数理解析研究所講究録
第 1570 巻 2007 年 68-75

SEQUENCE SPACES AND INCLUSION INDICES

LUZ M. FERNÁNDEZ-CABRERA*

ABSTRACT. Inclusion indices of quasi-Banach spaces have been studied by Cobos, Manzano, Martínez and the author (Bolletino U.M.I. **10-B** (2007), 99-117). We review their results on sequence spaces, providing proofs of results that were only stated in that paper.

0. INTRODUCTION.

Let E be a Banach space of sequences with $\ell_1 \hookrightarrow E \hookrightarrow \ell_\infty$, where $\hookrightarrow$ means continuous embedding. The *inclusion indices* of E are defined by

$$\delta_E = \sup\{p \geq 1 : \ell_p \hookrightarrow E\}\ ,\ \gamma_E = \inf\{p \leq \infty : E \hookrightarrow \ell_p\}.$$

Inclusion indices are useful in the research of properties of embeddings between sequence spaces (see, for example, [7], [8], [9] and [11]).

If E is symmetric then indices can be computed by using the fundamental function φ_E of E. Namely

$$\delta_E = \liminf_{n\to\infty} \frac{\log n}{\log \varphi_E(n)} \qquad \text{and} \qquad \gamma_E = \limsup_{n\to\infty} \frac{\log n}{\log \varphi_E(n)}. \tag{0.1}$$

Cobos, Manzano, Martínez and the author have studied in [5] inclusion indices of quasi-Banach spaces. Their results apply to function spaces, sequence spaces and to any intermediate space with respect to an ordered compatible couple. The aim of the present paper is to review their work on sequence spaces, providing proofs of results that were only stated in [5]. This is done in Section 2, while in Section 1 we recall some basic concepts on sequence spaces.

1. PRELIMINARIES

We denote by $\mathfrak{f}$ the set of all sequences $\xi = \{\xi_n\}$ which have a finite number of coordinates $\xi_n \neq 0$.

Following [14] we define the non-increasing rearrangement of a bounded sequence $\mu = \{\mu_n\} \in \ell_\infty$ as the sequence $\mu^* = \{s_n(\mu)\}$ given by

$$s_n(\mu) = \inf\{\|\mu - \tau\|_{\ell_\infty} : \tau = \{\tau_m\} \in \mathfrak{f},\ \operatorname{card}\{m \in \mathbb{N} : \tau_m \neq 0\} < n\}.$$

2000 *Mathematical Subject Classification*: 46A45, 46E30.

* The author has been supported in part by the Spanish Ministerio de Educación y Ciencia (MTM2004-01888).

Here card A stands for the cardinality of the set A. In the special case that ξ is a zero sequence, $\xi \in c_0$, then we have $s_n(\xi) = |\xi_n^*|$, where $\{\xi_n^*\}$ is the rearrangement of the elements of $\{\xi_n\}$ by magnitude of the absolute values, $|\xi_1^*| \geq |\xi_2^*| \geq \cdots$.

Given any subset $D \subseteq \mathbb{N}$, we put $e_D = \{\tau_n\}$ where $\tau_n = 1$ if $n \in D$ and $\tau_n = 0$ if $n \notin D$. If $\xi = \{\xi_n\}, \mu = \{\mu_n\}$ are bounded sequences, $\xi\mu$ denotes the sequence $\{\xi_n \mu_n\}$.

We say that a quasi-Banach lattice of bounded sequences E is *symmetric* (or *rearrangement invariant*) if E satisfies the following conditions:

(i) $e_{\{1\}}$ belongs to E with $\|e_{\{1\}}\|_E = 1$.

(ii) Whenever $\xi \in E$ and $\mu \in \ell_\infty$ with $\xi^* = \mu^*$, then $\mu \in E$ and $\|\xi\|_E = \|\mu\|_E$.

These conditions yield that $\mathfrak{f} \subseteq E$. On the other hand, we have $E \hookrightarrow \ell_\infty$ because for any $\xi = \{\xi_n\} \in E$ and any $n \in \mathbb{N}$

$$|\xi_n| = \|\xi_n e_{\{1\}}\|_E = \|\xi_n e_{\{n\}}\|_E \leq \|\xi_n\|_E.$$

The *fundamental function* of the symmetric sequence space E is defined by

$$\varphi_E(n) = \|e_{\{1,\cdots,n\}}\|_E.$$

The function φ_E is non-decreasing with $\varphi_E(1) = 1$. It is also clear that if $E = c_0$ or $E = \ell_\infty$ then $\lim_{n\to\infty} \varphi_E(n) = 1 < \infty$. Next we show that the converse of this statement holds.

Lemma 1.1. *If the fundamental function of a symmetric quasi-Banach sequence space E satisfies that $\lim_{n\to\infty} \varphi_E(n) = c < \infty$, then $E = c_0$ or $E = \ell_\infty$.*

Proof. Take any $\xi = \{\xi_n\} \in c_0$ and let $\eta_n = \xi e_{\{1,\cdots,n\}}$. For any $m > n$ we have

$$\begin{aligned} \|\eta_m - \eta_n\|_E &\leq \max\{|\xi_j| : n+1 \leq j \leq m\}\, \varphi_E(m-n) \\ &\leq c\max\{|\xi_j| : n+1 \leq j \leq m\} \longrightarrow 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\{\eta_n\}$ is a Cauchy sequence in E. This yields that $\xi \in E$ and that $c_0 \hookrightarrow E$. Consequently, $c_0 \hookrightarrow E \hookrightarrow \ell_\infty$. Now, using [14], Thm. 13.1.8, we conclude that $E = c_0$ or $E = \ell_\infty$. □

Important examples of symmetric quasi-Banach sequence spaces are ℓ_p and $\ell_{p,\infty}$. Recall that for $0 < p < \infty$ the *Lorentz sequence space* $\ell_{p,\infty}$ is formed by all bounded sequences $\xi = \{\xi_n\}$ having a finite quasi-norm

$$\|\xi\|_{\ell_{p,\infty}} = \sup_{n\in\mathbb{N}}\{n^{1/p} s_n(\xi)\}.$$

It is easy to check that

$$\varphi_{\ell_p}(n) = \varphi_{\ell_{p,\infty}}(n) = n^{1/p} \quad \text{for all} \quad n \in \mathbb{N}.$$

2. Indices of quasi-Banach sequence spaces

In this section we investigate the notion of inclusion indices of sequences spaces by using the whole scale of ℓ_p–spaces, that is $\{\ell_p\}_{p>0}$, and not only the Banach part where $1 \leq p \leq \infty$. The natural spaces to consider are quasi-Banach sequence spaces. Indices are defined as follows.

Definition 2.1. Let F be a quasi-Banach sequence space. We define the *lower inclusion index* of F by

$$\delta_F = \sup\{0 < p < \infty : \ell_p \hookrightarrow F\}.$$

If there is no $0 < p < \infty$ such that $\ell_p \hookrightarrow F$, we put $\delta_F = 0$.

The *upper inclusion index* of F is defined by

$$\gamma_F = \inf\{0 < p < \infty : F \hookrightarrow \ell_p\}.$$

If $F \not\hookrightarrow \ell_p$ for any $0 < p < \infty$, then we write $\gamma_F = \infty$.

Next we show that the formulae in (0.1) still hold for quasi-Banach sequence spaces. Note that the proof of (0.1) in the Banach case does not work in our setting because it is based on the fact that any symmetric Banach space X lies between the Lorentz and the Marcinkiewicz space with fundamental function φ_X (see [2] or [12]). For symmetric quasi-Banach spaces no similar result is known. Only for $p-$Banach spaces a partial result can be found in [1].

Theorem 2.2. *Let E be a symmetric quasi-Banach sequence space. Then*

$$\delta_E = \liminf_{n\to\infty} \frac{\log n}{\log \varphi_E(n)}.$$

Proof. Assume first that $\lim_{n\to\infty} \varphi_E(n) < \infty$. Then $\liminf_{n\to\infty}[\log n/\log \varphi_E(n)] = \infty$. On the other hand, using Lemma 1.1 we get that $E = c_0$ or $E = \ell_\infty$ and therefore $\delta_E = \infty$.

Assume now that $\lim_{n\to\infty} \varphi_E(n) = \infty$. If there is any $p > 0$ such that $\ell_p \hookrightarrow E$, then we can find $C > 0$ so that

$$\varphi_E(n) \le C n^{1/p} \quad \text{for any} \quad n \in \mathbb{N}.$$

Taking logarithms and lower limits we obtain $p \le \liminf_{n\to\infty}[\log n/\log \varphi_E(n)]$. This implies that

$$\delta_E \le \liminf_{n\to\infty} \frac{\log n}{\log \varphi_E(n)}.$$

If $\liminf_{n\to\infty}[\log n/\log \varphi_E(n)] = 0$, the previous argument shows that there is no $0 < p < \infty$ such that $\ell_p \hookrightarrow E$. Then, by Definition 2.1, we have that $\delta_E = 0$ and we are done.

In order to establish the remaining case, take any p with $0 < p < \liminf_{n\to\infty}[\log n/\log \varphi_E(n)]$ and let us check that $\ell_p \hookrightarrow E$. Since $\ell_p \hookrightarrow \ell_{p,\infty}$, it is enough to show that $\ell_{p,\infty} \hookrightarrow E$. A sufficient condition for the last embedding is that

$$\tau = \{n^{-1/p}\} \quad \text{belongs to} \quad E. \tag{2.1}$$

Indeed, if this is the case, for any $\xi \in \ell_{p,\infty}$ using that

$$s_n(\xi) = n^{-1/p}(n^{1/p}s_n(\xi)) \le n^{-1/p}\|\xi\|_{\ell_{p,\infty}},$$

we get

$$\|\xi\|_E = \|\xi^*\|_E \le \|\tau\|_E \|\xi\|_{\ell_{\ell_{p,\infty}}}.$$

To prove (2.1) take any q with $p < q < \liminf_{n\to\infty}[\log n/\log \varphi_E(n)]$. There exists $N \in \mathbb{N}$ such that $\varphi_E(n) < n^{1/q}$ for all $n \ge N$. Hence, we can find $M > 0$ such that

$$\varphi_E(n) \le M n^{1/q} \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.2}$$

Put

$$\eta_n = \tau e_{\{1,\cdots,2^n\}}.$$

Then $\{\eta_n\} \subseteq \mathfrak{f} \subseteq E$. We claim that $\{\eta_n\}$ is a Cauchy sequence in E. Indeed, let c be the constant in the triangle inequality of E and define ρ by the equation $(2c)^\rho = 2$. According to [3], Lemma 3.10.1 and (2.2) we derive for $n < m$

$$\begin{aligned}\|\eta_m - \eta_n\|_E^\rho &= \|\tau e_{\{2^n+1,\cdots,2^m\}}\|_E^\rho \\ &\le 2\sum_{j=n}^{m-1} \|\tau e_{\{2^j+1,\cdots,2^{j+1}\}}\|_E^\rho \\ &\le 2\sum_{j=n}^{m-1} 2^{-j\rho/p}\varphi_E(2^j)^\rho \\ &\le 2M^\rho \sum_{j=n}^{m-1} 2^{(1/q-1/p)\rho j} \longrightarrow 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Since the sequence formed by the n-th coordinates of $\eta_1, \eta_2, ..., \eta_m, ...$ converges to the n-th coordinate of τ, the limit of $\{\eta_m\}$ must be τ. Consequently, $\tau \in E$. This proves (2.1) and completes the proof. □

The corresponding formula for the upper index says the following.

Theorem 2.3. *Let E be a symmetric quasi-Banach sequence space. Then*

$$\gamma_E = \limsup_{n\to\infty} \frac{\log n}{\log \varphi_E(n)}.$$

Proof. If $E \hookrightarrow \ell_p$ for some $0 < p < \infty$, then we can find $C > 0$ such that $n^{1/p} \le C\varphi_E(n)$ for all $n \in \mathbb{N}$. Hence $\limsup_{n\to\infty}[\log n/\log\varphi_E(n)] \le p$. This implies that

$$\limsup_{n\to\infty} \frac{\log n}{\log \varphi_E(n)} \le \gamma_E.$$

If $\limsup_{n\to\infty}[\log n/\log\varphi_E(n)] = \infty$, there is no $0 < p < \infty$ such that $E \hookrightarrow \ell_p$. Then Definition 2.1 yields that $\gamma_E = \infty$ and we obtain the wanted equality.

To establish the equality in the remaining case where $\limsup_{n\to\infty}[\log n/\log\varphi_E(n)] < \infty$, we should show that $E \hookrightarrow \ell_p$ for all $p > \limsup_{n\to\infty}[\log n/\log\varphi_E(n)]$. With this aim, take any q with $p > q > \limsup_{n\to\infty}[\log n/\log\varphi_E(n)]$. There is $N \in \mathbb{N}$ such that $n^{1/q} < \varphi_E(n)$ for all $n \ge N$. Let $M > 0$ be such that

$$n^{1/q} \le M\varphi_E(n) \quad \text{for all} \quad n \in \mathbb{N}.$$

We claim that $E \hookrightarrow \ell_{q,\infty}$. Indeed, for any $\xi = \{\xi_n\} \in E$ and any $m \in \mathbb{N}$, we obtain

$$\|\xi\|_E = \|\xi^*\|_E \ge \|\xi^* e_{\{1,\cdots,m\}}\|_E \ge s_m(\xi)\varphi_E(m) \ge M^{-1}m^{1/q}s_m(\xi).$$

Hence $E \hookrightarrow \ell_{q,\infty}$. Now the result follows by using that $\ell_{q,\infty} \hookrightarrow \ell_p$. □

As an immediate consequences of Theorems 2.2 and 2.3 we obtain.

Corollary 2.4. *Let E be a symmetric quasi-Banach sequence space. Then*

$$\delta_E = \gamma_E \quad \textit{if and only if} \quad \lim_{n\to\infty} \frac{\log n}{\log \varphi_E(n)} \quad \textit{exists.}$$

Corollary 2.5. *Let E be a symmetric quasi-Banach sequence space. Assume that there is $0 < p < \infty$ such that for any $0 < \varepsilon < 1/p$, there are positive constants c_ε, C_ε so that*

$$c_\varepsilon n^{\frac{1}{p}-\varepsilon} \le \varphi_E(n) \le C_\varepsilon n^{\frac{1}{p}+\varepsilon} \quad \textit{for all} \quad n \in \mathbb{N}.$$

Then $\delta_E = \gamma_E = p$.

Next we show that the indices are equal if φ_E has regular variation at ∞.

Corollary 2.6. *Let E be a symmetric quasi-Banach sequence space. If $\lim_{n\to\infty}[\varphi_E(2n)/\varphi_E(n)]$ exists, then $\delta_E = \gamma_E$.*

Proof. Clearly, $\varphi_E(2n) \ge \varphi_E(n)$. So

$$\lim_{n\to\infty} \frac{\varphi_E(2n)}{\varphi_E(n)} = 2^\alpha \quad \text{for some} \quad 0 \le \alpha < \infty.$$

Assume $0 < \alpha < \infty$ and take any $0 < \varepsilon < \alpha$. There is $N \in \mathbb{N}$ such that

$$2^{\alpha-\varepsilon}\varphi_E(n) \le \varphi_E(2n) \le 2^{\alpha+\varepsilon}\varphi_E(n) \quad \text{for all} \quad n \ge N.$$

Let $k \in \mathbb{N}$ and take any $m \in \mathbb{N}$ with $2^k N \le m \le 2^{k+1}N$. We have

$$2^{k(\alpha-\varepsilon)}\varphi_E(N) \le \varphi_E(2^k N) \le \varphi_E(m) \le \varphi_E(2^{k+1}N) \le 2^{(k+1)(\alpha+\varepsilon)}\varphi_E(N).$$

Since $1/2N \le 2^k/m \le 1/N$, it follows that

$$\begin{aligned} \Big[\Big(\frac{1}{2N}\Big)^{\alpha-\varepsilon}\varphi_E(N)\Big]m^{\alpha-\varepsilon} &\le \Big(\frac{2^k}{m}\Big)^{\alpha-\varepsilon}\varphi_E(N)m^{\alpha-\varepsilon} \le \varphi_E(m) \\ &\le 2^{\alpha+\varepsilon}\Big(\frac{2^k}{m}\Big)^{\alpha+\varepsilon}\varphi_E(N)m^{\alpha+\varepsilon} \le \Big[2^{\alpha+\varepsilon}\Big(\frac{1}{N}\Big)^{\alpha+\varepsilon}\varphi_E(N)\Big]m^{\alpha+\varepsilon}. \end{aligned}$$

Put

$$C_1 = \Big(\frac{1}{2N}\Big)^{\alpha-\varepsilon}\varphi_E(N) \quad \text{and} \quad C_2 = 2^{\alpha+\varepsilon}\Big(\frac{1}{N}\Big)^{\alpha+\varepsilon}\varphi_E(N).$$

Then we obtain $C_1 m^{\alpha-\varepsilon} \le \varphi_E(m) \le C_2 m^{\alpha+\varepsilon}$ for all $m \ge 2N$, and so

$$\frac{1}{\alpha+\varepsilon} \le \liminf_{n\to\infty} \frac{\log n}{\log \varphi_E(n)} \le \limsup_{n\to\infty} \frac{\log n}{\log \varphi_E(n)} \le \frac{1}{\alpha-\varepsilon}.$$

Now, using Theorems 2.2 and 2.3, we conclude that $\delta_E = \gamma_E = 1/\alpha$.

The case $\alpha = 0$ can be treated analogously. □

Next we go on to work with spaces which are not symmetric. Then they do not have fundamental function and so indices should be computed in a different way.

Assume that F is a quasi-Banach sequence space such that $\ell_r \hookrightarrow F \hookrightarrow \ell_\infty$ for some $0 < r < \infty$. Then F can be regarded as an intermediate space with respect to the compatible couple (ℓ_r, ℓ_∞) and we can use ideas of interpolation theory to establish analytic formulae for computing the indices.

We recall that Peetre's K–functional and J–functional are defined by

$$K(t,\xi;\ell_r,\ell_\infty)=\inf\{\|\eta\|_{\ell_r}+t\|\mu\|_{\ell_\infty}\ :\xi=\eta+\mu,\ \eta\in\ell_r\,\mu\in\ell_\infty\},t>0,\xi\in\ell_\infty,$$

and

$$J(t,\xi;\ell_r,\ell_\infty)=\max\{\|\xi\|_{\ell_r},t\|\xi\|_{\ell_\infty}\},\quad t>0,\quad \xi\in\ell_r,$$

(see [2], [3] or [16]). Following [4], we put

$$\psi_F(t)=\sup\{K(t,\xi;\ell_r,\ell_\infty):\xi\in F,\|\xi\|_F=1\},$$

$$\rho_F(t)=\inf\{J(t,\xi;\ell_r,\ell_\infty):\xi\in\ell_r,\|\xi\|_F=1\}.$$

We refer, for instance, to [15], [13], [6], [7] and [8] for properties of these functions.

Indices of F are related to the functions ψ_F and ρ_F by means of the following analytic formulae:

Theorem 2.7. *Let $0<r<\infty$ and let F be a quasi-Banach sequence space with $\ell_r\hookrightarrow F\hookrightarrow\ell_\infty$. Then*

$$\delta_F=r\Big(1-\liminf_{t\to\infty}\frac{\log\rho_F(t)}{\log t}\Big)^{-1},$$

$$\gamma_F=r\Big(1-\limsup_{t\to\infty}\frac{\log\psi_F(t)}{\log t}\Big)^{-1}.$$

See [5] for the proof.

The next result shows a necessary and sufficient condition for equality of indices in terms of the functions ψ_F and ρ_F.

Theorem 2.8. *Let $0<r<\infty$ and F be a quasi-Banach sequence space with $\ell_r\hookrightarrow F\hookrightarrow\ell_\infty$. Then a necessary and sufficient condition for $\delta_F=\gamma_F$ is that the limits*

$$\lim_{t\to\infty}\frac{\log\rho_F(t)}{\log t}\quad\textit{and}\quad\lim_{t\to\infty}\frac{\log\psi_F(t)}{\log t}$$

exist and coincide.

Proof. If the limits exist and are equal, then Theorem 2.7 yields that the indices of F are equal.

Conversely, suppose that $\delta_F=\gamma_F$. Using again Theorem 2.7, we have

$$\liminf_{t\to\infty}\frac{\log\rho_F(t)}{\log t}=\limsup_{t\to\infty}\frac{\log\psi_F(t)}{\log t}.\tag{2.3}$$

On the other hand, we know from [3], Thm. 5.2.1 that

$$K(t,\xi;\ell_r,\ell_\infty)\sim\Big(\sum_{n=1}^{[t^r]}s_n^r(\xi)\Big)^{1/r}.$$

Here $[\cdot]$ is the greatest integer function. For $t\geq 1$, put $\xi_t=e_{\{1,\cdots,[t^r]\}}$. Let $C>0$ such that for any $t\geq 1$

$$K(t,\xi_t;\ell_r,\ell_\infty)\geq Ct.$$

We have

$$\rho_F(t) \le \frac{J(t,\xi_t;\ell_r,\ell_\infty)}{\|\xi_t\|_F} \le \frac{t}{\|\xi_t\|_F} \le \frac{K(t,\xi_t;\ell_r,\ell_\infty)}{C\|\xi_t\|_F} \le \frac{1}{C}\psi_F(t).$$

Therefore, using (2.3) we obtain

$$\limsup_{t\to\infty} \frac{\log \rho_F(t)}{\log t} \le \limsup_{t\to\infty} \frac{\log \psi_F(t)}{\log t} = \liminf_{t\to\infty} \frac{\log \rho_F(t)}{\log t}$$

and

$$\limsup_{t\to\infty} \frac{\log \psi_F(t)}{\log t} = \liminf_{t\to\infty} \frac{\log \rho_F(t)}{\log t} \le \liminf_{t\to\infty} \frac{\log \psi_F(t)}{\log t}.$$

Consequently, $\lim_{t\to\infty} [\log \psi_F(t)/\log t]$ and $\lim_{t\to\infty} [\log \rho_F(t)/\log t]$ exist and coincide. □

We end the paper with a result on the grade of proximity between sequence spaces. Recall that a bounded linear operator $T \in \mathcal{L}(X,Y)$ between two quasi-Banach spaces X and Y is called *strictly singular* if it fails to be an isomorphism on any infinite dimensional subspace (see [10] and [14]).

Theorem 2.9. *Let E and F be quasi-Banach sequence spaces with $\mathfrak{f} \subsetneq E \hookrightarrow F \hookrightarrow \ell_\infty$. Assume that $\delta_E = \gamma_E$ and $\delta_F = \gamma_F$. If the inclusion operator $E \hookrightarrow F$ is not strictly singular, then either:*

(i) $\mathfrak{f} \subsetneq E \subseteq F \subseteq \bigcap_{q>0} \ell_q$ *or*

(ii) $\bigcup_{q<\infty} \ell_q \subsetneq E \subseteq F \subsetneq \ell_\infty$ *or*

(iii) $\bigcup_{q<p} \ell_q \subsetneq E \subseteq F \subsetneq \bigcap_{q>p} \ell_q$ *for some* $1 < p < \infty$.

The proof can be found in [5].

Acknowledgements. The author would like to thank the organizer of the Symposium, Professor Kichi-Suke Saito, for his invitation, and to Professor Mikio Kato for his help during the stay in Japan.

References

[1] J. Bastero, H. Hudzik and A.M. Steinberg. On smallest and largest spaces among rearrangement-invariant p-Banach function spaces ($0 < p < 1$). *Indag. Mathem., N.S.* **2** (1991), 283-288.

[2] C. Bennett and R. Sharpley. *Interpolation of Operators* (Academic Press, 1988).

[3] J. Bergh and J. Löfström. *Interpolation Spaces. An introduction* (Springer, 1976).

[4] F. Cobos, M. Cwikel and P. Matos. Best possible compactness results of Lions-Peetre type. *Proc. Edinburgh Math. Soc.* **44** (2001), 153-172.

[5] F. Cobos, L.M. Fernández-Cabrera, A. Manzano and A. Martínez. Inclusion indices of quasi-Banach spaces. *Bolletino U.M.I.* (8) **10-B** (2007), 99-117.

[6] F. Cobos and E. Pustylnik. On strictly singular and strictly cosingular embeddings between Banach lattices of functions. *Math. Proc. Camb. Phil. Soc.* **133** (2002), 183-190.

[7] L.M. Fernández-Cabrera. Inclusion indices of function spaces and applications. *Math. Proc. Camb. Phil. Soc.* **137** (2004), 665-674.

[8] L.M. Fernández-Cabrera. On inclusion indices of function spaces. In: *Proc. Inter. Symposium on Banach and Function Spaces*, p. 185-203, (Yokohama Publishers, 2004).

[9] L.M. Fernández-Cabrera, F. Cobos, F.L. Hernández and V.M. Sánchez. Indices defined by interpolation scales and applications. *Proc. Royal Soc. Edinburgh* **134A** (2004), 695-717.

[10] S. Goldberg. *Unbounded Linear Operators* (McGraw-Hill, 1966).

[11] F.L. HERNÁNDEZ, V.M. SÁNCHEZ AND E. SEMENOV. Strictly singular and strictly co-singular inclusions between symmetric sequence spaces. *J. Math. Anal. Appl.* **291** (2004), 459-476.

[12] S.G. KREĬN, JU.I. PETUNIN AND E.M. SEMENOV *Interpolation of Linear Operators* (Amer. Math. Soc., Providence, R.I., 1982).

[13] L. MALIGRANDA AND M. MASTYLO. Notes on non-interpolation spaces. *J. Approx. Theory* **56** (1989), 333-347.

[14] A. PIETSCH. *Operator Ideals* (North-Holland, 1980).

[15] E. PUSTYLNIK. Embedding functions and their role in interpolation theory. *Abstract Appl. Analysis* **1** (1996), 305-325.

[16] H. TRIEBEL. *Interpolation Theory, Function Spaces, Differential Operators* (North-Holland, 1978).

Luz M. Fernández-Cabrera, DEPARTAMENTO DE MATEMÁTICA APLICADA, ESCUELA DE ESTADÍSTICA, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID. SPAIN

E-mail address: `luz_fernandez-c@mat.ucm.es`