数理解析研究所講究録
第 1570 巻 2007 年 83-94

COMMON FIXED POINT THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we assume that set of common fixed points of two asymptotically nonexpansive mappings is nonempty and one of these mappings is completely continuous. Then an iterative sequence $\{x_n\}$ converges strongly to some common fixed point of these mappings. If the mappings are not completely continuous but either the norm of the space is Fréchet differentiable or the dual of the space has Kadec-Klee property, then the iterative sequence $\{x_n\}$ converges weakly to some common fixed point of these mappings.

1. Introduction

Let C be a nonempty subset of a real Banach space E. A mapping $T : C \to C$ is : (i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;(ii) asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, we have $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for all $n \geq 1$;(iii) uniformly $L-$ Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in C$ and for all $n \geq 1$; (iv) completely continuous if $\{Tx_n\}$ has a convergent subsequence in C whenever $\{x_n\}$ is bounded in C.

It is obvious that nonexpansive mapping is asymptotically nonexpansive and asymptotically nonexpansive is uniformly $L-$ Lipschitzian but converses of these statements are not true, in general. Asymptotically nonexpansive mappings, since their introduction in 1972 by Goebel and Kirk [4] have remained under study by various authors. Goebel and Kirk [4] also proved: If C is a nonempty bounded closed

Key words and phrases. Noor Iterations, Asymptotically quasi-nonexpansive mapping, Common fixed point, Weak and Strong convergence.
2000 Mathematics Subject Classification: 47H09, 47H10.

convex subset of a uniformly convex Banach space E and $T : C \to C$ is an asymptotically nonexpansive mapping, then T has a fixed point. In recent years, Mann and Ishikawa iterative sequences have been studied extensively by many authors to solve one-parameter nonlinear operator equations as well as variational inequalities on a convex set C in Hilbert and Banach spaces (see, for example [8-11], [13],[14] and the references therein).

Finding common fixed points of a finite family $\{T_j : j = 1, 2, 3, ..., n\}$ of mappings acting on a Hilbert space is a problem that often arises in applied mathematics. Probably the most important case is the one where each mapping T_j is the metric projection onto some closed convex set C_j, under the assumption that intersection of all involved sets C_j is nonempty. In fact, many algorithms for solving "convex feasibility problem" connected to metric projections may be generalized to different classes of more general mappings having a nonempty set of common fixed points; for more details, see [12]. In 2001, Khan and Takahashi [6] introduced the following modified Ishikawa iterative scheme of two self mappings S, T on a convex set C :

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n T^n x_n + (1 - \beta_n) x_n, \\ x_{n+1} = \alpha_n S^n y_n + (1 - \alpha_n) x_n, \qquad n \geq 1, \end{cases} \tag{1.1}$$

where $0 < \delta \leq \alpha_n, \beta_n \leq 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$ and they approximated common fixed points of two asymptotically nonexpansive mappings through weak and strong convergence of the scheme. Their weak convergence result does not apply to L^p spaces with $p \neq 2$ because none of these spaces satisfy the Opial property while the strong convergence of the sequence has been proved under the assumption that domain of the mappings is compact. Moreover, the conditions on the iteration parameters α_n, β_n are also strong.

In this paper, by weaknening the conditions on the iteration parameters α_n, β_n, we, first, approximate common fixed points of two asymptotically nonexpansive mappings through weak convergence of the sequence (1.1) in the uniformly convex Banach space satisfying one of the conditions: (i) The space satisfy the Opial property; (ii) The norm of the space is Fréchet differentiable; (iii) The dual of the space has Kadec-Klee property. We also establish the strong convergence of the sequence (1.1) .

2. Prliminaries and Notations

A Banach space E is uniformly convex if for each $r \in (0,2]$, the modulus of convexity of E, given by

$$\delta(r) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge r\right\},$$

satisfies the inequality $\delta(r) > 0$. For a sequence, the symbol $\to$ (resp.$\rightharpoonup$) denotes norm (resp. weak) convergence. The space E is said to satisfy the *Opial condition* [7] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \ne x$. It satisfies the *Kadec-Klee property* if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $x_n \to x$ as $n \to \infty$.

Let $S = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. Then the norm of E is *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in S. Moreover, this norm is *Fréchet differentiable* if for each x in S, this limit is attained uniformly for $y \in S$. In the case of *Fréchet differentiable norm*, it has been obtained in [13] that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x+h\|^2 \le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|) (*)$$

for all x, h in E, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in X, \langle .,. \rangle$ is the pairing between E and E^* and b is a function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0} \frac{b(t)}{t} = 0$.

A mapping $T : C \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$. Throughout the paper, $F(T)$ denotes the set of fixed points of T.

We need the following useful lemmas for development of our convergence results.

Lemma 2.1[3]. *Let $\{r_n\}$ and $\{s_n\}$ be two nonnegative real sequences such that*

$$r_{n+1} \le (1+s_n)r_n \qquad \text{for all } n \ge 1.$$

If $\sum_{n=1}^{\infty} s_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 2.2[6]. Let E be a normed space and C be a nonempty closed convex subset of E. Let, for $L > 0, S$ and T be uniformly

$L-$Lipschitzian mappings of C into itself. Define a sequence $\{x_n\}$ as in (1.1). If

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0 = \lim_{n\to\infty} \|x_n - T^n x_n\|,$$

then

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|.$$

Lemma 2.3 [2]. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0.*

Lemma 2.4 [5]. *Let E be a reflexive Banach space such that E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in \omega_w(x_n)$(weak w-limit set of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.*

Lemma 2.5 [14]. *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda) \|y\|^p - \pi_p(\lambda) g(\|x - y\|)$$

for all $x, y \in B_r[0] = \{x \in E : \|x\| \leq r\}$, where $\pi_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$ for all $\lambda \in [0,1]$.

Lemma 2.6[1]. Let E be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of E. *Then there is a strictly increasing and continuous convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that, for every Lipschitzian continuous mapping* $T : C \to E$ and *for all $x, y \in C$ and $t \in [0,1]$*,the following inequality holds:

$$\|T(tx + (1-t)y) - (tTx + (1-t)Ty\| \leq Lg^{-1}\left(\|x - y\| - L^{-1}\|Tx - Ty\|\right),$$

where $L \geq 1$ is the Lipschitz constant of T.

3. Weak and Strong Convergence Results

We first prove the following helpful lemmas.

Lemma 3.*1*. *Let C be a nonempty closed convex subset of a normed space E and let $S, T : C \to C$ be asymptotically nonexpansive mappings both with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. If $F(S) \cap F(T) \neq \phi$ and the sequence$\{x_n\}$ is defined by (1.1), then*

$\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F(S) \cap F(T)$.

Proof. For any $p \in F(S) \cap F(T)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(S^n y_n - p) + (1-\alpha_n)(x_n - p)\| \\
&\leq \alpha_n k_n \|y_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \alpha_n k_n \|\beta_n(T^n x_n - p) + (1-\beta_n)(x_n - p)\| \\
&\quad +(1-\alpha_n)\|x_n - p\| \\
&\leq \alpha_n \beta_n k_n^2 \|x_n - p\| + \alpha_n(1-\beta_n)k_n \|x_n - p\| \\
&\quad +(1-\alpha_n)\|x_n - p\| \\
&\leq k_n^2 \|x_n - p\|.
\end{aligned}$$

By Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S) \cap F(T)$ as desired.

Lemma 3.2. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings both with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define sequences $\{x_n\}$* and $\{y_n\}$ by (1.1), *where* $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty, \liminf_{n\to\infty} \alpha_n > 0$ and $\beta_n \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. *If* $F(S) \cap F(T) \neq \phi$,then there exists a subsequence $\{x_i\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \|x_i - Sx_i\| = 0 = \lim_{i\to\infty} \|x_i - Tx_i\|.$$

Proof. For all $p \in F(S) \cap F(T)$, $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 3.1 and therefore $\{x_n - p\}$ is bounded. Consequently, $\{y_n - p\}, \{T^n x_n - p\}, \{T^n y_n - p\}$ are bounded. Therefore, we can obtain a closed ball $B_r[0]$ such that $\{x_n - p, y_n - p, T^n x_n - p, T^n y_n - p\} \subset B_r[0] \cap C$.

With the help of Lemma 2.5 and the scheme (1.1), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n(T^n x_n - p) + (1-\beta_n)(x_n - p)\|^2 \\
&\leq \beta_n \|T^n x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\
&\quad -\pi_2(\beta_n) g(\|x_n - T^n x_n\|) \\
&\leq k_n^2 \|x_n - p\|^2 - \pi_2(\beta_n) g(\|x_n - T^n x_n\|) \qquad (3.1)
\end{aligned}$$

Again by Lemma 2.5, the scheme (1.1) and the inequality (3.1), we infer that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|\alpha_n(S^n y_n - p) + (1-\alpha_n)(x_n - p)\|^2 \\
&\leq \alpha_n \|S^n y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad -\pi_2(\alpha_n) g(\|S^n y_n - x_n\|) \\
&\leq \alpha_n k_n^2 \|y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad -\pi_2(\alpha_n) g(\|S^n y_n - x_n\|) \\
&\leq \alpha_n k_n^4 \|x_n - p\|^2 - \alpha_n k_n^2 \pi_2(\beta_n) g(\|x_n - T^n x_n\|) \\
&\quad +(1-\alpha_n)\|x_n - p\|^2 - \pi_2(\alpha_n) g(\|S^n y_n - x_n\|) \\
&\leq k_n^4 \|x_n - p\|^2 - \alpha_n \pi_2(\beta_n) g(\|x_n - T^n x_n\|) \\
&\quad -\pi_2(\alpha_n) g(\|S^n y_n - x_n\|) \\
&\leq \|x_n - p\|^2 - \alpha_n \pi_2(\beta_n) g(\|x_n - T^n x_n\|) \\
&\quad -\pi_2(\alpha_n) g(\|S^n y_n - x_n\|) + (k_n^4 - 1)Q
\end{aligned}$$

where Q is a real number such that $\|x_n - p\|^2 \leq Q$.

From the above estimate, we obtain the following two important inequalities:

$$\begin{aligned}
\pi_2(\alpha_n) g(\|S^n y_n - x_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +(k_n^4 - 1)Q; \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
\alpha_n k_n^2 \pi_2(\beta_n) g(\|T^n x_n - x_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +(k_n^4 - 1)Q. \qquad (3.3)
\end{aligned}$$

Let m be any positive integer. Summing up the terms from 1 to m on both sides in the inequality (3.2), we have

$$\begin{aligned}
\sum_{n=1}^{m} \pi_2(\alpha_n) g(\|S^n y_n - x_n\|) &\leq \|x_1 - p\|^2 - \|x_{m+1} - p\|^2 + Q\sum_{n=1}^{m}(k_n^4 - 1) \\
&\leq \|x_1 - p\|^2 + Q\sum_{n=1}^{m}(k_n^4 - 1).
\end{aligned}$$

When $m \to \infty$ in the above inequality, we get

$$\sum_{n=1}^{\infty} \pi_2(\alpha_n) g(\|S^n y_n - x_n\|) < \infty$$

and hence

$$\liminf_{n\to\infty} g(\|S^n y_n - x_n\|) = 0.$$

By the properties of g, we have

$$\liminf_{n\to\infty} \|S^n y_n - x_n\| = 0.$$

Since $\liminf_{n\to\infty} \alpha_n > 0$, we have $\alpha_n > \alpha$ for all $n \geq n_0$. Also $\beta_n \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$.

Then the inequality (3.3) reduces to

$$\begin{aligned} \alpha\delta^2 \sum_{n=n_0}^{\infty} g\left(\|T^n x_n - x_n\|\right) &\leq \|x_{n_0} - p\|^2 + Q\sum_{n=n_0}^{\infty}(k_n^4 - 1) \\ &< \infty, \end{aligned}$$

which further, implies that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$$

Observe that

$$\begin{aligned} \|x_n - S^n x_n\| &\leq \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ & k_n \|x_n - y_n\| + \|S^n y_n - x_n\| \\ &\leq k_n(1-\delta)\|T^n x_n - x_n\| + \|S^n y_n - x_n\|. \end{aligned}$$

By lim inf on both sides in the above inequality, we get

$$\liminf_{n\to\infty} \|x_n - S^n x_n\| = 0.$$

Hence there exists a subsequence $\{x_i\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \left\|x_i - S^i x_i\right\| = 0 = \lim_{i\to\infty} \left\|x_i - T^i x_i\right\|.$$

Finally by Lemma 2.2, we get that

$$\lim_{i\to\infty} \|x_i - Sx_i\| = 0 = \lim_{i\to\infty} \|x_i - Tx_i\|.$$

Lemma 3.3. Let E be a uniformly convex Banach space and let C, S, T and $\{x_n\}$ be taken as in Lemma 3.1. If $F(S) \cap F(T) \neq \phi$,then for all $p_1, p_2 \in F(S) \cap F(T)$, $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$.

Proof. The sequence $\{x_n\}$ is bounded, since $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence we may assume C to be bounded. Let $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$. Then $a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$ exists as proved in Lemma 3.1. Define $W_n : C \to C$ by:

$$W_n x = \alpha_n S^n[\beta_n T^n x + (1-\beta_n)x] + (1-\alpha_n)x \quad \text{for all } x \in C.$$

Obviously $F(S) \cap F(T) \subseteq F(W_n)$. Also we can verify that

$$\|W_n x - W_n y\| \leq k_n^2 \|x - y\|.$$

Set

$$R_{n,m} = W_{n+m-1}W_{n+m-2}...W_{n,}\ m \geq 1 \text{ and}$$

$$b_{n,m} = \|R_{n,m}(tx_n + (1-t)p_1) - (tR_{n,m}x_n + (1-t)p_1)\|.$$

Then

$$\|R_{n,m}x - R_{n,m}y\| \leq \left(\prod_{j=n}^{n+m-1} k_j^2\right)\|x-y\|.$$

Since $R_{n,m}x_n = x_{n+m}$, we have

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\
&\leq b_{n,m} + \|R_{n,m}(tx_n + (1-t)p_1) - p_2\| \\
&\leq b_{n,m} + \left(\prod_{j=n}^{n+m-1} k_j^2\right) a_n(t) \\
&\leq b_{n,m} + H_n a_n(t), \text{ where } H_n = \prod_{j=n}^{\infty} k_j^2. \qquad (3.4)
\end{aligned}$$

By Lemma 2.6, there exists a strictly increasing continuous function $g : [0, \infty] \to [0, \infty]$ with $g(0) = 0$ such that

$$\begin{aligned}
b_{n,m} &\leq H_n g^{-1}\left(\|x_n - p_1\| - H_n^{-1}\|R_{n,m}x_n - p_1\|\right) \\
&= H_n g^{-1}\left(\|x_n - p_1\| - H_n^{-1}\|x_{n+m} - p_1\|\right) \qquad (3.5)
\end{aligned}$$

Combining (3.4) and (3.5), we get

$$a_{n+m}(t) \leq H_n g^{-1}\left(\|x_n - p_1\| - H_n^{-1}\|x_{n+m} - p_1\|\right) + H_n a_n(t)$$

Now fixing n and letting $m \to \infty$ in the above inequality, we have

$$\limsup_{m\to\infty} a_m(t) \leq \limsup_{m\to\infty} H_n g^{-1}\left(\|x_n - p_1\| - H_n^{-1}\lim_{m\to\infty}\|x_m - p_1\|\right) + H_n a_n(t)$$

and again letting $n \to \infty$, we get

$$\limsup_{m\to\infty} a_m(t) \leq g^{-1}(0) + \liminf_{n\to\infty} a_n(t) = \liminf_{n\to\infty} a_n(t).$$

This completes the proof.

Lemma 3.4. Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C, S, T and $\{x_n\}$ be as taken in Lemma 3.1. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty}\langle x_n, J(p_1 - p_2)\rangle$ exists for every $p_1, p_2 \in F(S) \cap F(T)$. Moreover $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the weak $\omega-$limit set of $\{x_n\}$.

Proof. Take $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality $(*)$, we have

$$\begin{aligned} & t\langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 \\ \leq\ & \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \\ \leq\ & t\langle x_n - p_1, J(p_1 - p_2)\rangle \\ & + \frac{1}{2}\|p_1 - p_2\| + b(t\|x_n - p_1\|). \end{aligned}$$

As $\sup_{n\geq 1}\|x_n - p_1\| \leq M$ for some $M > 0$,it follows from above the above inequality that

$$\begin{aligned} & t\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{1}{2}\|p_1 - p_2\|^2 \\ \leq\ & \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2 \\ \leq\ & t\liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle \\ & + \frac{1}{2}\|p_1 - p_2\|^2 + b(tM) \end{aligned}$$

That is,

$$\begin{aligned} \limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle \ \leq\ & \liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle \\ & + \frac{b(tM)}{tM}M. \end{aligned}$$

If $t \to 0$, then we see that $\lim_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F(S)\cap F(T)$. In particular, we have $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the weak ω−limit set of $\{x_n\}$.

Now, we are in a position to prove our convergence theorems.

Theorem 3.1. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings both with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define sequences $\{x_n\}$* and $\{y_n\}$ by (1.1), *where $\{\alpha_n\}, \{\beta_n\}$* are real sequences in $[0, 1]$ satisfying $\liminf_{n\to\infty}\alpha_n >$ 0, $\sum_{n=1}^{\infty}\alpha_n(1 - \alpha_n) = \infty$, and $\beta_n \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. *If $F(S)\cap F(T) \neq \phi$, then there exists a subsequence $\{x_i\}$* of $\{x_n\}$ *which converges weakly to a common fixed point of S and T provided that one of the following conditions holds:*

(i) E satisfies the Opial property;

(ii) E has a Frechet differentiable norm;

(iii) E^* has the Kadec-Klee property.

Proof. Let $p \in F(S) \cap F(T)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 3.1. Let $\{x_i\}$ be the subsequence as introduced in Lemma 3.2. Since E is reflexive, there exists a subsequence $\{x_j\}$ of $\{x_i\}$ converging weakly to some $z_1 \in C$. By Lemma 6, $\lim_{i\to\infty} \|x_i - Sx_i\| = 0 = \lim_{i\to\infty} \|x_i - Tx_i\|$ and $I - S, I - T$ are demiclosed at 0 by Lemma 2.3, therefore we obtain $Sz_1 = z_1$ and $Tz_1 = z_1$. That is, $z_1 \in F(S) \cap F(T)$. In order to show that $\{x_i\}$ converges weakly to z_1, take another subsequence $\{x_k\}$ of $\{x_i\}$ converging weakly to some $z_2 \in C$. Again in the same way, we can prove that $z_2 \in F(S) \cap F(T)$. Next, we prove that $z_1 = z_2$. Assume that (i) is given and suppose that $z_1 \neq z_2$, then by the Opial property

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{j\to\infty} \|x_j - z_1\| \\
&< \lim_{j\to\infty} \|x_j - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{k\to\infty} \|x_k - z_2\| \\
&< \lim_{k\to\infty} \|x_k - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This contradiction proves that $\{x_i\}$ converges weakly to a point in $F(S) \cap F(T)$.

Next suppose that (ii) is satisfied. From Lemma 3.4, we have that $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_i)$, where $\omega_w(x_i)$ denotes the weak ω−limit set of $\{x_i\}$. Now $\|z_1 - z_2\|^2 = \langle z_1 - z_2, J(z_1 - z_2)\rangle = 0$ gives that $z_1 = z_2$.Finally, let (iii) be given. As $\lim_{n\to\infty} \|tx_n + (1-t)z_1 - z_2\|$ exists, therefore by Lemma 2.4, we obtain $z_1 = z_2$.

If we replace the parametric conditions " $\liminf_{n\to\infty} \alpha_n > 0, \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ " by "$0 < \delta \leq \alpha_n \leq 1 - \delta < 1$ for some $\delta \in (0, \frac{1}{2})$" in Lemma 3.2, it becomes Lemma 3 of Khan and Takahashi[6]. Then the above theorem reduces to:

Theorem 3.2. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S, T : C \to C$ be asymptotically nonexpansive mappings both with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define sequences $\{x_n\}$ and $\{y_n\}$ by (1.1), where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$ such that $\delta \leq \alpha_n, \beta_n \leq 1 - \delta$ for some $\delta \in (0, \frac{1}{2})$. If $F(S) \cap F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T if one of the following*

conditions holds:
(i) E satisfies the Opial property;
(ii) E has a Frechet differentiable norm;
(iii) E^* has the Kadec-Klee property.

Next, we prove our strong convergence theorem.

Theorem 3.3. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S,T : C \to C$ be asymptotically nonexpansive mappings both with sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define sequences $\{x_n\}$ and $\{y_n\}$ by (1.1), where $\{\alpha_n\}$ and$\{\beta_n\}$ are real sequences in $[0,1]$ satisfying $\liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, and $\delta \le \beta_n \le 1-\delta$ for some $\delta \in (0, \frac{1}{2})$. If $F(S) \cap F(T) \neq \phi$ and either S or T is completely continuous, then* $\{x_n\}$ and $\{y_n\}$ *converge strongly to the same common fixed point of S and T.*

Proof. As proved in Lemma 3.2, there exists a subsequence $\{x_i\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \|x_i - Sx_i\| \;=\; 0 = \lim_{i\to\infty} \|x_i - Tx_i\|. \tag{3.6}$$

Since $\{x_i\}$ is bounded and S is completely continuous, so $\{Sx_i\}$ has a convergent subsequence $\{Sx_j\}$. Suppose $Sx_j \to z \in C$.

Then

$$\|x_j - z\| \le \|x_j - Sx_j\| + \|Sx_j - z\| \to 0.$$

Hence $x_j \to z$. Then (3.6) assures that z is a common fixed point of S and T. As $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S) \cap F(T)$, so $x_n \to z$. This completes the proof.

Remark 3.1. Our weak converence Theorems apply not only in Hilbert and L^p spaces $(1 < p < \infty)$ but also to the spaces whose dual has the Kadec-Klee property. Our strong convergence result improves Theorem 2 of Khan and Takahashi[6] due to the following reasons:

(i) Compactness of the domain is replaced by the complete continuity;

(ii) Conditions on iteration parameters are weaker than those used in [6].

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