(U(1)\times U(2))} S^3(\mathbb{R}^3 \oplus \mathbb{R})$ corresponds to the second case Lemma 2.2 (2) in [**PV99**], that is, $K_1 = K_2$ and it does not carry any positively curved $SU(3)$-invariant metric.

Finally in this subsection, we remark the following corollaries.

Because of the proofs of Proposition 1.1 and Lemma 1.3, the following corollaries can be shown.

COROLLARY 1.21. If M^7 has an $SU(3)$-action with codimension three principal orbits, then all orbits are principal orbits $\mathbb{C}P(2)$ and there is a fibration $\mathbb{C}P(2) \to M^7 \xrightarrow{\pi} \Sigma^3$ where π is a projection to the orbit space and the orbit space Σ^3 is a 3-dimensional manifols.

COROLLARY 1.22. There is no $SU(3)$-action on M^7 with codimension less than or equal to four orbits.

We omit the cases which satisfy there is codimension 2 dimensional orbits and the codimension 0 dimensional orbit (transitive case).

Therefore the next considering case is $\dim M = 8$.

2. The case whose dimension is 8

As an easy case, we assume M^8 is simply connected and has an $SU(3)(= G)$-action with codimension one orbits G/K. Then the following structure theorem holds (see **[U77]** and Section IV Theorem 8.2 in **[B72]**).

THEOREM 2.1. *Assume M is simply connected has a G-action with codimension one orbits G/K. Then G/K is a principal orbit and there are just two singular orbits G/K_1 and G/K_2. Moreover M is attaching two tubular neighborhoods X_1, X_2 of G/K_1, G/K_2 and their boundary $\partial X_1 = \partial X_2 = G/K$, that is,*

$$M = X_1 \cup X_2, \quad \partial X_1 = G/K = \partial X_2.$$

Moreover we have the following lemma (see **[U77]** or **[Ku]**).

LEMMA 2.2. *If $\dim M^8 - \dim G/K_1 > 2$, that is, $\dim G/K_1 < 6$ and M^8 is simply connected, then G/K_2 is simply connected, hence K_2 is connected.*

Put as follows:

$$SU(3)/S(U(1) \times U(2)) = \mathbb{P}, \quad SU(3)/SO(3) = \mathbb{L}, \quad SU(3)/N(SO(3)) = \mathbb{L}/\mathbb{Z}_3,$$
$$SU(3)/SU(2) = \mathbb{S}, \quad SU(3)/N(\mathbb{Z}_l \times U(2)) = \mathbb{S}/\mathbb{Z}_l,$$
$$SU(3)/T^2 = \mathbb{F}, \quad SU(3)/(\mathbb{Z}_2 \times T^2) = \mathbb{F}/\mathbb{Z}_2, \quad SU(3)/(\mathbb{Z}_3 \times T^2) = \mathbb{F}/\mathbb{Z}_3,$$
$$SU(3)/N(T^2) = \mathbb{F}/S_3,$$

where $l \geq 2$, $\mathbb{S} = S^5$ and $\mathbb{P} = \mathbb{C}P(2)$. First we prepare the following corollary, by Corollary 1.6, 1.13, Proposition 1.17 and Lemma 2.2 and because we can easily see that there is no fixed points.

COROLLARY 2.3. The pair $(G/K_1,\ G/K_2)$ is one of the following (we gather two cases (X, Y) and (Y, X)):

$$(\mathbb{P},\mathbb{P}),\ (\mathbb{P},\mathbb{L}),\ (\mathbb{P},\mathbb{S}),\ (\mathbb{P},\mathbb{F})$$
$$(\mathbb{L},\mathbb{L}),\ (\mathbb{L},\mathbb{S}),\ (\mathbb{L},\mathbb{F})$$
$$(\mathbb{S},\mathbb{S}),\ (\mathbb{S},\mathbb{F})$$
$$(\mathbb{F},\mathbb{L}/\mathbb{Z}_3),\ (\mathbb{F},\mathbb{S}/\mathbb{Z}_l),\ (\mathbb{F},\mathbb{F}),\ (\mathbb{F}/F,\mathbb{F}/F'),$$

where F and $F' = \mathbb{Z}_2$, $\mathbb{Z}_3$ or $S_3 = N(T^2)/T^2$.

We will consider each case (we will omit the case $((\mathbb{F}/F, \mathbb{F}/F'))$).

2.1. The case $(\mathbb{S},\mathbb{S})$. In this case singular orbits are $SU(2)$ and $\dim M^8 - \dim \mathbb{S} = 3$. Moreover we have the following lemma.

LEMMA 2.4. *If $K_i = SU(2)$, then the slice representation is the natural projection (double covering) $p_i : K_i \simeq SU(2) \to SO(3)$ and the tubular neighborhoods are unique. Fix $K_1 = SU(2) \subset SU(3)$ as the $(1,1)$ corrdinate of matrix is equal to 1. Then we can take the principal isotropy subgroup K as*

$$p_1^{-1}(SO(2)) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \in SU(2) = K_1 \subset SU(3) \;\middle|\; t \in U(1) \right\},$$

and $N(K) = T^2 \cup zT^2$ *where* $z \in \mathbb{Z}_2$ *so we can put* $N(K)/N(K)^o$ *as*

$$\left\{ I_3,\ \alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \right\}.$$

Because $\alpha \in SU(2)$, the following diagram is well-defined and commute:

$$\begin{array}{ccc} G \times_{K_1} K_1/K & \longrightarrow & G/K \\ 1 \times r_\alpha \downarrow & & \downarrow R_\alpha \\ G \times_{K_1} K_1/K & \longrightarrow & G/K, \end{array}$$

where the top and the bottom isomorphisms are defined by $[g, kK] = gkK$, $1 \times r_\alpha([g, kK]) = [g, k\alpha K]$ and $R_\alpha(gK) = g\alpha K$. Moreover $1 \times r_\alpha : \partial X_1 = G \times_{K_1} K_1/K \to \partial X_1$ can be equivariant extended to $X_1 = G \times_{K_1} D^3 \to X_1$. Hence the attaching map $R_\alpha : G/K \to G/K$ can be equivariant extended to $X_1 \to X_1$. Therefore we see thta two manifolds $M(I_3)$ and $M(\alpha)$ are equivariant diffeomorphic by the Uchida's criterion. Hence this case is unique and the following proposition holds.

PROPOSITION 2.5. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{S}, \mathbb{S})$, then M^8 is equivariant diffeomorphic to

$$SU(3) \times_{SU(2)} S^3(\mathbb{R}^3 \oplus \mathbb{R})$$

where $SU(2)$ acts on the $\mathbb{R}^3$-part in $S^3(\mathbb{R}^3 \oplus \mathbb{R}) \simeq S^3$ through the natural double covering $SU(2) \to SO(3)$.

2.2. The case $(\mathbb{L}, \mathbb{L})$. In this case singular orbits are $SO(3)$ and $\dim M^8 - \dim \mathbb{L} = 3$. Moreover we have the following lemma.

LEMMA 2.6. *If* $K_i = SO(3)$, *then the slice representation is the natural isomorphism* $\iota_i : K_i \simeq SO(3) \to SO(3)$ *and the tubular neighborhoods are unique. Fix* $K_1 = SO(3) \subset SU(3)$ *as the real part of* $SU(3)$. *Then we can take the principal isotropy subgroup* K *as*

$$\iota_1^{-1}(SO(2)) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in SO(3) = K_1 \subset SU(3) \;\middle|\; 0 \le \theta \le 2\pi \right\},$$

and $N(K) = T^2 \cup zT^2$ *where* $z \in \mathbb{Z}_2$ *so we can put* $N(K)/N(K)^o$ *as*

$$\left\{ I_3,\ \alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

Because $\alpha \in SO(3)$, the following diagram is well-defined and commute:

$$\begin{array}{ccc} G \times_{K_1} K_1/K & \longrightarrow & G/K \\ 1 \times r_\alpha \downarrow & & \downarrow R_\alpha \\ G \times_{K_1} K_1/K & \longrightarrow & G/K, \end{array}$$

where the top and the bottom isomorphisms are defined by $[g, kK] \to gkK$, $1 \times r_\alpha([g, kK]) = [g, k\alpha K]$ and $R_\alpha(gK) = g\alpha K$. Moreover $1 \times r_\alpha : \partial X_1 = G \times_{K_1} K_1/K \to \partial X_1$ can be equivariant extended to $X_1 = G \times_{K_1} D^3 \to X_1$ because $r_\alpha : \partial D^3 = K_1/K \to \partial D^3$ is an induced from the orthogonal map $D^3 \to D^3$. Hence the attaching map $R_\alpha : G/K \to G/K$ can be equivariant extended to $X_1 \to X_1$.

Therefore we see that two manifolds $M(I_3)$ and $M(\alpha)$ are equivariant diffeomorphic by the Uchida's criterion. Hence this case is unique and the following proposition holds.

PROPOSITION 2.7. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{L}, \mathbb{L})$, then M^8 is equivariant diffeomorphic to

$$SU(3) \times_{SO(3)} S^3(\mathbb{R}^3 \oplus \mathbb{R})$$

where $SO(3)$ acts on the $\mathbb{R}^3$-part in $S^3(\mathbb{R}^3 \oplus \mathbb{R}) \simeq S^3$ naturally.

2.3. The case $(\mathbb{L}, \mathbb{S})$. In this case we see the following proposition because of the same arguments in Section 2.1 and 2.2.

PROPOSITION 2.8. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{L}, \mathbb{S})$, then M^8 is equivariant diffeomorphic to

$$SU(3)$$

where $SU(3)$ acts on $SU(3)$ by $\varphi : SU(3) \times SU(3) \to SU(3)$ such that $\varphi(A, X) = AXA^t$.

2.4. The case $(\mathbb{P}, \mathbb{S})$. Since $G/K_2 = \mathbb{S}$, we see that the tubular neighborhood X_2 of $\mathbb{S}$ is unique and the principal isotropy group K is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \in SU(2) = K_2 \subset SU(3) \;\middle|\; t \in U(1) \right\}$$

by the same argument in Section 2.1.

Since $G/K_1 = \mathbb{P}$, we see that $K_1 \simeq S(U(1) \times U(2))$. We put $S(U(1) \times U(2))$ as

$$\left\{ \begin{pmatrix} t^{-2} & 0 \\ 0 & tA \end{pmatrix} \;\middle|\; t \in U(1),\ A \in SU(2) \right\}.$$

Since $\dim M^8 - \dim \mathbb{P} = 4$, K_1 acts on S^3 transitively and its isotropy group is conjugate to K. Hence the slice representation is unique and induced from $T^1 \times SU(2) (\simeq T^1 \times Sp(1))$ action on $S^3 \subset \mathbb{H}$ $((t, h) \cdot r = hrt^{-1})$.

Moreover we see that the attaching map is unique from the same argument in Section 2.1. Therefore we have the following proposition.

PROPOSITION 2.9. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{P}, \mathbb{S})$, then M^8 is equivariant diffeomorphic to

$$\mathbb{H}P(2) = Sp(3)/Sp(1) \times Sp(2)$$

where $SU(3)$ acts on $\mathbb{H}P(2)$ through the natural inclusion $SU(3) \to Sp(3)$.

2.5. The case $(\mathbb{P}, \mathbb{L})$. Since $G/K_2 = \mathbb{L}$, we see that the tubular neighborhood X_2 of $\mathbb{L}$ is unique and the principal isotropy group K is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \in SO(3) = K_2 \subset SU(3) \;\middle|\; 0 \leq \theta \leq 2\pi \right\}$$

by the same argument in Section 2.2.

Moreover the slice representation (the tubulaar neighborhood) is unique for G/K_1 and the attaching map is unique by the same argument in Section 2.4. Therefore we have the following proposition.

PROPOSITION 2.10. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{P}, \mathbb{L},)$, then M^8 is equivariant diffeomorphic to

$$G_2/SO(4)$$

where $SU(3)$ acts on $G_2/SO(4)$ through the natural inclusion $SU(3) \to G_2$.

2.6. The case $(\mathbb{P}, \mathbb{P})$. In this case $K_i \simeq S(U(1) \times U(2))$. Fix $K_1 = S(U(1) \times U(2))$ as

$$\left\{ \begin{pmatrix} t^{-2} & 0 \\ 0 & tA \end{pmatrix} \middle| \, t \in U(1), \ A \in SU(2) \right\}.$$

Then the slice representation of K_1 is induced from $T^1 \times SU(2) (\simeq T^1 \times Sp(1))$ action on $S^3 \subset \mathbb{H}$ $((t, h) \cdot r = hrt^{-l}$, where $l \in \mathbb{N})$. Therefore the principal isotropy group is

$$\left\{ \begin{pmatrix} \lambda^{-2}t^{-2} & 0 & 0 \\ 0 & \lambda t^{l+1} & 0 \\ 0 & 0 & \lambda t^{-l+1} \end{pmatrix} \middle| \, t \in U(1), \ \lambda \in \mathbb{Z}_l \right\},$$

where $\mathbb{Z}_1 = \{1\}$. Hence we see that the slice representation of $K_2 \simeq S(U(1) \times U(2))$ is unique up to $l \in \mathbb{N}$ which is induced by K_1.

If $l = 1$, then there are two attaching map by $|N(K)/N(K)^o| = 2$ and the Uchida's criterion. If $l \neq 1$, then there is unique attaching map by $N(K) = N(K)^o$ and the Uchida's criterion.

PROPOSITION 2.11. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{P}, \mathbb{P})$, then M^8 is equivariant diffeomorphic to one of the followings:

$$Q_4 (= SO(6)/(SO(2) \times SO(4))),$$
$$SU(3) \times_{S(U(1) \times U(2))} S^4(\mathbb{C}_l^2 \oplus \mathbb{R})$$

where in the first case $SU(3)$ acts on Q_4 through the natural inclusion $SU(3) \to SO(6)$ and in the second case $S(U(1) \times U(2))$ acts on $\mathbb{C}_l^2$-part in $S^4(\mathbb{C}_l^2 \oplus \mathbb{R}) \simeq S^4$ by the representation $\rho_l : S(U(1) \times U(2)) \to U(2)$ $(l \in \mathbb{N})$.

2.7. The case $(\mathbb{F}, \mathbb{S}/\mathbb{Z}_l)$ $(l \geq 1)$. Since $G/K_2 = \mathbb{S}/\mathbb{Z}_l$ (where $\mathbb{Z}_1 = \{1\}$), we can fix $K_2 = S(\mathbb{Z}_l \times U(2))$. Since we can easily show that there is unique slice representation of K_2, there is a unique tubular neighborhood X_2 of G/K_2. Then we see tha principal isotorpy group is as follows:

$$\left\{ \begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda t^{-1} & 0 \\ 0 & 0 & \lambda t \end{pmatrix} \middle| \, \lambda \in \mathbb{Z}_l, \ t \in U(1) \right\}$$

Moreover we see that the slice representation of $K_2 = T^2$ is unique up to $l \in \mathbb{N}$ which is induced by K_1, and the attaching map is unique for each $l \in \mathbb{N}$ by the same argument in Section 2.1. Therefore we have the following proposition.

PROPOSITION 2.12. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{F}, \mathbb{S}/\mathbb{Z}_l)$ $(l \geq 1)$, then M^8 is equivariant diffeomorphic to

$$SU(3) \times_{S(U(1) \times U(2))} S^4(\mathbb{C}_l \oplus \mathbb{R}^3)$$

where $S(U(1)\times U(2))$ acts on $\mathbb{C}_l$-part in $S^4(\mathbb{C}_l\oplus\mathbb{R}^3)\simeq S^4$ by the representation τ_l : $S(U(1)\times U(2))\to U(1)$ $(l\in\mathbb{N})$ and on $\mathbb{R}^3$-part in $S^4(\mathbb{C}_l\oplus\mathbb{R}^3)$ by the representation $\sigma: S(U(1)\times U(2))\to SO(3)$.

2.8. The cases $(\mathbb{L},\mathbb{F})$ **and** $(\mathbb{F},\mathbb{L}/\mathbb{Z}_3)$. Since $G/K_2=\mathbb{L}$ or $\mathbb{L}/\mathbb{Z}_3$, we have $K_2=SO(3)$ or $\mathbb{Z}_3\times SO(3)$ where $\mathbb{Z}_3$ is the center of $SU(3)$. For each case there is a unique slice representation of K_2 and the tubular neightborhood X_2 of G/K_2 is unique. And we have the principal isotropy group is as follows:

$$\left\{\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}\in SO(3)\ \middle|\ 0\le\theta\le 2\pi\right\},$$

$$\left\{\begin{pmatrix}\lambda&0&0\\0&\lambda\cos\theta&-\lambda\sin\theta\\0&\lambda\sin\theta&\lambda\cos\theta\end{pmatrix}\in \mathbb{Z}_3\times SO(3)\ \middle|\ 0\le\theta\le 2\pi,\ \lambda\in\mathbb{Z}_3\right\}.$$

Therefore we also have the tubular neighborhood of G/K_1 is unique each case and the attaching map is unique by the same argument in Section 2.2. Hence we have the following propositions.

PROPOSITION 2.13. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{F},\mathbb{L}/\mathbb{Z}_3)$, then M^8 is equivariant diffeomorphic to

$$N=\Delta\backslash SO(6)/(SO(3)\times SO(3))$$

where $SU(3)$ acts on N through the natural inclusion $SU(3)\to U(3)\to SO(6)$ and Δ is the center of $U(3)$.

PROPOSITION 2.14. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{L},\mathbb{F})$, then M^8 is equivariant diffeomorphic to

$$\widetilde{N}$$

where $\widetilde{N}$ is the universal (three folds) covering of N.

2.9. The case $(\mathbb{P},\mathbb{F})$. Now we have the principal isotropy group is as follows from $K_1=S(U(1)\times U(2))$ and Section 2.6:

$$\left\{\begin{pmatrix}\lambda^{-2}t^{-2}&0&0\\0&\lambda t^{l+1}&0\\0&0&\lambda t^{-l+1}\end{pmatrix}\ \middle|\ t\in U(1),\ \lambda\in\mathbb{Z}_l\right\},$$

where $\mathbb{Z}_1=\{1\}$. Hence X_2 is unique for each $l\in\mathbb{N}$.

If $l=1$ then there are two attaching maps, and if $l\neq 1$ then there is unique attaching map. Therefore we have the following proposition.

PROPOSITION 2.15. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{P},\mathbb{F})$, then M^8 is equivariant diffeomorphic to one of the followings:

$$\mathbb{C}P(2)\times\mathbb{C}P(2),$$
$$SU(3)\times_{S(U(1)\times U(2))}\mathbb{P}(\mathbb{C}_l^2\oplus\mathbb{C})$$

where in the first case $SU(3)$ acts on $\mathbb{C}P(2)\times\mathbb{C}P(2)$ diagonally and in the second case $S(U(1)\times U(2))$ acts on $\mathbb{C}_l^2$-part in $\mathbb{P}(\mathbb{C}_l^2\oplus\mathbb{C})\simeq\mathbb{C}P(2)$ through the representation $\rho_l: S(U(1)\times U(2))\to U(2)$ $(l\in\mathbb{N})$.

2.10. The case $(\mathbb{F},\mathbb{F})$. Since $G/K_i \simeq \mathbb{F}$, we can put $K_1 = T^2 = K_2$. The slice representation $K_1 = T^2 \to U(1) \simeq SO(2) \subset O(2)$ is as follows:

$$\begin{pmatrix} t_1^{-1}t_2^{-1} & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & t_2 \end{pmatrix} \to t_1^p t_2^q.$$

We can put $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ up to equivalence of the representation and the conjugation of K_1. The principal isotropy group is

$$\left\{ \begin{pmatrix} \lambda^{-1}\omega^{-1}t^{-1+\frac{q}{p}} & 0 & 0 \\ 0 & \lambda t^{-\frac{q}{p}} & 0 \\ 0 & 0 & \omega t \end{pmatrix} \middle| \lambda \in \mathbb{Z}_p,\ \omega \in \mathbb{Z}_q \right\}$$

where $\mathbb{Z}_{\pm 1} = \{1\} = \mathbb{Z}_0$. Therefore the slice repesentation of K_2 is same as above the slice representation of K_1. Moreover we see that there are two attaching maps for $p = q$ and there is a unique attaching map for $p \neq q$. Hence we have the following proposition.

PROPOSITION 2.16. If M^8 has $SU(3)$-action with codimension one orbits and two singular orbits $(\mathbb{F},\mathbb{F})$, then M^8 is equivariant diffeomorphic to one of the followings:

$$SU(3) \times_{S(U(1)\times U(2))} H_{2k+1},$$
$$SU(3) \times_{T^2} S^2(\mathbb{C}_{(p,q)} \oplus \mathbb{R})$$

where in the first case $S(U(1)\times U(2))$ acts on the Hirzebruch surface H_{2k+1} induced by the line bundle over $\mathbb{C}P(1)$ whose first chern class is odd (also see [**Ku07**]), and in the second case T^2 acts on $\mathbb{C}_{(p,q)}$-part in $S^2(\mathbb{C}_{(p,q)} \oplus \mathbb{R})$ through the representation $\tau_{(p,q)} : T^2 \to U(1)$ $(p, q \in \mathbb{N})$.

REMARK 2.17. $SU(3) \times_{S(U(1)\times U(2))} H_{2k+1}$ is one of the $p = q$ cases. If $p \neq q$ then a manifold is $SU(3) \times_{T^2} S^2(\mathbb{C}_{(p,q)} \oplus \mathbb{R})$. If $p = q$ then we can consider $SU(3) \times_{T^2} S^2(\mathbb{C}_{(p,p)} \oplus \mathbb{R})$ as $SU(3) \times_{S(U(1)\times U(2))} H_{2k}$ where the Hirzebruch surface $H_{2k} (\simeq \mathbb{C}P(1) \times \mathbb{C}P(1))$ induced by the line bundle over $\mathbb{C}P(1)$ whose first chern class is even.

We omit the case $((\mathbb{F}/F, \mathbb{F}/F'))$.

Finally we remark the following corollaries.

Because of the proofs of Proposition 1.1 and Lemma 1.3, the following corollaries can be shown.

COROLLARY 2.18. If M^8 has an $SU(3)$-action with codimension four principal orbits, then all orbits are principal orbits $\mathbb{C}P(2)$ and there is a fibration $\mathbb{C}P(2) \to M^8 \xrightarrow{\pi} \Sigma^4$ where π is a projection to the orbit space and the orbit space Σ^4 is a 4-dimensional manifols.

COROLLARY 2.19. There is no $SU(3)$-action on M^8 with codimension less than or equal to five orbits.

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