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Linear and topological properties of a sequence space defined by an L_p-function

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Abstract

We introduce a sequence space $\Lambda_p(f)$ defined by an L_p-function $f(\neq 0)$ for $1 \leq p < +\infty$ by

$$\Lambda_p(f) := \{\boldsymbol{a} \in \mathbf{R}^\infty : \Psi_p(\boldsymbol{a} : f) < +\infty\},$$

where

$$\begin{aligned}\Psi_p(\boldsymbol{a} : f) &:= \Big(\sum_n \int_{-\infty}^{+\infty} |f(x-a_n) - f(x)|^p \, dx\Big)^{\frac{1}{p}} \\ &= \Big(\sum_n \|f(\cdot - a_n) - f(\cdot)\|_{L_p}^p\Big)^{\frac{1}{p}},\end{aligned}$$

and discuss the linear and topological properties of $\Lambda_p(f)$, that is, the linearity, the relations with ℓ_p, the linear topological property of the metric $d_p(\boldsymbol{a}, \boldsymbol{b}) = \Psi_p(\boldsymbol{a} - \boldsymbol{b} : f)$ on $\Lambda_p(f)$, the completeness, and so on.

In the case where $p = 2, \Lambda_2(\sqrt{f})$ is studied in the theory of translation equivalence of the infinite product measure $\boldsymbol{\mu} = \otimes_1^\infty f(x)dx$ on $I\!R^\infty$. In fact, if $f(x) > 0$ a.e.(x), then $\boldsymbol{a} \in \Lambda_2(\sqrt{f})$ if and only if the translation $\boldsymbol{\mu_a}$ is equivalent to $\boldsymbol{\mu}$, see Kakutani[3], Shepp[4].

1 Introduction

Let $f(\neq 0)$ be an L_p-function on the real line $\mathbf{R}$.
For $1 \leq p < +\infty$ and for a real sequence $\boldsymbol{a} = \{a_n\} \in \mathbf{R}^\infty$, we set

$$\begin{aligned} \Psi_p(\boldsymbol{a} : f) &:= \Big(\sum_n \int_{-\infty}^{+\infty} |f(x-a_n) - f(x)|^p \, dx\Big)^{\frac{1}{p}} \\ &= \Big(\sum_n \|f(\cdot - a_n) - f(\cdot)\|_{L_p}^p\Big)^{\frac{1}{p}}, \end{aligned}$$

and define $\Lambda_p(f)$ by

$$\Lambda_p(f) := \{\boldsymbol{a} \in \mathbf{R}^\infty : \Psi_p(\boldsymbol{a} : f) < +\infty\}.$$

By the triangular inequality of L_p-norm, we have

$$\Psi_p(\boldsymbol{a} - \boldsymbol{b} : f) \leq \Psi_p(\boldsymbol{a} : f) + \Psi_p(\boldsymbol{b} : f),$$

which implies that $\Lambda_p(f)$ is an additive subgroup of $\mathbf{R}^\infty$.

Define a metric on $\Lambda_p(f)$ by

$$d_p(\boldsymbol{a}, \boldsymbol{b}) := \Psi_p(\boldsymbol{a} - \boldsymbol{b} : f).$$

Then $(\Lambda_p(f), d_p(\boldsymbol{a}, \boldsymbol{b}))$ becomes a topological group.

In this talk, we are concerned with the following problems:

1. the linearity of $\Lambda_p(f)$,
2. the relations between $\Lambda_p(f)$ and ℓ_p, and
3. the linear topological property of the metric $d_p(\boldsymbol{a}, \boldsymbol{b})$ on $\Lambda_p(f)$,
4. the completeness of $(\Lambda_p(f), d_p)$.

2 Linearity of $\Lambda_p(f)$

The function f is called unimodal at α if there exists $\alpha \in \mathbb{R}$ such that $f(x)$ is non-decreasing on $(-\infty, \alpha)$ and non-increasing on $(\alpha, +\infty)$.

Theorem 1 ([1]) Assume the L_p-function $f(\neq 0)$ is unimodal. Then we have

$$\Psi_p(t\boldsymbol{a} : f) \leq \Psi_p(\boldsymbol{a} : f), \ 0 < t \leq 1$$

for any $\boldsymbol{a} \in \Lambda_p(f)$. In particular, $\Lambda_p(f)$ is a linear space.

3 Relations between $\Lambda_p(f)$ and ℓ_p

We say $I_p(f) < +\infty$ if $f(x)$ is absolutely continuous on $\mathbb{R}$ and the *p-integral* defined by

$$I_p(f) := \int_{-\infty}^{+\infty} |f'(x)|^p \, dx$$

is finite. In particular $I_2(\sqrt{f})$, where f is a probability density function on $\mathbb{R}$, coincides with the Shepp's integral(Shepp[4]).

Theorem 2 ([2]) Let $1 \leq p < +\infty$ and let $f(\neq 0)$ be an L_p-function on $\mathbb{R}$. Then $\Lambda_p(f) \subset \ell_p$

Theorem 3 ([2]) Let $1 < p < +\infty$ and $f(\neq 0)$ be an L_p-function on $\mathbb{R}$. Then $\Lambda_p(f) = \ell_p$ if and only if $I_p(f) < +\infty$.

4 Linear topological properties of $\Lambda_p(f)$

If $I_p(f) < +\infty$, then $\Lambda_p(f) = \ell_p$ as a sequence space. We shall show in this case the ℓ_p-norm $\| \; \|_p$ is stronger than the metric d_p,

Theorem 4 Assume $I_p(f) < +\infty$. Then the ℓ_p-norm is stronger than the metric d_p on $\Lambda_p(f) = \ell_p$.

Proof. Since $\Psi_p(\boldsymbol{a} : f)$ is lower semi-continuous on ℓ_p, by the Baire's category theorem, there exists N such that the set $L_N := \{\boldsymbol{a} \in \Lambda_p(f) = \ell_p : \Psi_p(\boldsymbol{a} : f) \leq N\}$ has an interior point with respect to the ℓ_p-norm. So that there exists $\boldsymbol{a}_0 \in L_N$ and $\delta > 0$ such that $\|\boldsymbol{a} - \boldsymbol{a}_0\|_p \leq \delta$ implies $\Psi_p(\boldsymbol{a} : f) \leq N$, which implies

$$\|\boldsymbol{a}\|_p \leq \delta \Rightarrow \Psi_p(\boldsymbol{a} : f) \leq \Psi_p(\boldsymbol{a} + \boldsymbol{a}_0 : f) + \Psi_p(\boldsymbol{a}_0 : f) \leq 2N.$$

and

$$\|\boldsymbol{a}\|_p \leq K \Rightarrow \Psi_p(\boldsymbol{a} : f) \leq 2\Big(\Big[\frac{K}{\delta}\Big] + 1\Big)N.$$

By Xia[5], Lemma I.2.2, there exists $\boldsymbol{b}_0$ such that $\Psi_p(\cdot \; : f)$ is ℓ_p-continuous at $\boldsymbol{b}_0$. So that for every $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$\|\boldsymbol{b} - \boldsymbol{b}_0\|_p \leq \lambda \Rightarrow |\Psi_p(\boldsymbol{b} : f)^p - \Psi_p(\boldsymbol{b}_0 : f)^p| \leq \varepsilon.$$

Now we shall show $\Psi_p(\cdot \; : f)$ is ℓ_p-continuous at $\boldsymbol{0}$. For every $\boldsymbol{b}$ with $\|\boldsymbol{b}\| \leq \lambda$, and for every natural numbers n and N, we set

$$\boldsymbol{b}(m, N) := \Big(b_1^0, \cdots, b_N^0, b_{N+1}^0 + b_1, \cdots, b_{N+m}^0 + b_m, b_{N+m+1}^0, \cdots\Big),$$

where $\boldsymbol{b}_0 = \{b_i^0\}$. Then we have

$$\|\boldsymbol{b}(m,N) - \boldsymbol{b}_0\|_p = (\sum_{i=1}^{m} b_i^p)^{\frac{1}{p}} \leq \lambda,$$

which implies

$$|\Psi_p(\boldsymbol{b}(m,N):f)^p - \Psi_p(\boldsymbol{b}:f)^p| = \sum_{i=1}^{m} \int_{-\infty}^{+\infty} |f(x - b_{N+i}^0 - b_i) - f(x)|^p dx \leq \varepsilon.$$

Letting $N \to +\infty$, we have

$$\sum_{i=1}^{m} \int_{-\infty}^{+\infty} |f(x - b_i) - f(x)|^p dx \leq \varepsilon,$$

for every m, and

$$\Psi_p(\boldsymbol{b}:f)^p = \sum_{i=1}^{+\infty} \int_{-\infty}^{+\infty} |f(x - b_i) - f(x)|^p dx \leq \varepsilon,$$

which shows $\Psi_p(\cdot\ : f)$ is ℓ_p-continuous at $\mathbf{0}$.

We can now easily deduce the continuity of $\Psi_p(\cdot\ : f)$ at any point $\boldsymbol{c}_0$ as follows. If $\|\boldsymbol{c} - \boldsymbol{c}_0\|_p \leq \lambda$, then we have

$$|\Psi_p(\boldsymbol{c}:f) - \Psi_p(\boldsymbol{c}_0:f)| \leq \Psi_p(\boldsymbol{c} - \boldsymbol{c}_0 : f) \leq \varepsilon^{\frac{1}{p}}.$$

Theorem 5 If $f(x)$ is unimodular, then the metric d_p is the vector topology on $\Lambda_p(f)$.

Proof. By Theorem 1, the scalar multiplication is continuous.

We consider the largest linear subspace $\Sigma_p(f)$ of $\Lambda_p(f)$ after Yamasaki[6] as follows. Define

$$\Sigma_p(f) := \{\boldsymbol{a} \in \Lambda_p(f) : t\boldsymbol{a} \in \Lambda_p(f) \text{ for every t} \in \mathrm{R}\}.$$

Lemma 6 If $\boldsymbol{a}(\neq \mathbf{0}) \in \Sigma_p(f)$, then the real function $\varphi(t:\boldsymbol{a}) = \Psi_p(t\boldsymbol{a}:f)^p$ is continuous on the real line $\mathbb{R}$. Moreover, the metric

$$\rho(s,t) = \Psi_p((t-s)\boldsymbol{a}:f)$$

gives the equivqlent metric with the usual metric $|s-t|$.

Proof. The continuity of $\varphi(t : \boldsymbol{a})$ is proved by the similar way to Theorem 5. Since $\boldsymbol{a} \neq \boldsymbol{0}$, there exists $a_k \neq 0$. If

$$\int_{-\infty}^{+\infty} |f(x - t_n a_k) - f(x)|^p dx \to 0 \text{ as n} \to +\infty,$$

then it follows that $t_n \to 0$, see the proof of Theorem 2. This proves the second assertion.

Let $V_\varepsilon = \{\boldsymbol{a} \in \Sigma_p(f) : \Psi_p(\boldsymbol{a} : f) \le \varepsilon\}$. Then for every $\boldsymbol{x} \in \Sigma_p(f)$, we can find $\delta > 0$ such that

$$t\boldsymbol{x} \in V_\varepsilon \text{ for every } -\delta < \mathrm{t} < \delta.$$

Consequently we can linearize d_p as follows, see Yamasaki[6], p.185, Xia[5], Lemma I.1.2. The linearization $\sigma_p(\boldsymbol{a}, \boldsymbol{b})$ of $d_p(\boldsymbol{a}, \boldsymbol{b})$ is defined by

$$\sigma_p(\boldsymbol{a}, \boldsymbol{b}) := \sup_{|t| \le 1} d_p(t\boldsymbol{a}, t\boldsymbol{b})$$

for $\boldsymbol{a}, \boldsymbol{b} \in \Sigma_p(f)$.

Theorem 7 $(\Sigma_p(f), \sigma_p(\boldsymbol{a}, \boldsymbol{b}))$ is a topological vector space.

5 Completeness of $\Lambda_p(f)$

Theorem 8 ([1]) Let $f(\neq 0)$ be an L_p-function. Then $\Lambda_p(f)$ is complete with respect to d_p for $1 \le p < +\infty$.

Theorem 9 $(\Sigma_p(f), \sigma_p(\boldsymbol{a}, \boldsymbol{b}))$ is complete.

6 Examples

Example 10 Define $f(x) := \max\{1 - |x|, 0\}$. Then we have
(1) *for* $1 \le p < 2$, $\Lambda_p(f) = \ell_p$,
(2) $\Lambda_2(f) = \left\{\boldsymbol{a} = (a_n) \in \mathbb{R}^\infty \;\middle|\; \sum_n a_n^2 \left(1 + \left|\log |a_n|\right|\right) < +\infty\right\}$, and
(3) *for* $p > 2, \Lambda_p(f) = \ell_2$.

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