数理解析研究所講究録
第 1570 巻 2007 年 123-136

Two New Nonexpansive Mappings and Geometry of Banach Spaces

東京工業大学・大学院情報理工学研究科
髙橋渉 (Wataru Takahashi)
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology

Abstract. Our purpose in this article is to discuss new nonlinear operators in a Banach space which are related to nonexpansive mappings and to obtain convergence theorems for the operators. We first deal with a nonlinear operator called a relatively nonexpansive mapping which generalizes a nonexpansive mapping in a Hilbert space. Using this operator, we prove a strong convergence theorem which generalizes Nakajo and Takahashi [29]. We also obtain another theorem for relatively nonexpansive mappings which is connected with Reich's theorem [33]. Next, we define another nonlinear operator in a Banach space called a generalized nonexpansive mapping. This mapping also generalizes a nonexpansive mapping in a Hilbert space. Using this mapping, we also get a strong convergence theorem which is related to Nakajo and Takahashi [29] and is different from the theorem above. Further, we obtain a weak convergence theorem of Reich's type. Finally, we prove a strong convergence theorem for nonexpansive mappings in a Banach space which is closedly related to Nakajo and Takahashi [29].

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then, a mapping T of C into itself is called nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for all $x,y\in C$. We denote by $F(T)$ the set of fixed points of T.

Mann [22] introduced the following iterative sequence to approximate a fixed point of a nonexpansive mapping: $x_1=x$ and

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)Tx_n,\quad n=1,2,\ldots,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Reich [33] proved the following weak convergence theorem for such a sequence. For the proof, see Takahashi [46].

Theorem 1.1 (Reich [33]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x\in C$ and let $\{x_n\}$ be a sequence defined by $x_1=x$ and*

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)Tx_n,\quad n=1,2,\ldots,$$

where $\{\alpha_n\} \subset [0,1]$ *satisfies*

$$0 \le \alpha_n < 1 \text{ and } \sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty.$$

Then, $\{x_n\}$ *converges weakly to* $z \in F(T)$*, where* $z = \lim_{n\to\infty} Px_n$.

Reich [33] proved really such a theorem in a uniformly convex Banach space whose norm is a Fréchet differentiable. On the other hand, we know many problems in nonlinear analysis and optimization which are formulated as follows: Find

$$u \in H \text{ such that } 0 \in Au, \tag{1.1}$$

where A is a maximal monotone operator from H to H. Such $u \in H$ is called a zero point (or a zero) of A. A well-known method for solving (1.1) in a Hilbert space H is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n}x_n, \quad n = 1,2,\ldots, \tag{1.2}$$

where $\{r_n\} \subset (0,\infty)$ and $J_r = (I+rA)^{-1}$ for all $r > 0$. This algorithm was first introduced by Martinet [23]. In [39], Rockafellar proved that if $\liminf_{n\to\infty} r_n > 0$ and $A^{-1}0 \ne \emptyset$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a solution of (1.1). Motivated by Rockafellar's result, Kamimura and Takahashi [16] proved the following convergence theorem.

Theorem 1.2 (Kamimura and Takahashi [16]). *Let* H *be a Hilbert space and let* $A \subset H \times H$ *be a maximal monotone operator. Let* $J_r = (I+rA)^{-1}$ *for all* $r > 0$ *and let* $\{x_n\}$ *be a sequence defined as follows:* $x_1 = x \in H$ *and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, \quad n = 1,2,\ldots,$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \liminf_{n\to\infty} r_n > 0.$$

If $A^{-1}0 \ne \emptyset$*, then the sequence* $\{x_n\}$ *converges weakly to an element* v *of* $A^{-1}0$*, where* $v = \lim_{n\to\infty} Px_n$ *and* P *is the metric projection of* H *onto* $A^{-1}0$.

Solodov and Svaiter [41] also proved the following strong convergence theorem by the hybrid method in mathematical programming.

Theorem 1.3 (Solodov and Svaiter [41]). *Let* H *be a Hilbert space and let* $A \subset H \times H$ *be a maximal monotone operator. Let* $x \in H$ *and let* $\{x_n\}$ *be a sequence defined by*

$$\begin{cases} x_1 = x \in H, \\ 0 = v_n + \dfrac{1}{r_n}(y_n - x_n), \ v_n \in Ay_n, \\ H_n = \{z \in H : \langle z - y_n, v_n\rangle \le 0\}, \\ W_n = \{z \in H : \langle z - x_n, x_1 - x_n\rangle \le 0\}, \\ x_{n+1} = P_{H_n \cap W_n}x_1, \ n = 1,2,\ldots, \end{cases}$$

where $\{r_n\}$ *is a sequence of positive numbers. If* $A^{-1}0 \ne \phi$ *and* $\liminf_{n\to\infty} r_n > 0$*, then* $\{x_n\}$ *converges strongly to* $P_{A^{-1}0}x_1$.

Motivated by Solodov and Svaiter [41], Nakajo and Takahashi [29] proved the following strong convergence teorem by using the hybrid method for nonexpansive mappings in a Hilbert space.

Theorem 1.4 (Nakajo and Takahashi [29]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, \dots, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$ and $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to $Px_1 \in F(T)$.

After Nakajo and Takahashi [29], many reseachers have studied such theorems by hybrid methods in a Hilbert space; see, for instance, [14, 24, 42, 55]. However, we can not find a theorem for nonexpansive mappings in a Banach space which generalizes Nakajo and Takahashi [29].

Our purpose in this article is to consider new nonlinear operators in a Banach space for extending Nakajo and Takahashi's result [29] in a Hilbert space to that in a Banach space.

In Section 3, we deal with a nonlinear operator in a Banach space called a relatively nonexpansive mapping which generalizes a nonexpansive mapping in a Hilbert space. We know that a relatively nonexpansive mapping in a Banach space is completely different from a nonexpansive mapping in a Banach space. In this section, we state a strong convergence theorem for relatively nonexpansive mappings which generalizes Nakajo and Takahashi [29]. We also obtain another theorem for relatively nonexpansive mappings which is connected with Reich's theorem [33].

In Section 4, we define another nonlinear operator in a Banach space which generalizes a nonexpansive mapping in a Hilbert space. We call such a nonlinear operator a generalized nonexpansive mapping. In this section, we obtain a strong convergence theorem which is related to Nakajo and Takahashi [29] and is different from the result in Section 3. Further, we obtain a weak convergence theorem of Reich's type. Finally, in Section 5, we prove a strong convergence theorem for nonexpansive mappings in a Banach space which is closedly related to Nakajo and Takahashi [29].

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. If E is uniformly convex, then δ satisfies that $\delta(\epsilon/r) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \le r\left(1-\delta\left(\frac{\epsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ and $\|x-y\| \ge \epsilon$. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \le \|x-y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the metric projection of E onto C. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. We know the following result: Let E be a smooth Banach space. Let C be a nonempty closed convex subset of E and $x_1 \in E$. Then, $x_0 = P_C(x_1)$ if and only if

$$\langle x_0 - y, J(x_1 - x_0)\rangle \ge 0$$

for all $y \in C$, where J is the duality mapping of E.

A Banach space E is said to satisfy Opial's condition [31] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial's condition.

Let C be a closed convex subset of E. A mapping $T\colon C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x-y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be sunny if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \ge 0$. A mapping P of C into C is said to be a retraction if $P^2 = P$. We denote the closure of the convex hull of D by $\overline{\text{co}}D$.

A multi-valued operator $A\colon E \to E^*$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \ge 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; for instance, see [46].

Theorem 2.1. *Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Theorem 2.2. *Let E be a strictly convex and smooth Banach space and let $x, y \in E$. If $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

A duality mapping J of a smooth Banach space is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $Jx_n \overset{*}{\rightharpoonup} Jx$, where $\overset{*}{\rightharpoonup}$ means the weak* convergence.

3 Relatively nonexpansive mappings

In this section, we first deal with a strong convergence theorem in a Banach space which generalizes Nakajo and Takahashi's theorem (Theorem 1.4) in a Hilbert space.

Let E be a reflexive, strictly convex and smooth Banach space. The function $\phi\colon E\times E \to (-\infty,\infty)$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [18]. Let C be a nonempty closed convex subset of E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that

$$\phi(x_0, x) = \inf\{\phi(z,x) : z \in C\}. \tag{3.1}$$

Now, we define the mapping Q_C of E onto C by $Q_C x = x_0$, where x_0 is defined by (3.1). Such Q_C is called the generalized projection of E onto C. It is easy to see that in a Hilbert space, the mapping Q_C is coincident with the metric projection.

Lemma 3.1. *Let E be a smooth Banach space, let C be a nonempty closed convex subset of E, let $x \in E$ and let $x_0 \in C$. Then, the following (1) and (2) are equivalent:*
(1) $\phi(x_0, x) = \min_{y\in C}\phi(y,x)$;
(2) $\langle x_0 - y, Jx - Jx_0\rangle \geq 0$ for all $y \in C$.

Let E be a smooth Banach space. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [36] if C contains a sequence $\{x_n\}$ which converges weakly to p and the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.

The following is a strong convergence theorem for relatively nonexpansive mappings in a Banach space which generalizes Nakajo and Takahashi's theorem [29] in a Hilbert space.

Theorem 3.2 (Matsushita and Takahashi [26]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself with $F(T) \neq \phi$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_1 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = Q_{H_n\cap W_n} x \end{cases}$$

for all $n = 1, 2, \dots,$ where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $Q_{F(T)}x$, where $Q_{F(T)}$ is the generalized projection from C onto $F(T)$.

Using Theorem 3.2, we can prove Nakajo and Takahashi's theorem (Theorem 1.4) as follows: To show Nakajo and Takahashi's theorem, it is sufficient to prove that if T is nonexpansive, then T is relatively nonexpansive. It is obvious that $F(T) \subset \hat{F}(T)$. If $u \in \hat{F}(T)$, then there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. Since T is nonexpansive, T is demiclosed. So, we have $u = Tu$. This implies $F(T) = \hat{F}(T)$. Further, in a Hilbert space H, we know that

$$\phi(x, y) = \|x - y\|^2$$

for every $x, y \in H$. So, $\|Tx - Ty\| \leq \|x - y\|$ is equivalent to $\phi(Tx, Ty) \leq \phi(x, y)$. Therefore, T is relatively nonexpansive. Using Theorem 3.2, we obtain the desired result.

Using Theorem 3.2, we can prove a strong convergence threorem for maximal monotone operators in a Banach space. Before stating the theorem, we define the following resolvents for maximal monotone operators in a Banach space. Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Theorem 2.1 and the strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r. \tag{3.2}$$

If $Q_r x = x_r$, then we can define a single valued mapping $Q_r : E \to D(A)$ by $Q_r = (J + rA)^{-1}J$ and such Q_r is called the relative resolvent of A. We know that $A^{-1}0 = F(Q_r)$ for all $r > 0$; see [45, 46] for more details.

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^*, let Q_r be the relative resolvent of A, where $r > 0$. If $A^{-1}0$ is nonempty, then Q_r is a relatively nonexpansive mapping on E.*

Using this result and Theorem 3.2, we prove a strong convergence theorem for relative resolvents of maximal monotone operators in a Banach space.

Theorem 3.4. *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^*, let Q_r be the relative resolvent of A, where $r > 0$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_1 = x \in E, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JQ_r x_n), \\ H_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x \end{cases}$$

for all $n = 1, 2, \ldots$, where J is the duality mapping on E. If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $Q_{A^{-1}0}x$, where $Q_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.

Next, we obtain a weak convergence theorem for relatively nonexpansive mappings in a Banach space which is connected with Reich [33], Browder and Petryshyn's theorem [6] and Rockafellar's theorem [39]. Before proving it, we need the following proposition.

Proposition 3.5 (Matsushita and Takahashi [25]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a* *sequence of real numbers such that* $0 \le \alpha_n \le 1$. *Let* $x_1 \in C$ *and let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = Q_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n)$$

for $n = 1, 2, \ldots$. *Then* $\{Q_{F(T)} x_n\}$ *converges strongly to a fixed point of* T, *where* $Q_{F(T)}$ *is the generalized projection from* C *onto* $F(T)$.

Using Proposition 3.5, we can prove the following weak convergence theorem.

Theorem 3.6 (Matsushita and Takahashi [25]). *Let* E *be a uniformly convex and uniformly smooth Banach space, let* C *be a nonempty closed convex subset of* E, *and let* T *be a relatively nonexpansive mapping from* C *into itself such that* $F(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that*

$$0 \le \alpha_n \le 1 \quad \text{and} \quad \liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0.$$

Let $x_1 \in C$ *and let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = Q_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n)$$

for $n = 1, 2, \ldots$. *If* J *is weakly sequentially continuous, then* $\{x_n\}$ *converges weakly to* u, *where* $u = \lim_{n\to\infty} Q_{F(T)} x_n$ *and* $Q_{F(T)}$ *is the generalized projection from* C *onto* $F(T)$

Using Theorem 3.6, we can prove the following two weak convergence theorems.

Theorem 3.7 ([6]). *Let* C *be a nonempty closed convex subset of a Hilbert space* H, *let* T *be a nonexpansive mapping from* C *into itself such that* $F(T) \neq \emptyset$ *and let* λ *be a real number such that* $0 < \lambda < 1$. *Let* $x_1 \in C$ *and let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = \lambda x_n + (1-\lambda) T x_n$$

for $n = 1, 2, \ldots$. *Then* $\{x_n\}$ *converges weakly to* u, *where* $u = \lim_{n\to\infty} P_{F(T)} x_n$ *and* $P_{F(T)}$ *is the metric projection from* C *onto* $F(T)$

Theorem 3.8. *Let* E *be a uniformly convex and uniformly smooth Banach space, let* A *be a maximal monotone operator from* E *to* E^* *such that* $A^{-1}0 \neq \emptyset$, *let* Q_r *be the relative resolvent of* A *where* $r > 0$, *and let* $\{\alpha_n\}$ *be a sequence of real numbers such that*

$$0 \le \alpha_n \le 1 \quad \text{and} \quad \liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0.$$

Let $x_1 \in E$ *and let* $\{x_n\}$ *be the sequence defined by*

$$x_{n+1} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J Q_r x_n)$$

for $n = 1, 2, \ldots$. *If* J *is weakly sequentially continuous, then* $\{x_n\}$ *converges weakly to* u *in* $A^{-1}0$, *where* $u = \lim_{n\to\infty} Q_{A^{-1}0} x_n$ *and* $Q_{A^{-1}0}$ *is the generalized projection from* E *onto* $A^{-1}0$.

Kamimura and Takahashi [18] extended Solodov and Svaiter's result [41] to the following theorem by using Lemma 3.1 and the resolvents defined by (3.2).

Theorem 3.9 (**[18]**). *Let E be a uniformly convex and uniformly smooth Banach space and let A be a maximal monotone operator from E into E^* such that $A^{-1}0 \neq \phi$. Let $Q_r = (J+rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = Q_{r_n} x_n, \\ H_n = \{z \in E : \langle z - y_n, Jx_n - Jy_n \rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_1, \ n = 1, 2, \ldots, \end{cases}$$

where $\{r_n\}$ is a sequence of positive numbers such that $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $Q_{A^{-1}0}x_1$, where $Q_{A^{-1}0}$ is the generalized projection of E onto $A^{-1}0$.

Kamimura, Kohsaka and Takahashi [15] also proved a weak convergence theorem of Mann's type for maximal monotone operators in a Banach space. Before stating the theorem, we need the following strong convergence theorem.

Theorem 3.10 (**[15]**). *Let E be a smooth and uniformly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0$ is nonempty, let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $Q_{A^{-1}0}$ be the generalized projection of E onto $A^{-1}0$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n)J(Q_{r_n} x_n)), \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$. Then, the sequence $\{Q_{A^{-1}0}(x_n)\}$ converges strongly to an element of $A^{-1}0$, which is a unique element $v \in A^{-1}0$ such that

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y \in A^{-1}0} \lim_{n\to\infty} \phi(y, x_n).$$

Using Theorem 3.10, we can prove the following theorem in a Banach space which generalizes the results of Rockafellar [39] and Kamimura and Takahashi [16] in a Hilbert space.

Theorem 3.11 (**[15]**). *Let E be a smooth and uniformly convex Banach space whose duality mapping J is weakly sequentially continuous. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0$ is nonempty, let $Q_r = (J+rA)^{-1}J$ for all $r > 0$ and let $Q_{A^{-1}0}$ be the generalized projection of E onto $A^{-1}0$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n)J(Q_{r_n} x_n)), \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then, $\{x_n\}$ converges weakly to an element v of $A^{-1}0$, where $v = \lim_{n\to\infty} Q_{A^{-1}0}(x_n)$.

As a direct consequence of Theorem 3.11, we obtain the following:

Theorem 3.12. *Let E be a smooth and uniformly convex Banach space whose duality mapping J is weakly sequentially continuous. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0$ is nonempty, let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $Q_{A^{-1}0}$ be the generalized projection of E onto $A^{-1}0$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$x_{n+1} = Q_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then, the sequence* $\{x_n\}$ *converges weakly to an element* v *of* $A^{-1}0$, *where* $v = \lim_{n\to\infty} Q_{A^{-1}0}(x_n)$.

Problem. If E and E^* are uniformly convex Banach spaces, does Theorem 3.12 hold without assumming that J is weakly sequentially continuous ?

4 Generalized nonexpansive mappings

Let E be a smooth Banach space and let D be a nonempty closed convex subset of E. A mapping $R : D \to D$ is called generalized nonexpansive if $F(R) \neq \emptyset$ and

$$\phi(Rx, y) \leq \phi(x, y), \quad \forall x \in D, \forall y \in F(R),$$

where $F(R)$ is the set of fixed points of R. A point p in C is said to be a generalized asymptotic fixed point of T [13] if C contains a sequence $\{x_n\}$ such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and the strong $\lim_{n\to\infty}(Jx_n - JTx_n) = 0$. The set of generalized asymptotic fixed points of T will be denoted by $\check{F}(T)$.

Let E be a reflexive and smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. For each $\lambda > 0$ and $x \in E$, consider the set

$$R_\lambda x := \{z \in E : x \in z + \lambda BJ(z)\}.$$

Then $R_\lambda x$ consists of one point. We also denote the domain and the range of R_λ by $D(R_\lambda) = R(I + \lambda BJ)$ and $R(R_\lambda) = D(BJ)$, respectively. Such R_λ is called the generalized resolvent of B and is denoted by

$$R_\lambda = (I + \lambda BJ)^{-1}.$$

We get some properties of R_λ and $(BJ)^{-1}0$.

Proposition 4.1 ([12]). *Let E be a reflexive and strictly convex Banach space with a Fréchet differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

1. $D(R_\lambda) = E$ *for each* $\lambda > 0$;
2. $(BJ)^{-1}0 = F(R_\lambda)$ *for each* $\lambda > 0$, *where* $F(R_\lambda)$ *is the set of fixed points of* R_λ;
3. $(BJ)^{-1}0$ *is closed;*
4. R_λ *is generalized nonexpansive for each* $\lambda > 0$.

Proposition 4.2 ([13]). *Let E be a smooth and uniformly convex Banach space, let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, and let R_λ be the generalized resolvent of B for $\lambda > 0$. Then $\check{F}(R_\lambda) = F(R_\lambda)$.*

Next, we get the following result for generalized nonexpansive mappings.

Proposition 4.3. *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_C be a retraction of E onto C. Then R_C is sunny and generalized nonexpansive if and only if*

$$\langle x - R_C x, J(R_C x) - J(y)\rangle \geq 0$$

for each $x \in E$ and $y \in C$.

Let E be a smooth and strictly convex Banach space and let C be a nonempty closed subset of E. Then, a sunny generalized nonexpansive retraction of E onto C is unique. In fact, let R, S be two sunny generalized nonexpansive retractions of E onto C. Then, by Proposition 4.3, for each $x \in E$, we have

$$\langle x - Rx, J(Rx) - J(y)\rangle \geq 0, \ \langle x - Sx, J(Sx) - J(y)\rangle \geq 0, \ \forall y \in C.$$

From $Rx, Sx \in C$, we get

$$\langle x - Rx, J(Rx) - J(Sx)\rangle \geq 0, \ \langle x - Sx, J(Sx) - J(Rx)\rangle \geq 0.$$

From these inequalities, we have

$$\langle Sx - Rx, J(Rx) - J(Sx)\rangle \geq 0.$$

Since E is strictly convex, we get $Sx = Rx$.

Before showing an example of sunny generalized nonexpansive retractions, we recall the following theorem.

Theorem 4.4 ([34]). *Let E be a Banach space and let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. If E^* is strictly convex and has a Fréchet differentiable norm. Then, for each $x \in E$, $\lim_{\lambda\to\infty}(J + \lambda A)^{-1}J(x)$ exists and belongs to $A^{-1}0$.*

Using Theorem 4.4, we get the following result.

Theorem 4.5 ([12]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

1. *For each $x \in E$, $\lim_{\lambda\to\infty} R_\lambda x$ exists and belongs to $(BJ)^{-1}0$;*
2. *If $Rx := \lim_{\lambda\to\infty} R_\lambda x$ for each $x \in E$, then R is a sunny generalized nonexpansive retraction of E onto $(BJ)^{-1}0$.*

Next, we discuss proximal point algorithms for generalized resolvents of a maximal monotone operator $B \subset E^* \times E$. We start with the following lemma. Compare this lemma with the results in Kamimura and Takahashi [18], and Kohsaka and Takahashi [20].

Lemma 4.6. *Let E be a reflexive, strictly convex, and smooth Banach space, let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, and $R_r = (I + rBJ)^{-1}$ for all $r > 0$. Then*

$$\phi(x, R_r x) + \phi(R_r x, u) \leq \phi(x, u)$$

for all $r > 0$, $u \in (BJ)^{-1}0$, and $x \in E$.

The following is a strong convergence theorem for generalized nonexpansive mappings in a Banach space which is related to Nakajo and Takahashi's theorem [29] in a Hilbert space.

Theorem 4.7 (Ibaraki and Takahashi [13]). *Let E be a uniformly convex and uniformly smooth Banach space, let T be a generalized nonexpansive mapping from E into itself with $F(T) \neq \phi$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_1 = x \in E, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n), \\ H_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} x \end{cases}$$

for all $n = 1, 2, \ldots,$ *where* J *is the duality mapping on* E. *If* $\check{F}(T) = F(T)$, *then* $\{x_n\}$ *converges strongly to* $R_{F(T)}x$, *where* $R_{F(T)}$ *is the sunny generalized nonexpansive retraction from* C *onto* $F(T)$.

We can also prove the following weak convergence theorem, which is a generalization of Kamimura and Takahashi's weak convergence theorem (Theorem 1.2).

Theorem 4.8. *Let* E *be a smooth and uniformly convex Banach space whose duality mapping* J *is weakly sequentially continuous. Let* $B \subset E^* \times E$ *be a maximal monotone operator, let* $R_r = (I + rBJ)^{-1}$ *for all* $r > 0$ *and let* $\{x_n\}$ *be a sequence defined as follows:* $x_1 = x \in E$ *and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) R_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy*

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \liminf_{n\to\infty} r_n > 0.$$

If $B^{-1}0 \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges weakly to an element of* $(BJ)^{-1}0$.

5 Concluding remarks

Recently, Matsushita and Takahashi [27] proved the following strong convergence theorem for nonexpansive mappings in a Banach space which is related to Nakajo and Takahashi's theorem [29].

Theorem 5.1 (Matsushita and Takahashi [27]). *Let* E *be a uniformly convex and smooth Banach space, let* C *be a nonempty bounded closed convex subset of* E *and let* T *be a nonexpansive mapping from* C *into itself. Let* $\{x_n\}$ *be a sequence in* C *defined by*

$$\begin{cases} x_1 = x \in C, \\ C_n = \overline{\text{co}}\{z \in C : \|z - y_n\| \leq \|z - x_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x \end{cases}$$

for all $n = 1, 2, \ldots,$ *where* $P_{C_n \cap D_n}$ *is the metric projection from* E *onto* $C_n \cap D_n$ *and* $\{t_n\}$ *is a sequence in* $(0,1)$ *with* $t_n \to 0$. *Then* $\{x_n\}$ *converges strongly to the element* $P_{F(T)}x$, *where* $P_{F(T)}$ *is the the metric projection from* E *onto* $F(T)$.

For the proof of Theorem 5.1, Matsushita and Takahashi [27] used essentially the following Bruck's theorem [7]:

Theorem 5.2 (Bruck [7]). *Let* C *be a closed convex subset of a uniformly convex Banach space* E. *Then for each* $r > 0$, *there exists a strictly increasing convex continuous function* $\lambda : [0, \infty) \to [0, \infty)$ *such that* $\lambda(0) = 0$ *and*

$$\lambda\left(\left\|T\left(\sum_{j=0}^{n} \lambda_j x_j\right) - \sum_{j=0}^{n} \lambda_j T x_j\right\|\right) \leq \max_{0 \leq j < k \leq n} (\|x_j - x_k\| - \|Tx_j - Tx_k\|)$$

for all $n \in \mathbb{N}$, $\{\lambda_j\} \in \Delta^n$, $\{x_j\} \subset C \cap B_r$ *and* $T \in Lip(C, 1)$, *where* $\Delta^n = \{\{\lambda_0, \lambda_1, \ldots, \lambda_n\} : 0 \leq \lambda_j$ *and* $\sum_{j=0}^{n} \lambda_j = 1\}$, $B_r = \{z \in E : \|z\| \leq r\}$ *and* $Lip(C, 1)$ *is the set of all nonexpansive mappings of* C *into* E.

Problem. Can we prove Theorem 5.1 under assuming that C is a closed and convex subset of E and $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$?

References

[1] Y. I. Alber, *Metric and generalized projections in Banach spaces: Properties and applications*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 15–20.

[2] K. Aoyama and W. Takahashi, *Strong convergence theorems for a family of relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory **8** (2007), to appear.

[3] H. Brèzis, Opérateurs maximaux monotones, Mathematics Studies No. 5, North-Holland, Amsterdam, 1973.

[4] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA **54** (1965), 1041–1044.

[5] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[6] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

[7] R. E. Bruck, *On the convex approximation property and the asymptotic behaviour of nonlinear contractions in Bnach spaces*, Israel J. Math. **38** (1981), 304–314.

[8] J. Diestel, *Geometry of Banach spaces, Selected Topics*, Lecture Notes in Mathematics **485**, Springer, Berlin, 1975.

[9] O. Güler,*On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29** (1991), 403–419.

[10] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[11] T. Ibaraki and W. Takahashi, *Weak and strong convergence theorems for new resolvents of maximal monotone operators in Banach spaces*, Advances in Mathematical Economics, **10** (2007), 51–64.

[12] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory, to appear.

[13] T. Ibaraki and W. Takahashi, *Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces*, to appear.

[14] H. Iiduka and W. Takahashi, *Strong convergence theorems by a hybrid method for nonexpansive mappings and inverse-strongly-monotone mappings*, in Fixed Point Theory and Applications (J. G. Falset, E. L. Fuster and B. Sims Eds.) , Yokohama Publishers, Yokohama, 2004, pp. 81-94.

[15] S. Kamimura, F. Kohsaka and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal. **12** (2004), 417–429.

[16] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.

[17] S. Kamimura and W. Takahashi, *Weak and strong convergence of solutions to accretive operator inclusions and applications*, Set-Valued Anal. **8** (2000), 361–374.

[18] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach apace*, SIAM J. Optim. **13** (2002), 938–945.

[19] M. Kikkawa and W. Takahashi, *Strong convergence theorems by the viscosity approximation methods for nonexpansive mappings in Banach spaces*, in Convex Analysis and

Nonlinear Analysis (W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2007, pp. 227–238.

[20] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal. **2004** (2004), 239-249.

[21] F. Kohsaka and W. Takahashi, Weak and strong convergence theorems for minimax problems in Banach spaces, in Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka, Eds.), Yokohama Publishers, 2004, pp. 203–216.

[22] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[23] B. Martinet, *Regularisation, d'inèquations variationelles par approximations succesives*, Revue Francaise d'Informatique et de Recherche Operationelle, 1970, pp. 154–159.

[24] C. Martinez-Yanes and H. K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal. **64** (2006), 2400-2411.

[25] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2004** (2004), 37-47.

[26] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.

[27] S. Matsushita and W. Takahashi, *Approximating fixed points of nonexpansive mappings in a Banach space by metric projections*, Applied Math. Compution, in press.

[28] J. J. Moreau, *Proximité et dualité dans un espace Hilberien*, Bull. Soc. Math., France **93** (1965), 273–299.

[29] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372-378.

[30] S. Ohsawa and W. Takahashi, *Strong convergence theorems for resolvento of maximal monotone operator*, Arch. Math. **81** (2003), 439–445.

[31] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[32] S. Plubtieng and K. Ungchittrakool, *Strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal., to appear.

[33] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[34] S. Reich, *Constructive techniques for accretive and monotone operators*, Applied Nonlinear Analysis (V. Lakshmikan, ed.), Academic Press, New York, 1979, pp. 335–345.

[35] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.

[36] S. Reich, *A weak convergence theorem for the alternative method with Bregman distance*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 313-318.

[37] R. T. Rockafellar *Characterization of the subdifferentials of convex functions*, Pacific J. Math. **17** (1966), 497–510.

[38] R. T. Rockafellar *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[39] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.

[40] M. V. Solodov and B. F. Svaiter, *A hybrid projection – proximal point algorithm*, J. Convex Anal. **6** (1999), 59–70.

[41] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. **87** (2000), 189–202.

[42] A. Tada and W. Takahashi, *Weak and Strong convergence theorems for a nonexpansive*

mapping and an equilibrium problem, J. Optim. Theory Appl. **133** (2007), 359–370.

[43] W. Takahashi, *Fan's existence theorem for inequalities concerning convex functions and its applications*, in Minimax Theory and Applications (S. Simons and B. Ricceri, Eds.), Kluwer Academic Publishers, 1998, pp. 241-260.

[44] W. Takahashi, *Iterative methods for approximation of fixed points and their applications*, J. Oper. Res. Soc. Japan **43** (2000), 87–108.

[45] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[46] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).

[47] W. Takahashi, *Fixed point theorems and proximal point algorithms*, in Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka, Eds.), Yokohama Publishers, 2003, pp. 471–481.

[48] W. Takahashi, *Weak and strong convergence theorems for nonlinear operators of accretive and monotone type and applications*, in Nonlinear Analysis and Applications (R. P. Agarwal and D. O'Regan, Eds.), Kluwer Academic Publishers, 2003, pp. 891-912.

[49] W. Takahashi, *Convergence theorems for nonlinear projections in Banach spaces*, in RIMS Kokyuroku 1396 (M. Tsukada, Ed.), 2004, pp. 49–59.

[50] W. Takahashi, *Convergence theorems and nonlinear projections in Banach spaces*, in Banach and Function Spaces (M. Kato and L. Maligranda, Eds.), Yokohama Publishers, Yokohama, 2004, pp. 145–174.

[51] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2005 (Japanese).

[52] W. Takahashi, *Weak and strong convergence theorems for nonlinear operators and their applications*, in RIMS Kokyuroku 1443 (T. Maruyama, Ed.), 2005, pp. 1–14.

[53] W. Takahashi, *Viscosity approximation methods for resolvents of accretive operators in Banach spaces*, J. Fixed Point Theory Appl. **1** (2007), 135–147.

[54] W. Takahashi and G. E. Kim, *Approximating fixed points of nonexpansive mappings in Banach spaces*, Math. Japon. **48** (1998), 1–9.

[55] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, to appear.

[56] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl. **104** (1984), 546–553.

[57] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.