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Some results on von Neumann-Jordan type constants of Banach Spaces

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In a recent paper [3] J. Gao introduced the parameter $E(X)$ for a Banach space X by

$$E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2 : \|x\| = \|y\| = 1\}$$

and investigated some sufficient conditions for X to have normal structure in terms of $E(X)$. In this short note we shall present several recent results of the authors on the parameter $E(X)$, especially in connection with the von Neumann-Jordan type constant $C_t(X)$ and the James type constant $J_{X,t}(\tau)$. Some sufficient conditions so that a Banach space X has normal structure will be given. We shall also consider relation between $E(X)$ and $E(X^*)$, where X^* is the dual space of X.

Let X be a Banach space with $\dim X \geq 2$ and let $-\infty \leq t < \infty$.

(i) The *James type constant* $J_{X,t}(\tau)$, $\tau \geq 0$, is defined by

$$J_{X,t}(\tau) = \begin{cases} \sup\left\{\left(\dfrac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2}\right)^{1/t} : \|x\| = \|y\| = 1\right\} & \text{if} -\infty < t < \infty, \\ \sup\left\{\min(\|x+\tau y\|, \|x-\tau y\|) : \|x\| = \|y\| = 1\right\} & \text{if } t = -\infty. \end{cases}$$

(ii) The *von Neumann-Jordan type constant* $C_t(X)$ is defined by

$$C_t(X) = \sup\left\{J_{X,t}(\tau)^2/(1+\tau^2) : 0 \leq \tau \leq 1\right\}.$$

The following well-known constants are expressed by these constants.

James contant: $J(X) = J_{X,-\infty}(1)$

von Neumann-Jordan constant: $C_{NJ}(X) = C_2(X)$

modulus of smoothness: $\rho_X(\tau) = J_{X,1}(\tau) - 1$

Gao's parameter : $E(X) = 2J_{X,2}(1)^2$

Zbăganu constant: $C_Z(X) = C_0(X)$.

If $-\infty \leq t \leq s < \infty$, then $J_{X,t}(\tau) \leq J_{X,s}(\tau)$ for all $\tau \geq 0$ and $C_t(X) \leq C_s(X)$. Recall that X is said to be *uniformly non-square* if $J(X) < 2$. It is well-known that X is uniformly non-square if and only if X^* is uniformly non-square.

Theorem 1. *Let $-\infty \leq t < \infty$. Then the following are equivalent.*
(i) X *is uniformly non-square.*
(ii) $J_{X,t}(1) < 2$.
(iii) $J_{X,t}(\tau) < 1 + \tau$ *for some* $0 < \tau < 1$.
(iv) $C_t(X) < 2$.
(v) $E(X) < 8$.
(vi) $\rho'_X(0) = \lim_{\tau \to +0} \rho_X(\tau)/\tau < 1$.

It is easy to see that for any Banach space X

$$2J(X)^2 \leq 2J_{X,t}(1)^2 \leq E(X) \leq 4C_{NJ}(X) \quad \text{if } -\infty \leq t \leq 2,$$

where we have equality in all the inequalities if X is an L_p-space, $1 \leq p \leq \infty$.

Theorem 2. *For any Banach space X*

$$\frac{(1+\rho_X(1))^2}{2} \leq \frac{E(X)}{4} \leq 1 + \rho_X(1)^2. \tag{1}$$

Remark 1. In the first and second inequalities in (1) equality attains with an ℓ_2-ℓ_∞ space and an ℓ_2-ℓ_1 space, respectively. Equality attains in the both inequalites in (1) if and only if X is not uniformly non-square.

Theorem 3. *For any Banach space X*

$$1 + \rho'_X(0)^2 \leq \frac{E(X^*)}{4} \leq 1 + \rho_X(1)^2. \tag{2}$$

Remark 2. If X is uniformly smooth (i.e., $\rho'_X(0) = 0$), then X is a Hilbert space if and only if equality holds in the first inequality in (2). In this case the second inequality is strict. Note that equality holds in the both inequalities of (2) if and only if $\rho_X(\tau) = \rho_X(1)\tau$ for all $0 \leq \tau \leq 1$: ℓ_2-ℓ_∞ and ℓ_∞-ℓ_1 spaces are such examples with this condition.

Theorem 4. *For any Banach space X*

$$\frac{C_1(X)}{2} + \sqrt{C_1(X) - 1} \leq \frac{E(X)}{4} \leq 1 + (\sqrt{2C_1(X)} - 1)^2. \tag{3}$$

Remark 3. In the first and second inequalities of (3) we have equality with an ℓ_2-ℓ_∞ space and an ℓ_2-ℓ_1 space, respectively. We have equality in the both inequalities of (3) if and only if X is not uniformly non-square.

Theorem 5. *For any Banach space* X

$$\frac{C_1(X)}{2} + \sqrt{C_1(X) - 1} \leq C_1(X) \leq \frac{E(X^*)}{4} \leq 1 + (\sqrt{2C_1(X)} - 1)^2. \tag{4}$$

Remark 4. If X is an ℓ_2-ℓ_1 space, we have equality in the second inequality of (4). It is easy to see that if equality holds in the first inequality of (4), X is not uniformly non-square and hence we have equality in the other inequalities.

Corollary 1. *The following are equivalent.*
(i) X *is a Hilbert space.*
(ii) $E(X) = 4$.
(iii) $C_1(X) = 1$.

Theorem 6. *For any Banach space* X

$$E(X^*) \leq 4 + (\sqrt{2E(X)} - 2)^2. \tag{5}$$

Remark 5. If X is an ℓ_2-ℓ_∞ space, then $E(X) = 3 + 2\sqrt{2}$ and $E(X^*) = 6$, whence we have equality in (5). Note that since $E(X^{**}) = E(X)$, we also have the estimate of $E(X^*)$ from below by $E(X)$:

$$E(X^*) \geq (2 + \sqrt{E(X) - 4})^2/2. \tag{6}$$

Of course we have equality if X is an ℓ_2-ℓ_1 space.

A Banach space X is said to have *normal structure* (resp. *weak normal structure*) if $r(K) < \mathrm{diam}(K)$ for every non-singleton closed bounded convex subset (resp. weakly compact convex subset) K of X, where $\mathrm{diam}(K) := \sup\{\|x - y\| : x, y \in K\}$ and $r(K) := \inf\{\sup\{\|x - y\| : y \in K\} : x \in K\}$. It is clear that if X is reflexive and has weak normal structure, then X has normal structure. The *normal structure coefficient* of X is the number:

$$N(X) = \inf\{\mathrm{diam}(K)/r(K) : K \subset X \text{ bounded and convex, } \mathrm{diam}(K) > 0\}.$$

Obviously $1 \le N(X) \le 2$. X is said to have *uniform normal structure* if $N(X) > 1$. As is well-known, if $\rho'_X(0) = \lim_{\tau \to +0} \rho_X(\tau)/\tau < 1/2$, or, if $\delta_X(1) > 0$, then X has uniform normal structure (cf. [5]). Since $\rho'_X(0) < 1/2$ if and only if $\delta_{X^*}(1) > 0$, we have

Theorem 7. *Let $1 \le t \le 2$. Then, $J'_{X,t}(0) = \rho'_X(0)$. Hence, if $J'_{X,t}(0) < 1/2$, both X and X^* have uniform normal structure.*

Gao [3] proved that if $E(X) < 5$, X has uniform normal structure. We note that if $E(X) < 5$, then by Theorem 3 $\rho'_{X^*}(0) < 1/2$ and both X and X^* have uniform normal structure. We shall give an improvement of this result.

Theorem 8. *Let $C_1(X) < (3+\sqrt{5})/4$. Then both X and X^* have uniform normal structure. In particular if $E(X) < 3+\sqrt{5}$, both X and X^* have uniform normal structure.*

Remark 6. For any Banach space X, $C_1(X) \le C_{NJ}(X)$; these two constants are different in general. For example, if X is an ℓ_∞-ℓ_1 space, then $C_1(X) = 5/4$ and $C_{NJ}(X) = (3+\sqrt{5})/4$. Therefore Theorem 8 may be considered as an improvement of a result in [2] which assert that if $C_{NJ}(X) < (3+\sqrt{5})/4$, then both X and X^* have uniform normal structure. On the other hand Theorem 8 can be proved by using a result in [1] which assert that if $J(X) < (1+\sqrt{5})/2$, X has uniform normal structure. Let us mention that if $E(X) < 3+\sqrt{5}$, then $C_1(X^*) < (3+\sqrt{5})/4$ by Theorem 5; the converse is not true in general.

In [8] B. Sims gave a sufficient condition for the normal structure of a Banach space X by means of the modulus of convexity $\delta_X(\epsilon)$ and the *coefficient of weak orthogonality* $w(X)$, which is defined to be the supremum of the set of all real numbers $\lambda > 0$ such that

$$\lambda \liminf_{n\to\infty} \|x + x_n\| \le \liminf_{n\to\infty} \|x - x_n\|$$

for all $x \in X$ and for all weakly null sequences (x_n) in X. As was pointed out in Jiménez-Melado, Llorens-Fuster and Saejung [6], Sims' result is equivalent to the statement that any Banach space X with $J(X) < 2w(X)$ has normal structure. They showed in [6] that if $J(X) < 1 + w(X)$, X has normal structure. Note that since $1/3 \le w(X) \le 1$, the condition $J(X) < 2w(X)$ implies that $J(X) < 1 + w(X)$.

Recently Gao [4] also showed that if $E(X) < 1 + 2w(X) + 5(w(X))^2$, X has normal structure. It is easy to see that if $E(X) < 1 + 2w(X) + 5(w(X))^2$, then $J(X) < 1 + w(X)$.

Theorem 9. *Let $E(X) < 1 + 2w(X) + 5(w(X))^2$. Then both of X and X^* have normal structure.*

Remark 7. X is uniformly non-square if and only if $E(X) < 8$ (see Theorem 1). Hence if X is uniformly non-square and $w(X) = 1$, both of X and X^* have normal structure.

References

[1] S. Dhompongsa, A. Kaewkhao and S. Tasena, On a generalized James constant, J. Math. Anal. Appl. **285** (2003), 419-435.

[2] S. Dhompongsa, P. Piraisangjun and S. Saejung, Generalized Jordan-von Neumann constants and uniform normal structure, Bull. Austral. Math. Soc. **67** (2003), 225-240.

[3] J. Gao, A Pythagorean approach in Banach spaces, J. Inequal. Appl. **2006**: Article ID 94982 (2006), 1-11.

[4] J. Gao, On some geometric parameters in Banach spaces, J. Math. Anal. Appl. **334** (2007), 114-122.

[5] K. Goebel and W. A. Kirk, Topics in metric fixed point theory, Cambridge University Press, 1990.

[6] A. Jiménez-Melado, E. Llorens-Fuster and S. Saejung, The von Neumann-Jordan constant, weak orthogonality and normal structure in Banach spaces, Proc. Amer. Math. Soc. **134** (2006), 355-364.

[7] M. Kato, L. Maligranda and Y. Takahashi, On James, Jordan-von Neumann constants and the normal structure coefficients of Banach spaces, Studia Math. **144** (2001), 275-295.

[8] B. Sims, A class of spaces with weak normal structure, Bull. Austral. Math. Soc. **50** (1994), 523-528.

[9] Y. Takahashi, Some geometric constants of Banach spaces – A unified approach, to appear in: "Banach and Function Spaces II", Proceedings of the Second International Symposium on Banach and Function Spaces, Kitakyushu, Japan, 2006, eds. M. Kato and L. Maligranda, Yokohama Publishers.

[10] C. Yang and F. Wang, On a new geometric constant related to the von Neumann-Jordan constant, J. Math. Anal. Appl. **324** (2006), 555-565.