Boundedness of $\gamma$-Cesàro means ($\gamma > 0$) of operators

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In this talk I would like to report some recent results on the boundedness properties of $\gamma$-Cesàro means of operators, where $\gamma > 0$. The results are taken from joint works with Jeng-Chung Chen, Yuan-Chuan Li, and Sen-Yen Shaw (cf. [1], [3]).

1. The discrete case. Let $T : X \rightarrow X$ be a bounded linear operator on a Banach space $X$. The Cesàro means of order $\gamma$ (or $\gamma$-Cesàro means) of $T$, where $\gamma \geq 0$, are defined by

$$C_n^\gamma = C_n^\gamma(T) := \frac{1}{\sigma_n^\gamma} \sum_{k=0}^{n} \sigma_{n-k}^{\gamma-1}T^k,$$

where $\sigma_n^\beta = \binom{\beta+n}{n}$ for $n \geq 1$, and $\sigma_0^\beta = 1$ (see Zygmund [5, Chapter 3]). Among them are the following two particular means: $C_n^0 = C_n^0(T) = T^n$, and $C_n^1 = C_n^1(T) = (n+1)^{-1} \sum_{k=0}^{n} T^k$.

The Abel means of $T$ are the operators $A_r = A_r(T) := (1-r) \sum_{k=0}^{\infty} r^kT^k$, defined for $0 < r < 1/r(T)$, where $r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$ denotes the spectral radius of $T$. Clearly, $r(T) \leq 1$ if and only if $A_r$ exists for all $0 < r < 1$. (Moreover, in this case, we have $A_r = (1-r)(I-rT)^{-1}$ for each $0 < r < 1$.) The following is well-known (cf. [5]):

If $0 < \gamma < \beta < \infty$, then

1. $\sup_{0< r< 1} \|A_r\| \leq \sup_{n \geq 0} \|C_n^\beta\| \leq \sup_{n \geq 0} \|C_n^\gamma\| \leq \sup_{n \geq 0} \|C_n^0\| = \sup_{n \geq 0} \|T^n\|)$;

in particular, if $T$ is a positive linear operator on a Banach lattice $X$, then

2. $\sup_{0< r< 1} \|A_r\| < \infty \iff \sup_{n \geq 0} \|C_n^1\| < \infty$ (cf. Emilion [2]).
In connection with these relations, two questions come up naturally:

(A) If $T$ is positive, then does the implication 
\[ \sup_{0<r<1} |A_r| < \infty \Rightarrow \sup_{n \geq 1} \| C_n^\gamma \| < \infty \]
hold for a certain constant $\gamma$, with $0 < \gamma < 1$?

(B) If $T$ is not assumed to be positive, then does the implication 
\[ \sup_{0<r<1} |A_r| < \infty \Rightarrow \sup_{n \geq 1} \| C_n^\gamma \| < \infty \]
hold for a certain constant $\gamma$, with $\gamma \geq 1$?

Our answers are as follows.

**Theorem 1.** For any $\gamma$, with $0 < \gamma < 1$, there exists a positive linear operator $T$ on an $L_1$-space such that \( \sup_{n \geq 1} \| C_n^\beta \| < \infty \) for all $\beta > \gamma$, but \( \sup_{n \geq 1} \| C_n^\beta \| = \infty \).

**Theorem 2.** There exists a positive linear operator $T$ on an $L_1$-space such that \( \sup_{n \geq 1} \| C_n^\beta \| < \infty \) for all $\beta > 0$, but \( \sup_{n \geq 1} \| T^n \| = \infty \).

**Theorem 3.** Let $\dim X = \infty$. Then the following hold:

(i) For any integer $k \geq 1$, there exists a bounded linear operator $T$ on $X$ such that \( \sup_{n \geq 1} \| C_n^k \| < \infty \), but \( \sup_{n \geq 1} \| C_n^\beta \| = \infty \) for all $\beta$, with $0 \leq \beta < k$.

(ii) There exists a bounded linear operator $T$ on $X$, with $r(T) = 1$, such that \( \sup_{0<r<1} |A_r| < \infty \), but \( \sup_{n \geq 1} \| C_n^\beta \| = \infty \) for all $\beta \geq 0$.

**2. The continuous case.** Let $T(\cdot)$ be a $C_0$-semigroup of bounded linear operators on a Banach space $X$. The $\gamma$-th Cesàro means of $T(\cdot)$, where $\gamma \geq 0$, are defined as 
\[ C^\gamma_0 = C^\gamma_0 (T(\cdot)) := T(0) \]
and, for $t > 0$,
\[ C^\gamma_t = C^\gamma_t (T(\cdot)) := \begin{cases} T(t) & \text{if } \gamma = 0, \\ \gamma t^{-\gamma} \int_0^t (t-u)^{\gamma-1} T(u) \, du & \text{if } \gamma > 0. \end{cases} \]

The Abel means of $T(\cdot)$ are the operators
\[ A_\lambda = A_\lambda (T(\cdot)) := \lambda \int_0^\infty e^{-\lambda u} T(u) \, du = \lim_{t \to \infty} \lambda \int_0^t e^{-\lambda u} T(u) \, du, \]
defined for $\lambda > 0$ if the limit exists. As in the discrete case, we have (cf. [4]):

If $0 < \gamma < \beta < \infty$, then
\[
\sup_{0 < \lambda < \infty} \| A_\lambda \| \leq \sup_{t > 0} \| C^\beta_t \| \leq \sup_{t > 0} \| C^\gamma_t \| \leq \sup_{t > 0} \| C^0_t \| \leq \sup_{t > 0} \| T(t) \| ;
\]
in particular, if $T(\cdot)$ is a positive $C_0$-semigroup on a Banach lattice $X$, then

$$ \sup_{0<\lambda<\infty} \|A_\lambda\| < \infty \iff \sup_{t>0} \|C_t^1\| < \infty \quad (\text{cf. [2]}). $$

The following are the continuous case results:

**Theorem 1'.** For any $\gamma$, with $0 < \gamma < 1$, there exists a positive $C_0$-semigroup $T(\cdot)$ on an $L_1$-space such that $\sup_{t>0} \|C_t^\beta\| < \infty$ for all $\beta > \gamma$, but $\sup_{t>0} \|C_t^\gamma\| = \infty$.

**Theorem 2'.** There exists a positive $C_0$-semigroup $T(\cdot)$ on an $L_1$-space such that $\sup_{t>0} \|C_t^\beta\| < \infty$ for all $\beta > 0$, but $\sup_{t>0} \|T(t)\| = \infty$.

**Theorem 3'.** Let $\dim X = \infty$. Then the following hold:

(i) For any integer $k > 1$, there exists a $C_0$-semigroup $T(\cdot)$ on $X$ such that $\sup_{t>0} \|C_t^k\| < \infty$, but $\sup_{t>0} \|C_t^\beta\| = \infty$ for all $\beta$, with $0 \leq \beta < k$.

(ii) There exists a $C_0$-semigroup $T(\cdot)$ on $X$ such that $\sup_{0<\lambda<\infty} \|A_\lambda\| < \infty$, but $\sup_{t>0} \|C_t^\beta\| = \infty$ for all $\beta \geq 0$.

**References**


