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The geometry and topology of arithmetic hyperbolic 3-manifolds

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1 Introduction

This paper is based on three lectures given by the author at the RIMS Symposium, "Topology, Complex Analysis and Arithmetic of Hyperbolic Spaces" held at the Research Institute for Mathematical Sciences, Kyoto University, in December 2006. The goal of the lectures was to describe recent work on understanding topological and geometric properties of arithmetic hyperbolic 3-manifolds, and the connections with number theory. This is the theme of the paper.

Our discussion of topological aspects of arithmetic hyperbolic 3-orbifolds is motivated by the following central conjectures from 3-manifold topology:

Conjecture 1.1. *Let M be a closed hyperbolic 3-manifold, then M is virtually Haken; ie M has a finite sheeted cover N which contains embedded incompressible surface (necessarily of genus at least 2).*

Conjecture 1.2. *Let M be a closed hyperbolic 3-manifold, then M has a finite sheeted cover N for which $b_1(N) > 0$ (where $b_1(N)$ is the first Betti number of N).*

We can also rephrase Conjecture 1.2 to say that $\mathrm{vb}_1(\mathrm{M}) > 0$ where $\mathrm{vb}_1(\mathrm{M})$ is defined by

$$\mathrm{vb}_1(\mathrm{M}) = \sup\{\mathrm{b}_1(\mathrm{N}) : \mathrm{N} \text{ is a finite cover of } \mathrm{M}\},$$

and is called *the virtual first Betti number of M.*

Conjecture 1.3. *Let M be a closed hyperbolic 3-manifold, then* $\mathrm{vb}_1(\mathrm{M}) = \infty$.

Conjecture 1.4. *Let M be a closed hyperbolic 3-manifold, then $\pi_1(M)$ is large; that is to say, some finite index subgroup of $\pi_1(M)$ admits a surjective homomorphism onto a non-abelian free group.*

Now it is clear that Conjecture 1.4 implies Conjecture 1.3 implies Conjecture 1.2, and standard 3-manifold topology shows that Conjecture 1.2 implies Conjecture 1.1. Our interest here is in recent work towards reversing these implications.

Our geometric discussion is centered around the set of lengths of closed geodesics, as well as the set of geodesics themselves, and in particular on how these force a certain rigidity on commensurability classes. For example, the *length spectrum* $\mathcal{L}(M)$ of a hyperbolic 3-manifold M is the set of all lengths of closed geodesics on M counted with multiplicities. A question that has attracted some attention is whether hyperbolic 3-manifolds with the same length spectra are commensurable. We discuss this and other related questions in this paper.

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Given these preliminary remarks, the paper is organized as follows. In §2 we recall the definition of arithmetic Fuchsian and Kleinian groups, arithmetic hyperbolic 2 and 3-orbifolds together with some other terminology and notation that we will use. In §3, we discuss a geometric characterization of arithmetic hyperbolic 3-manifolds which will be used in §4 to show that for arithmetic hyperbolic 3-manifolds, Conjecture 1.2 implies Conjecture 1.4. In §4, we also discuss how certain conjectures from automorphic forms implies Conjectures 1.1–1.4. This is well-known to the experts, and our purpose is simply to sketch some of the ideas involved. In the opposite direction, in §6 we discuss how widely distributed arithmetic rational homology 3-spheres are. We motivate this in §5 by discussing a similar question in dimension 2 that offers useful comparisons for dimension 3. Finally, in §7 we discuss the geometric properties of arithmetic hyperbolic 2 and 3-orbifolds.

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2 Preliminaries

For more details on the topics covered in this section see [29] for number theoretic background and [39] for details on quaternion algebras and arithmetic groups.

2.1

By a number field k we mean a finite extension of $\mathbf{Q}$. The ring of integers of k will be denoted R_k. A place ν of k will be one of the canonical absolute values of k. The finite places of k correspond bijectively to the prime ideals of R_k. An infinite place of k is either real, corresponding to an embedding of k into $\mathbf{R}$, or complex, corresponding to a pair of distinct complex conjugate embeddings of k into $\mathbf{C}$. We denote by k_ν the completion of of k at a place ν. When ν is an infinite place, k_ν is isomorphic to $\mathbf{R}$ or $\mathbf{C}$ depending on whether ν is real or complex.

If $\mathcal{A}$ is an ideal of R_k, the norm of $\mathcal{A}$ is the cardinality of the quotient ring $R_k/\mathcal{A}$ and will be denoted $\mathrm{N}\mathcal{A}$.

2.2

Let k be a field of characteristic different from 2. The standard notation for a quaternion algebra over k is the following. Let a and b be non-zero elements of k, then $\left(\frac{a,b}{k}\right)$ (known as the Hilbert Symbol) denotes the quaternion algebra over k with basis $\{1, i, j, ij\}$ subject to $i^2 = a$, $j^2 = b$ and $ij = -ji$.

Let k be a number field, and ν a place of k. If B is a quaternion algebra defined over k, the classification of quaternion algebras $B_\nu = B \otimes_k k_\nu$ over the local fields k_ν is quite simple. If ν is complex then B_ν is isomorphic to $M(2, k_\nu)$ over k_ν. Otherwise there is, up to isomorphism over k_ν, a unique quaternion division algebra over k_ν, and B_ν is isomorphic over k_ν to either this division algebra or to $M(2, k_\nu)$.

Let B be a quaternion algebra over the number field k. B is *ramified* at a place ν of k if B_ν is a division algebra. Otherwise we say B is *unramified* at ν. We shall denote the set of places (resp. finite places) at which B is ramified by Ram B (resp. $\mathrm{Ram_f}$ B). The *discriminant* of B is the R_k-ideal $\prod_{\nu \in \mathrm{Ram_f}\, B} \mathcal{P}_\nu$ where $\mathcal{P}_\nu$ is the prime ideal associated to the place ν.

We summarize for convenience the classification theorem for quaternion algebras over number fields (see [39] Chapter 7).

Theorem 2.1. • *The set* Ram B *is finite, of even cardinality and contains no complex places.*

- *Conversely, suppose S is a finite set of places of k which has even cardinality and which contains no complex places. Then there is a quaternion algebra B over k with* Ram B = S, *and this B is unique up to isomorphism over k.*
- *B is a division algebra of quaternions if and only if* Ram B $\neq \emptyset$. □

2.3

We next recall the definition of arithmetic Fuchsian and Kleinian groups.

Let k be a totally real number field, and let B be a quaternion algebra defined over k which is ramified at all infinite places except one. Let $\rho: B \rightarrow M(2,\mathbf{R})$ be an embedding, $\mathcal{O}$ be an order of B, and $\mathcal{O}^1$ the elements of norm one in $\mathcal{O}$. Then $\mathrm{P}\rho(\mathcal{O}^1) < \mathrm{PSL}(2,\mathbf{R})$ is a finite co-area Fuchsian group, which is co-compact if and only if B is not isomorphic to $M(2,\mathbf{Q})$. A Fuchsian group Γ is defined to be arithmetic if and only if Γ is commensurable with some such $\mathrm{P}\rho(\mathcal{O}^1)$.

Notation: *Let B/k be as above and $\mathcal{O}$ be an order of B. We will denote the group $\mathrm{P}\rho(\mathcal{O}^1)$ by $\Gamma^1_{\mathcal{O}}$.*

Arithmetic Kleinian groups are obtained in a similar way. In this case we let k be a number field having exactly one complex place, and B a quaternion algebra over k which is ramified at all real places of k. As above, if $\mathcal{O}$ is an order of B and $\rho: \mathcal{O}^1 \hookrightarrow \mathrm{SL}(2,\mathbf{C})$, then $\Gamma^1_{\mathcal{O}}$ is a Kleinian group of finite co-volume. An *arithmetic Kleinian group* Γ is a subgroup of $\mathrm{PSL}(2,\mathbf{C})$ commensurable with a group of the type $\Gamma^1_{\mathcal{O}}$. An arithmetic Kleinian group is cocompact if and only if the quaternion algebra B as above is a division algebra.

In both the Fuchsian and Kleinian cases, the isomorphism class of the quaternion algebra B/k determines a wide commensurability class of groups in $\mathrm{PSL}(2,\mathbf{R})$ and $\mathrm{PSL}(2,\mathbf{C})$ respectively (see [39] Chapter 8). By Theorem 2.1 the isomorphism classes of such quaternion algebras will be completely determined by the finite set of places of k at which B is ramified.

A hyperbolic orbifold $\mathbf{H}^2/\Gamma$ or $\mathbf{H}^3/\Gamma$ will be called arithmetic if Γ is an arithmetic Fuchsian or Kleinian group.

Recall that if Γ is a Kleinian group, the invariant trace-field of Γ is the field

$$k\Gamma = \mathbf{Q}(\{\mathrm{tr}^2\gamma : \gamma \in \Gamma\})$$

and the invariant quaternion algebra

$$A\Gamma = \{\Sigma a_j\gamma_j : a_j \in k\Gamma, \gamma_j \in \Gamma^{(2)}\}.$$

As discussed in [39] these are invariants of commensurability. When Γ is arithmetic, the field k and algebra B coincide with $k\Gamma$ and $A\Gamma$

Arithmetic Fuchsian or Kleinian groups form a small but interesting subclass of Fuchsian or Kleinian groups. For example it is known that there are only finitely many conjugacy classes of arithmetic Kleinian groups whose volume is bounded above by some constant K. A similar statement holds for arithmetic Fuchsian groups and this also shows that there are only finitely many conjugacy classes of arithmetic Fuchsian groups of a fixed signature (see [39] Chapter 11).

Notation: *We shall call a group G is derived from a quaternion algebra if G is a subgroup of some $\Gamma^1_{\mathcal{O}}$, regardless of whether G has finite index in $\Gamma^1_{\mathcal{O}}$.*

2.4

We set up some notation and collect some information on 3-orbifolds that we will make use of.

Let Q be a compact orientable 3-orbifold. We shall denote by $\mathrm{sing}(O)$ its singular locus, and $|Q|$ the underlying 3-manifold. Let $\mathrm{sing}^0(Q)$ and $\mathrm{sing}^-(Q)$ denote the components of the singular locus with, respectively, zero and negative Euler characteristic. For any prime p, let $\mathrm{sing}_p(Q)$ denote the union of the arcs and circles in $\mathrm{sing}(Q)$ with singularity order that is a multiple of p. Let $\mathrm{sing}_p^0(Q)$ and $\mathrm{sing}_p^-(Q)$ denote those components of $\mathrm{sing}_p(Q)$ with zero and negative Euler characteristic.

When $Q = \mathbf{H}^3/\Gamma$ is a closed orientable hyperbolic 3-orbifold the nature of the singular locus Σ is completely understood. Σ is a link or graph in $|Q|$ and the local groups are either cyclic, dihedral, A_4, S_4 or A_5. Note the dihedral group of order 4 is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. The cases of the dihedral groups, A_4, S_4 and A_5 arise when there are vertices in the singular locus.

It can be shown that one can always pass to a finite cover that has no vertices but does have non-empty singular locus (cf. the proof of Proposition 4.4 of [28]). On the other hand, it is shown (and used in a crucial way) in [28] that for certain orbifolds with vertices in the singular locus, then the number of vertices can be increased without bound in passage to finite sheeted covers.

2.5

Let Γ be a non-cocompact Kleinian (resp. Fuchsian) group acting on $\mathbf{H}^3$ (resp. $\mathbf{H}^2$) with finite co-volume. Let $\mathcal{U}(\Gamma)$ denote the subgroup of Γ generated by parabolic elements of Γ. Note that $\mathcal{U}(\Gamma)$ is visibly a normal subgroup of Γ, and we may define:

$$V(\Gamma) = (\Gamma/\mathcal{U}(\Gamma))^{ab} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Setting $r(\Gamma) = \dim_{\mathbf{Q}}(V(\Gamma))$, then it follows from standard arguments that $r(\Gamma)$ denotes the dimension of the space of non-peripheral homology.

Example 1: Since $\mathrm{PSL}(2,\mathbf{Z}) = \mathrm{P} < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} >$, it follows that $r(\mathrm{PSL}(2,\mathbf{Z})) = 0$.

More generally if Γ is a non-cocompact Fuchsian group such that $\mathbf{H}^2/\Gamma$ has underlying space of genus g, then $r(\Gamma) = 2g$.

Example 2: If L is a link in S^3 with $S^3 \setminus L = \mathbf{H}^3/\Gamma$ then $r(\Gamma) = 0$, since Γ is generated by meridians.

2.6

Let p be a prime, and let $\mathbf{F}_p$ denote the field of order p. If X is a group, space or orbifold, let $d_p(X)$ be the dimension of $H_1(X;\mathbf{F}_p)$. With this we make the following definition that will be useful in what follows.

Definition: *Let X be a group, space or orbifold and let p be a prime. Then a collection $\{X_i\}$ of finite index subgroups or finite-sheeted covers of X with index or degree $[X : X_i]$ is said to have linear growth of mod p homology if*

$$\inf_i \frac{\mathrm{d_p(X_i)}}{[\mathrm{X : X_i}]} > 0.$$

Much of the discussion in §4 is motivated by the extra structure that having non-empty singular locus provides. An indication of this is the following result proved in [26] (see also [28] in the case of $p = 2$ for a proof that does not use the Golod-Shafarevich inequality).

Theorem 2.2. *Let Q be a compact orientable 3-orbifold with non-empty singular locus and a finite-volume hyperbolic structure. Let p be a prime that divides the order of an element of $\pi_1^{\mathrm{orb}}(Q)$. Then Q has a tower of finite-sheeted covers $\{Q_i\}$ that has linear growth of mod p homology.*

3 A geometric characterization of arithmeticity

A fundamental dichotomy of Margulis [40] (which holds much more generally) asserts that a Kleinian group Γ of finite co-volume is arithmetic if and only if the *commensurator*

$$\mathrm{Comm}(\Gamma) = \{\mathrm{x} \in \mathrm{PSL}(2, \mathbf{C}) \ : \ \mathrm{x}\Gamma\mathrm{x}^{-1} \text{ is commensurable with } \Gamma\}$$

is dense in $\mathrm{PSL}(2, \mathbf{C})$. Moreover, when Γ is non-arithmetic the commensurator is also a Kleinian group of finite co-volume, which is the unique maximal element in the commensurability class of Γ. In this section we interpret this geometrically. In particular we will prove:

Theorem 3.1. *Let $M = \mathbf{H}^3/\Gamma$ be an orientable finite volume hyperbolic 3-manifold. Then M is arithmetic if and only if the following condition holds:*

Let $\gamma \subset M$ be a closed geodesic. Then there exists a finite sheeted cover $M_\gamma \to M$ such that M_γ admits an orientation-preserving involution τ such that the fixed point set of τ image of γ.

Proof: That this condition is satisfied when Γ is arithmetic is proved in [12] (and implicit in [28]). We give the proof for completeness. We can assume without loss of generality that Γ is derived from a quaternion algebra B/k. Let $a \in \Gamma$ be a hyperbolic element whose axis A_a projects to γ. Let $b \in \Gamma$ be chosen so that its axis A_b is disjoint from A_a. Now the Lie product $ab - ba$ defines an involution $\tau_{a,b}$ for which the axis of rotation is the perpendicular bisector of A_a and A_b in $\mathbf{H}^3$. Denote this geodesic by δ. As shown in [28] (see Proposition 2.4), there is an order $\mathcal{O}$ of B for which $\tau_{a,b}$ lies in the image in $\mathrm{PSL}(2, \mathbf{C})$ of the normalizer of $\mathcal{O}$ in B. This is an arithmetic Kleinian group commensurable with Γ (see [28] or [39] Chapter 6). Hence there is a hyperbolic element $g \in \Gamma$ whose axis A_g is the geodesic δ.

It follows that A_a is now the perpendicular bisector of the axes A_g and aA_g. Repeating the argument of the previous paragraph provides an involution fixing A_a (namely arising from the Lie product of the elements g and aga^{-1}) and lies in an arithmetic Kleinian group Δ commensurable with Γ. To complete the proof, take the core of $\Gamma \cap \Delta$ in Δ, and let M_γ be the corresponding cover of M

For the converse, assume the manifold $M = \mathbf{H}^3/\Gamma$ is non-arithmetic and satisfies the condition. Then by the result of Margulis mentioned above, $\mathrm{Comm}(\Gamma)$ is the unique maximal element in the commensurability class of Γ. We can assume without loss of generality that $\Gamma = \mathrm{Comm}(\Gamma)$ (note that we now allow the existence of elements of finite order $\Gamma = \mathrm{Comm}(\Gamma)$).

Let $\{\delta_n\}$ be a collection of hyperbolic elements in Γ that lie in distinct cyclic subgroups of Γ up to conjugacy in Γ. We are assuming that the condition holds, and so for each n we can construct an involution τ_n that normalizes a finite index subgroup of Γ and which has axis of rotation in $\mathbf{H}^3$ the axis of δ_n. Let Δ_n be the subgroups constructed above which are normalized by τ_n. Since Δ_n is commensurable with Γ and Γ is the unique maximal element it follows that $\tau_n \in \Gamma$ for all n.

Now Γ contains only a finite number of cyclic groups of order 2 up to conjugacy. Hence infinitely many of these are conjugate in Γ, and in particular there is a Γ conjugacy that takes the axis of τ_n to that of τ_1 (say). Hence there is a Γ conjugacy that takes the axis of γ_n to that of a fixed γ_1. However, this contradicts how these elements were chosen. □

A consequence of Theorem 3.1 is that every arithmetic Kleinian group is commensurable with one containing an element of order 2. Indeed more is true, the following is shown in [28].

Theorem 3.2. *Let Γ be an arithmetic Kleinian group. Then Γ is commensurable with a Kleinian group Γ_0 such that Γ_0 contains a group isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.*

Remarks:(1) The $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ constructed in Theorem 3.2 is related to a Hilbert Symbol for the invariant quaternion algebra. More precisely, if $\left(\frac{a,b}{k}\right)$ is a Hilbert Symbol, then notice that the projection of the group generated by i and j to $\mathrm{PGL}(2,\mathbf{C}) \cong \mathrm{PSL}(2,\mathbf{C})$ is simply a copy of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

(2) In §7 we mention (see the proof of Theorem 7.9) another geometric characterization of $\mathrm{Comm}(\Gamma)$. In this case, it is the axes themselves that are used rather than the involutions.

4 Towards Conjectures 1.1–1.4 for arithmetic hyperbolic 3-manifolds

Conjectures 1.1, 1.2, 1.3 and 1.4 still seem rather intractible for an arbitrary closed hyperbolic 3-manifold. However, recent evidence points to the situation being somewhat better for those closed hyperbolic 3-manifolds commensurable with a hyperbolic 3-orbifold with non-empty singular locus. This is evident in recent work of M. Lackenby, [26] and [27] as well as in [28]. A philosophical reason that this case may be more amenable to study is that it is known from [11] that if M is a finite volume non-compact hyperbolic 3-manifold, then $\pi_1(M)$ is large, so the strongest of the conjectures discussed in §1 holds in this setting. Having non-trivial singular locus seems to be an aid that helps replace the existence of the cusp (which is the crucial thing in the methods [11]) in the cusped setting. As shown in §3, arithmetic Kleinian groups are always commensurable with groups containing non-trivial elements of finite order, and so fit into this picture. On the other hand, "most" closed hyperbolic 3-manifolds are never commensurable with an orbifold as above.

4.1

Here we sketch the proof of the following theorem.

Theorem 4.1. *Conjecture 1.2 implies Conjecture 1.4 for arithmetic hyperbolic 3-manifolds.*

This theorem will follow immediately from Theorem 3.2 and the next two results proved in [12] and [28] respectively. We can assume the manifold is closed by [11].

Theorem 4.2. *Conjecture 1.2 implies Conjecture 1.3 for closed arithmetic hyperbolic 3-manifolds.*

Theorem 4.3. *Let $Q = \mathbf{H}^3/\Gamma$ be a 3-orbifold (with possibly empty singular locus) commensurable with a closed orientable hyperbolic 3-orbifold that contains $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ in its orbifold fundamental group. Suppose that $\mathrm{vb}_1(Q) \geq 4$. Then Γ is large.*

We will now sketch some of the ideas in the proofs of these results.

Sketch proof of Theorem 4.2:

Let M be an arithmetic hyperbolic 3-manifold which is assumed to have positive first Betti number.

An important ingredient in the proof of Theorem 4.2 is the following general geometric result proved in [12].

Theorem 4.4. *Let N be a closed hyperbolic 3-manifold. Then there is a closed geodesic η with the property that it has a non-right angle transverse intersection with every least area surface in N.*

Given the geodesic $\eta \subset M$ provided by the above theorem, Theorem 3.1 provides a finite cover $\widetilde{M}$ of M such that $\widetilde{M}$ admits an orientation-preserving involution τ such that the fixed point set of τ contains a component of the preimage of η. The key point now is to show that $b_1(\widetilde{M}) > b_1(M)$.

Suppose to the contrary they have the same rank, so that $p_* : H_2(\widetilde{M}) \longrightarrow H_2(M)$ is a rational isomorphism. Pick a connected embedded surface F, whose homology class represents an eigenvalue ± 1 for the action of τ_* on $H_2(\widetilde{M})$. Consider the class $p_*[F] \in H_2(M)$, this might not be primitive, so take a least area embedded surface G in M representing the primitive class. Hence we may write $p_*[F] = a[G]$ for some integer a. The following two lemmas are proved in [12].

Lemma 4.5. *Let $p : \widetilde{M} \longrightarrow M$ be a finite sheeted covering and suppose that $H_2(\widetilde{M})$ has the same rank as $H_2(M)$.*

Fix a connected embedded surface F in M representing some nonzero class in $H_2(M)$ and let $\widetilde{F_1}, \ldots, \widetilde{F_k}$ be the components of $p^{-1}(F)$.

Then for every i, j, $[\widetilde{F_i}] = \pm[\widetilde{F_j}]$ in $H_2(\widetilde{M})$.

Lemma 4.6. *In the notation of Lemma 4.5, suppose in addition that F is least area for $[F]$.*

Then every $\widetilde{F_i}$ has the same area and this is least area for the class $[\widetilde{F_i}]$.

Given these lemmas, the sketch of the proof is finished as follows. First, it can be argued that the surface G is connected (otherwise $\pi_1(M)$ can be shown to be large). Now $p^{-1}(G)$ consists of components each of which is an embedded surface and therefore a primitive class in $H_2(\widetilde{M})$. Furthermore, since p_* is a rational isomorphism, $[F] = \pm[G^*]$, where G^* is any choice of a component of $p^{-1}(G)$. It follows we have that $\tau_*[G^*] = \pm[G^*]$. Moreover, by Lemma 4.6, any such G^* is a least area surface in the homology class $[G^*] = \pm[F]$.

By choice of η, there is at least one component of the preimage of G which has a nonright angle transverse intersection with $\tilde{\eta}$. Make this choice for G^*. However, this surface can be used to violate a result of Hass on intersections of least area surfaces [21] to obtain a contradiction. Briefly, the surfaces G^* and $\tau(G^*)$ are homologous up to orientation, least area, yet they meet without coinciding for angle reasons using the involution. Repeated application of this increases the first Betti number without bound. □

Remarks:(1) An interesting feature of this argument is that although it uses arithmetic in an essential way, it is geometric, in that arithmeticity is used to produce involutions.

(2) Theorem 4.2 was proved in the case of *congruence* arithmetic hyperbolic 3-manifolds by Borel in [2]. This method of proof was recently generalized to give different proofs of Theorem 4.2 in [1] and [53].

Sketch proof of Theorem 4.3:

Key to this proof is work of Lackenby, [25] and [26]. In particular the following result of Lackenby gives a method of proving largeness of an arbitrary finitely presented group (which is a consequence of a stronger result in [25]).

Proposition 4.7. *Let G be a finitely presented group, and let $\phi : G \to \mathbf{Z}$ be a surjective homomorphism. Let $G_i = \phi^{-1}(i\mathbf{Z})$, and suppose that, for some prime p, $\{G_i\}$ has linear growth of mod p homology. Then G is large.*

The crucial hypothesis in Proposition 4.7 is the linear growth of mod p homology. In the context of orbifolds, as noted in §2, Theorem 2.2 provides a tower with linear growth in mod p homology for some prime p. However, Proposition 4.7 uses a very specific tower. This can also be established, for in [28] it is shown that:

Proposition 4.8. *Let Q be a compact orientable 3-orbifold, and let C be a component of* $\mathrm{sing}_p^0(Q)$ *for some prime p. Let $p_i\colon |Q_i| \to |Q|$ $(i \in \mathbf{N})$ be distinct finite sheeted covering spaces of $|Q|$ such that the restriction of p_i to each component of $p_i^{-1}(C)$ is a homeomorphism onto C. Let Q_i be the corresponding covering spaces of Q. Then $\{Q_i\}$ has linear growth of mod p homology.*

Proof. A standard 3-manifold argument (see [28] Proposition 3.1) shows that

$$d_p(Q_i) \geq |\mathrm{sing}_p^0(Q_i)|.$$

By assumption $p_i^{-1}(C)$ is a homeomorphism onto C and so it follows that

$$|\mathrm{sing}_p^0(Q_i)| \geq [Q_i : Q].$$

Hence $\inf_i d_p(Q_i)/[Q : Q_i] \geq 1$ as required. □

It quickly follows from Proposition 4.7 and Proposition 4.8 that we have.

Corollary 4.9. *Suppose Q be a compact orientable 3-orbifold with non-empty singular locus containing a circle component. Assume that that $b_1(Q) \geq 2$, then $\pi_1^{\mathrm{orb}}(Q)$ is large.*

The proof of Theorem 4.3 is now completed as follows. By hypothesis, Q has a finite cover Q' such that $b_1(Q') \geq 4$. Let Q'' be the hyperbolic orbifold, commensurable with Q, containing $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ in its fundamental group. Now, Q' and Q'' are commensurable, and hence they have a common cover Q''', say. Since Q''' is hyperbolic, it has a manifold cover M. We may assume that M regularly covers O''. Now, b_1 does not decrease under finite covers, and so $b_1(M) \geq 4$. Since $M \to Q''$ is a regular cover, it has a group of covering transformations G

Now, $\pi_1^{\mathrm{orb}}(Q'')$ contains $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, and hence some singular point of Q'' has local group that contains $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. It follows that G contains $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Let h_1 and h_2 be the commuting covering transformations of M corresponding to the generators of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. These are involutions. Let h_3 be the composition of h_1 and h_2, which also is an involution. For $i = 1, 2$ and 3, let Q_i be the quotient M/h_i. Since h_i has non-empty fixed point set, $\mathrm{sing}(O_i)$ is a non-empty collection of simple closed curves with order 2.

It is shown in [28] that for at least one $i \in \{1,2,3\}$, $b_1(Q_i) \geq 2$. So, Corollary 4.9 now shows $\pi_1^{\mathrm{orb}}(Q_i)$ is large for some i, and hence so is Γ. □

4.2

Here we summarize what us known to us at this time about when Conjecture 1.4 holds for arithmetic hyperbolic 3-manifolds (using Theorem 4.2). We only consider closed manifolds, since as mentioned above, the finite volume non-compact case is completely understood by [11].

Throughout the discussion $M = \mathbf{H}^3/\Gamma$ is an arithmetic hyperbolic 3-manifold with invariant quaternion algebra B/k. Note that any proper subfield of k is totally real.

k has a real subfield of index 2: We distinguish two cases; when there are totally geodesic surfaces and when there are not.

In general the existence of a totally geodesic surface in a closed hyperbolic 3-manifold allows one to deduce that Conjecture 1.4 holds (see [32] and [37]). In the case when M is an arithmetic hyperbolic 3-manifold, the existence of a totally geodesic surface forces conditions on B/k. The number field k satisfies $[k : k \cap \mathbf{R}] = 2$ and B has a Hilbert symbol of the form $\left(\frac{a,b}{k}\right)$, where a and b are non-zero elements of $k \cap \mathbf{R}$ (see [39] Chapter 9.5).

If one considers the case when only $[k : k \cap \mathbf{R}] = 2$, forgetting the condition on B, it is also known that $\mathrm{vb}_1(\Gamma) > 0$. This is proved by various authors ([24], [31] and [38]).

In some of the cases when $[k : k \cap \mathbf{R}] = 2$ but no totally geodesic surface is present, there is also geometric/topological proofs that $\mathrm{vb}_1(\Gamma) > 0$. This happens when there is an orientation-reversing involution on some manifold in the commensurability class of $\mathbf{H}^3/\Gamma$.

Other conditions on subfields: The paper [24] can be applied in some other cases as we now describe.

Suppose $\ell \subset k$ is a proper subfield and there is a tower of intermediate fields:

$$\ell = \ell_0 \subset \ell_1 \subset \ldots \subset \ell_{m-1} \subset \ell_m = k$$

where each extension ℓ_i/ℓ_{i-1} is either a cyclic extension of prime degree or a non-Galois cubic extension. Then $\mathrm{vb}_1(\Gamma) > 0$.

In a similar spirit it is shown in [45] that if k is contained inside a solvable Galois extension of ℓ, then $\mathrm{vb}_1(\Gamma) > 0$. In particular a corollary of these results is the following.

Corollary 4.10. *Suppose $[k : \mathbf{Q}] \leq 4$, then Γ is large.*

Proof: Any such extension has solvable Galois group over $\mathbf{Q}$. □

Clozel's result: Perhaps the most general method known at present for proving Conjecture 1.2 for arithmetic hyperbolic 3-manifolds is [9]. In particular this applies to every field k with one complex place, but there is a condition on the quaternion algebra. We state Clozel's result here in the notation of this paper. Some topological consequences are discussed below.

Theorem 4.11. *Let Γ be an arithmetic Kleinian group as above. Assume that for every place $\nu \in \mathrm{Ram}_f\, B$, k_ν contains no quadratic extension of $\mathbf{Q}_p$ where p is a rational prime and $\nu | p$. Then $\mathrm{vb}_1(\Gamma) > 0$.*

4.3

There are situations when Theorem 4.11 can be used together with the existence of certain finite subgroups in arithmetic Kleinian groups to prove Conjecture 1.4. For example, we have the following from [28].

Theorem 4.12. *Let Γ be an arithmetic Kleinian group commensurable with a Kleinian group containing A_4, S_4 or A_5 or a finite dihedral group that is derived from a quaternion algebra. Then Γ is large.*

The proof of this follows from Theorem 4.11 on noticing that the existence of these finite subgroups places conditions on the invariant trace-field and quaternion algebras of the arithmetic Kleinian groups (indeed this is true arithmetic or not). For example in the case of A_4, S_4 or A_5 the invariant quaternion algebra is isomorphic to $\left(\frac{-1,-1}{k}\right)$, and this can be shown to be unramified at all finite places.

The conditions forced on the algebra are reminiscent of how the existence of a totally geodesic surface in arithmetic hyperbolic 3-manifold places conditions on the invariant trace-field and quaternion algebra (recall §4.2)

Remark: The discussions above, together with the discussion in §4.2, motivate the following. This provides a uniform setting for how the existence of a totally geodesic surface or a finite group of the form A_4, S_4 or A_5 has on the invariant quaternion algebra, and is perhaps more approachable than the general problem.

Challenge: *Let k be a field with one complex place, and ℓ a (necessarily) totally real subfield. Suppose Γ is an arithmetic Kleinian group with invariant trace-field k and invariant quaternion algebra B that satisfies:*

$$B \cong A \otimes_\ell k$$

for some quaternion algebra A over ℓ (ie. B lies in the image of the natural map of Brauer groups $\mathrm{Br}(\ell) \to \mathrm{Br}(\mathrm{k})$*). Show that Γ is large.*

Note also, that removing the hypothesis in Theorem 4.12 that the dihedral subgroup is derived from a quaternion algebra would prove that all arithmetic Kleinian groups are large. It would be interesting to give a geometric proof of Theorem 4.12. Thus we pose as a warm-up challenge to the general situation, and which should be more amenable.

Conjecture 4.13. *Let Γ be a Kleinian group of finite co-volume containing A_4, S_4 or A_5. Then Γ is large.*

Another interesting topological application of Clozel's result is that it applies to show that if Σ is an arithmetic integral homology 3-sphere then $\mathrm{vb}_1(\Sigma) > 0$, and hence $\pi_1(\Sigma)$ is large. For in this case, the invariant trace-field of Σ has even degree over $\mathbf{Q}$, and the invariant quaternion algebra of Σ is unramified at all finite places (see for example [39] Theorem 6.4.3). So from the perspective of Conjectures 1.1–1.4, in the arithmetic setting, integral homology spheres are easier to handle than rational homology 3-spheres.

We close this subsection with two examples on the limitation of current techniques regarding the arithmetic methods mentioned here.

Example 1: This is taken from [28], and is a commensurability class of arithmetic 3-orbifolds for which none of the methods discussed applies to provide a cover with positive first Betti number. We do this in the first possible degree, namely 5 (see Corollary 4.10).

Let $p(x) = x^5 - x^3 - 2x^2 + 1$. Then p has three real roots and one pair of complex conjugate roots. Let t be a complex root and let $k = \mathbf{Q}(t)$. Now k has one complex place and its Galois group is S_5. There is a unique prime $\mathcal{P}$ of norm 11^2 in k. It follows that $k_{\mathcal{P}}$ is a quadratic extension of $\mathbf{Q}_{11}$. Take B ramified at the real embeddings and the prime $\mathcal{P}$. Then it is unknown whether any arithmetic Kleinian group arising from B has a cover with positive first Betti number.

Briefly, if Γ is any group in the commensurability class, then since k has odd degree, the discussion in §4.2 shows that there are no non-elementary Fuchsian subgroups. The result of Clozel does not apply by the condition on $\mathcal{P}$, and none of the other work discussed above applies since $[k : \mathbf{Q}] = 5$ and the Galois group is S_5. For this final part, note that if k were contained in a solvable extension L of $\mathbf{Q}$, then the Galois closure K of k would be a subfield of L. However, this implies that the Galois group $\mathrm{Gal}(K/k)$ is a quotient of $\mathrm{Gal}(L/k)$, which is solvable, and this is a contradiction.

One can also adjoin other primes to $\mathrm{Ram_f B}$ that constructs quaternion algebras for which $\Gamma^1_{\mathcal{O}}$ are torsion free.

In the next example, we construct a family of arithmetic orbifolds $Q_p = \mathbf{H}^3/\Gamma_p$ (p a prime congruent to 1 mod 4) where Γ_p contains elements of order p and for which $\mathrm{vb}_1(\Gamma_p)$ is unknown to be positive. Note that the invariant trace-fields of the orbifolds Q_p have degree going to infinity (compare with Corollary 5.6).

Example 2: Let p be a prime congruent to 1 mod 4, and let $\ell_p = \mathbf{Q}(\cos \pi/p)$. Note that ℓ_p contains the quadratic extension $\mathbf{Q}(\sqrt{p})$.

Claim: *(1) There exists a number field k_p with one complex place such that $[k_p : \ell_p] = 5$ and* $\mathrm{Gal}(k_p/\ell_p) \cong S_5$.

(2) There exist infinitely many rational primes q such that if ν is a place of k_p with $\nu|q$ then k_{p_ν} contains a quadratic extension of $\mathbf{Q}_q$.

(3) Let ζ_p denote a primitive p-th root of unity. Then, the primes ν constructed in (2) *do not split in $k_p(\zeta_p)/k_p$.*

Given these three conditions we can construct orbifolds as follows. Let p and k_p be as in (1) above, let B_p/k_p be a quaternion algebra ramified at all real places, and let ν be a place of k_p as in (2). Since k_p has even degree we insist that B_p is ramified at another place ν' which we also assume is as in (2). Since ν and ν' do not split in $k_p(\zeta_p)/k_p$, it follows that $k_p(\zeta_p)$ embeds in B_p (see [39] 7.3.3 for example) and indeed, for any maximal order $\mathcal{O}_p \subset B_p$ the group $\Gamma^1_{\mathcal{O}_p}$ contains an element of order p (see for example [39] Theorem 12.5.4). Let $Q_p = \mathbf{H}^3/\Gamma^1_{\mathcal{O}_p}$.

As in Example 1 above, it is unknown using the methods described in this paper that $\mathrm{vb}_1(\mathrm{Q_p}) > 0$. Briefly, although $[k_p : \mathbf{Q}]$ has even degree there are no totally geodesic surfaces, since ℓ_p is the maximal real subfield of k_p which is of index 5. For if there were totally geodesic surfaces, then k_p would contain a totally real subfield of index 2 (see §4.3) which would contain ℓ_p and this is false. As in Example 1, condition (1) above ensures that k_p is not contained in any solvable extension of ℓ_p, and the condition on the prime ν ensures that Clozel's condition fails.

Proof of Claim:

To prove (1), note that given the existence of a field k_p with $[k_p : \ell_p] = 5$ and with one complex place, then $\mathrm{Gal}(k_p/\ell_p) \cong S_5$. This follows for example using the arguments of §3 of [8]. The existence of such a field can be argued as follows. We seek to build a monic irreducible polynomial $p(x) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$ ($a_i \in \ell_p$), such that $p(x)$ has 1 pair of complex conjugate roots and 3 real roots, and furthermore, for every Galois embedding $\sigma : \ell_p \to \mathbf{R}$ the polynomial $p^\sigma(x)$ obtained by applying σ to the coefficients of p has only real roots. Now ℓ_p embeds in $\mathbf{R}^{\frac{p-1}{2}}$ as a dense subset, so that ℓ_p^5 embeds in $\mathbf{R}^{\frac{5(p-1)}{2}}$ as a dense subset. Hence a polynomial p as above gives a vector in $\mathbf{R}^{\frac{5(p-1)}{2}}$ and since the condition on the roots is an open condition it follows that we can find a a polynomial satisfying the condition on the roots. Moreover, by a similar argument, the polynomial can be assumed irreducible.

To establish (2) and (3) we argue as follows. The Galois group of $\mathbf{Q}(\zeta_p)/\mathbf{Q}$ is cyclic of order $p-1$, so the Cebotarev Density Theorem gives infinitely many rational primes q such that q is totally inert in $\mathbf{Q}(\zeta_p)$ (i.e. the inertial degree is precisely $p-1$). As remarked above, ℓ_p contains $\mathbf{Q}(\sqrt{p})$, and so for any prime q as above, if ω is the $\mathbf{Q}(\sqrt{p})$-prime lying over q, and ν is a k_p-prime over q, then k_ν contains $\mathbf{Q}(\sqrt{p})_\omega$. By properties of the inertial degree, $[\mathbf{Q}(\sqrt{p})_\omega : \mathbf{Q}_q] = 2$. This proves (2).

Now (3) also follows, since choosing a prime q as in (2), and ν a k_p-prime lying over q, then ν cannot split in $k_p(\zeta_p)$. For if so, then it follows that for the unique (from the previous paragraph) ℓ_p-prime μ with $\nu|\mu$, that μ must split in $\mathbf{Q}(\zeta_p)$, and this is false from above.

4.4

The methods of [9], [31], [24], and [31] are those of automorphic forms and automorphic representations (see also [48] for a survey of some of this). In particular, these methods involve applying some type of "functoriality"; eg Base Change, or the Jacquet-Langlands correspondence (see more on this below). Perhaps the biggest challenge from the geometric or topological perspective is to understand geometric features of incompressible surfaces represented by duals to the cohomology classes produced by these methods. In some cases, totally geodesic surfaces can be recognized in the application of Base Change in quadratic extensison, but this is not always the case.

To complete the discussion about the connection between the topology of finite covers of arithmetic hyperbolic 3-manifolds and the theory of automorphic forms, we should mention that Conjecture 1.2 is a consequence of various conjectures in automorphic forms and the Langlands Program. One such conjecture is mentioned [9], here we briefly discuss another that relates to elliptic curves. Our intention here is not to give a detailed exposition of all the details, but simply to indicate to a topologist, the gist of the connection, and the powerful implications that the theory of automorphic forms has in the setting of arithmetic groups. Because we will only give a brief discussion, we refer the reader to [17] and [23] for more details regarding the theory of automorphic representations.

We emphasize that what we discuss below is well-known to the experts in the theory of automorphic representations, our purpose is merely to advertise the connections.

We begin with Base Change for GL(2). Let k be a number field, $\mathbf{A}_k$ the adeles of k and denote by $\mathcal{A}(k)$ the set of equivalence classes of automorphic representations of $\mathrm{GL}(2,\mathbf{A}_k)$. Let $\mathcal{A}_c$ denote the set of equivalence classes of cuspidal automorphic representations of $\mathrm{GL}(2,\mathbf{A}_k)$.

Suppose that K/k is a finite extension of number fields, then part of the Langlands Program predicts that if $[\pi] \in \mathcal{A}(k)$ then there is a canonically associated $[\pi'] \in \mathcal{A}(K)$, called the *the Base Change lift of* π. Furthermore if π is cuspidal, then "typically", π' is cuspidal. The existence of this Base Change lift has been established in very few cases, and are the source of the non-vanishing of vb_1 in the case of a field with one complex place having certain proper subfields as discussed in §4.3.

For our purposes, we make the following assumption:

Assumption BC: *Let k be a field with one complex place, then Base Change lifts exist from $\mathcal{A}(\mathbf{Q})$ to $\mathcal{A}(k)$.*

Given this we sketch the proof of:

Theorem 4.14. *Let M be an arithmetic hyperbolic 3-manifold. Under Assumption BC,* $\mathrm{vb}_1(\mathrm{M}) > 0$, *and hence $\pi_1(M)$ is large.*

Sketch Proof: We can assume that $M = \mathbf{H}^3/\Gamma$ is closed. Denote the invariant trace-field of Γ by k and B respectively. Assume that $\mathrm{Ram}_f\mathrm{B} = \{\nu_1, \ldots, \nu_s\}$. To show that $\mathrm{vb}_1(\mathrm{M}) > 0$, we shall construct a congruence subgroup of some group $\Gamma^1_{\mathcal{O}}$ in the commensurability class of Γ and for which the first Betti number is non-zero.

The connection with the cohomology is provided by Matsushima's theorem [41] which says in this setting, that if $X = \mathbf{H}^3/\Delta$ is a closed orientable hyperbolic 3-manifold then $b_1(X)$ can be computed as the multiplicity of a certain irreducible unitary representation of $\mathrm{SL}(2,\mathbf{C})$ occuring in the decomposition of $L^2(\mathrm{SL}(2,\mathbf{C})/\Delta)$ under the right regular representation. We shall denote this by π_0. The goal is to show that the multiplicity of π_0 can be arranged to be non-zero by passage to a congruence subgroup. This is achieved using Assumption BC together with the Jacquet-Langlands Correspondence, the main result of which we state below in a somewhat abbreviated form (see [17] and [23]). To state this, we denote by $B_{\mathbf{A}_k}$ the adeles of the division algebra B (see [39] Chapter 7 for example) and $B^*_{\mathbf{A}_k}$ the invertible adeles. Note that irreducible unitary representations of these adelic groups have the form $\otimes\pi_v$ (the completed restricted tensor product), where π_v is an irreducible unitary representation of the local group at the place ν.

Theorem 4.15. *There is a correspondence which associates to each irreducible unitary representation π' of $B_{\mathbf{A}_k}$ (which is not 1-dimensional) an irreducible unitary representation π of* $\mathrm{GL}(2, \mathbf{A}_k)$. *Indeed, if $\pi = \otimes\pi_v$ the correspondence $\pi' \longrightarrow \pi$ is one-to-one onto the collection of cusp forms of* $\mathrm{GL}(2, \mathbf{A}_k)$ *with π_ν a discrete series representation for each $\nu \in$* Ram B.

The goal is to produce a cuspidal automorphic representation π of $\mathrm{GL}(2, \mathbf{A}_k)$ such that at the infinite places this representation is *of the right type*; i.e has the form $\pi_0 \otimes \mathbf{C} \otimes \ldots \otimes \mathbf{C}$ (the $\mathbf{C}$ factors corresponding to the trivial representation of the invertible elements in the Hamiltonian quaternions). To this end, let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ denote the primes associated to the finite places in $\mathrm{Ram}_f\mathrm{B}$, and assume that $\mathrm{N}\mathcal{P}_j = q_j = p_j^{t_j}$.

Denoting by $N = \prod p_j$, let E be an elliptic curve defined over $\mathbf{Q}$ with j-invariant $1/N$ (see [50] for details about elliptic curves). That such an elliptic curve exists can be seen explicitly from the following construction on setting $j = 1/N$.

Let $j \in \mathbf{Q}$, $A = 27j/(1728 - j)$ and $B = 2A$. Then $y^2 = x^3 + Ax + B$ is an elliptic curve defined over $\mathbf{Q}$ with j-invariant j. Now $E/\mathbf{Q}$ is modular, and the level can be shown to be NC for some integer C. In particular, modularity implies that E is attached to a cusp form of weight 2 (which is a newform) for $\Gamma_0(NC)$, which in turn provides a cuspidal automorphic representation π with $[\pi] \in \mathcal{A}_c(\mathbf{Q})$. We remark that the identification of this cuspidal automorphic representation from the newform provides a cuspidal automorphic representation which is discrete series at the infinite place of $\mathbf{Q}$ and at those primes dividing NC (see [17]).

Assumption BC provides a Base Change lift π', which is cuspidal in this case because of the j-invariant, which guarantees that the elliptic curve will have potentially multiplicative reduction at primes dividing N over k. It can also be shown that, since at the infinite place of $\mathbf{Q}$, π is discrete series, then at the infinite places of k this representation has the right type. Furthermore, again from the behavior of π, at the places in $\mathrm{Ram}_f\mathrm{B}$, and k-primes lying over primes dividing C, the representation is necessarily discrete series. Theorem 4.15 now provides a cuspidal automorphic representation for which π_0 occurs at the complex place. A more detailed analysis of the correspondence actually shows that this provides a cusp form on a congruence subgroup of some $\Gamma^1_{\mathcal{O}}$ where the level corresponds to k-primes lying over primes dividing C and k-primes lying over the primes p_j. □

4.5

We now discuss some other more general situations where the existence of orbifolds in a commensurability class are helpful. Firstly, there are other situations where the existence of certain 2-dimensional orbifolds can be used to prove Conjecture 1.1, 1.2, 1.3 and 1.4; eg as above when the 2-orbifold is totally geodesic by [32] (see also [38]). Here we indicate another class, which again provides evidence that hyperbolic 3-manifolds commensurable with orbifolds do provide a more reasonable class to work with.

Theorem 4.16. *Let $Q = \mathbf{H}^3/\Gamma$ be a closed hyperbolic 3-orbifold for which $\pi_1(|Q|)$ is infinite. Assume that Q contains an essential 2-orbifold with underlying space the 2-sphere or the 2-torus. Suppose that M is a hyperbolic 3-manifold commensurable with Q, then M is virtually Haken.*

Proof: We can assume that Q is orientable, otherwise, by [34], Q has a finite sheeted cover which is a non-orientable hyperbolic 3-manifold, and it is well known that such manifolds have positive first Betti number (see [22] Chapter 6). Hence $\mathrm{vb}_1(\mathrm{M}) > 0$.

Letting N denote the normal closure of all the elliptic elements of Γ, then $\pi_1(|Q|) = \Gamma/N$. Note that the solution to the Geometrization Conjecture ensures that $\pi_1(|Q|)$ is residually finite. Let $\phi : \Gamma \to \Gamma/N$ be the quotient homomorphism.

We deal first with the case of the sphere. Thus suppose that $S \hookrightarrow Q$ is an embedded essential 2-sphere with cone points. Since the orbifold group F, of S, is generated by elliptic elements it follows that $\phi(F) = 1$.

Now assume that Γ_0 is a torsion free subgroup of finite index in Γ. Then $F_0 = F \cap \Gamma_0$ is a surface group of genus at least 2. Restricting ϕ to F_0 we have that $\phi(F_0) = 1$. Hence, since $\pi_1(|Q|)$ is infinite, and residually finite, we deduce that F is contained in infinitely many subgroups of finite index in Γ, and hence F_0 is contained in infinitely many subgroups of finite index in Γ_0. It is a result of Jaco (see [49] Corollary 2.3) that $\mathbf{H}^3/\Gamma_0$, and therefore M, is virtually Haken.

The argument in the case when S is the 2-torus with cone points is handled in a similar way. However, here note that $\phi(F)$ need not be trivial, but it is a quotient of $\mathbf{Z} \times \mathbf{Z}$. Thus $\phi(F)$ is an Abelian subgroup of $\pi_1(|Q|)$ of rank at most 2. This is either trivial or isomorphic to one of $\mathbf{Z}$, $\mathbf{Z}^2$, $\mathbf{Z}/n\mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ (see [22] Chapter 9). If $\phi(F)$ is trivial we argue as above. Thus assume that $A = \phi(F)$ is one of the non-trival Abelian groups listed above. The case of $A = \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ is ruled out since the existence of such a subgroup forces $|Q|$ to be non-orientable (see [22] Theorem 9.12) contrary to the assumption.

Suppose first that $\pi_1(|Q|)$ is irreducible. Note that A has infinite index in $\pi_1(|Q|)$, and since $\pi_1(|Q|)$ is infinite, it follows from [22] Chapter 9 that A is isomorphic to $\mathbf{Z}$ or $\mathbf{Z}^2$. In these cases [20] shows that A is a separable subgroup of $\pi_1(|Q|)$ (assuming the Geometrization conjecture). We now argue as above using the result of Jaco.

If $|Q|$ is not irreducible, then either $|Q| = S^2 \times S^1$ or $|Q|$ decomposes as a non-trivial connect sum. In the former case we obtain a surjection onto $\mathbf{Z}$, and we may deduce that M is virtually Haken from this. In the latter case, the argument above will deal with all cases other than $A = \mathbf{Z}/n\mathbf{Z}$. However, in this case, [22] Theorem 9.8 gives a decomposition $\pi_1(|Q|) = \pi(X) * G$, where X is a closed 3-manifold, $|G| < \infty$ and A is conjugate into G. Now $\pi_1(|Q|)$ is infinite, so $\pi_1(X) \neq 1$. Hence Γ surjects a non-trivial free product, and so this determines a non-trivial action of Γ on a tree. It follows that $\pi_1(M)$ must contain a finite index subgroup that admits a non-trivial action on a tree, and so M is virtually Haken. □

We remark that it is shown in [28] that any arithmetic hyperbolic 3-orbifold always admits a finite cover which is also an orbifold with non-trivial singular locus and satisfies the condition on the fundamental group of the underlying space given in Theorem 4.16.

A well-known approach to proving Conjecture 1.1 is to show that a closed hyperbolic 3-manifold has a finite sheeted cover for which the character variety (of (P)SL(2, $\mathbf{C}$) representations) contains a positive dimensional component. For a closed hyperbolic 3-manifold even constructing "new" characters in a finite sheeted cover seems hard. We now address this issue exploiting orbifolds.

Note that if $M = \mathbf{H}^3/\Gamma$ with $k = \mathbf{Q}(\mathrm{tr}\,\Gamma)$, then any Galois embedding σ of k defines a character χ_σ obtained by applying σ to the character of a faithful discrete representation.

Definition: *Let $M = \mathbf{H}^3/\Gamma$ be a closed orientable hyperbolic 3-manifold. Say that M has extra characters if there exists an infinite irreducible* (P)SL(2, $\mathbf{C}$) *representation of Γ that is different from χ_σ for any Galois embedding of* $\mathbf{Q}(\mathrm{tr}\,\Gamma)$.

In addition we shall say that M virtually has extra characters if Γ contains a finite index subgroup Δ such that $\mathbf{H}^3/\Delta$ has extra characters.

Theorem 4.17. *Let $M = \mathbf{H}^3/\Gamma$ be a closed orientable hyperbolic 3-manifold commensurable with an orbifold commensurable that contains $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ in its orbifold fundamental group. Then* *either* $\mathrm{vb}_1(\mathrm{M}) > 0$ *or* M *virtually has extra characters.*

Proof: Theorem 8.1 of [28] shows that, under the hypothesis of Theorem 4.17, M is commensurable with an orbifold Q for which $\pi_1(|Q|)$ is infinite. It follows that Γ has a finite index subgroup Γ_0 such that Γ_0 surjects $\pi_1(X)$ where X is a closed orientable 3-manifold with $\pi_1(X)$ infinite. Assuming the Geometrization Conjecture, we can assume that X is hyperbolic, otherwise it is easily seen that $\mathrm{vb}_1(\mathrm{X}) > 0$ and so $\mathrm{vb}_1(\mathbf{H}^3/\Gamma_0) > 0$.

When X is hyperbolic, the faithful discrete representation of $\pi_1(X)$ provides an extra character (it is different from χ_σ because this representation is not faithful on Γ_0). □

Together with Theorem 4.1 we have.

Corollary 4.18. *Let M be an arithmetic hyperbolic 3-manifold. Then either $\pi_1(M)$ is large or M virtually has extra characters.*

This also gives.

Corollary 4.19. *Let M be an arithmetic hyperbolic 3-manifold. Then M has a finite sheeted cover N for which the character variety of N contains at least one more component of characters of irreducible representations.*

Proof: By Corollary 4.18 either $\pi_1(M)$ is large, so that $\pi_1(M)$ contains a finite index subgroup surjecting a free non-abelian group or there is a finite index subgroup with an extra character. Both cases provide extra components in the $(\mathrm{P})\mathrm{SL}(2, \mathbf{C})$ character variety. □

4.6

Another well-known conjecture in the same spirit as Conjecture 1.1–1.4 is the virtual fibering conjecture of Thurston.

Conjecture 4.20. *Let M be a finite volume hyperbolic 3-manifold. Then M has a finite sheeted cover that fibers over S^1.*

Unlike Conjectures 1.1–1.4 where there is a considerable amount of supporting evidence (even in the arithmetic case), little is known about Conjecture 4.20. For the complements of knots and links there is work of [4], [15], [30], and [57] that provides some evidence, but for closed manifolds almost nothing is known. However, in the arithmetic setting, many well-known examples are known to be virtually fibered (see [5] and [47]). Arithmeticity is used in that the pair $(k\Gamma, A\Gamma)$ is a complete invariant of the commensurability class, and so one can construct bundles (for example by surgery on fibered knot complements) and just compare arithmetic data (see [5] and [47] for more details).

5 The 2-dimensional situation

In this section we address the following question. By saying a *Fuchsian group* Γ *is of genus 0*, we mean that the underlying space of the orbifold $\mathbf{H}^2/\Gamma$ has genus 0.

Question 5.1. *Are there infinitely many commensurability classes of arithmetic Fuchsian groups of genus 0?*

The motivation for this question is the situation in dimension 3 (discussed in §6). Note that if Γ is a cocompact Fuchsian group of genus 0, then Γ^{ab} is finite, and so the orbifold $\mathbf{H}^2/\Gamma$ is a "rational homology 2-sphere". Conjectures 1.2–1.4 predict that on passage to certain finite covers, arithmetic hyperbolic 3-manifolds stop being rational homology 3-spheres. Thus it seems interesting to discuss how "widely distributed" arithmetic rational homology 3-spheres are. This is address in §6 when we discuss an analogue of Question 5.1 for arithmetic rational homology 3-spheres.

The answer to Question 5.1 is provided by the following result from [36].

Theorem 5.2. *The answer to Question 5.1 is no.*

In contrast, it is easy to construct infinitely many cocompact arithmetic Fuchsian groups of genus 0 (see [36] Corollary 4.10). Before discussing the proof we recall some notation and discuss a classical situation which Theorem 5.2 generalizes.

5.1

Recall that a *congruence subgroup* of $\mathrm{PSL}(2,\mathbf{Z})$ is any subgroup of $\mathrm{PSL}(2,\mathbf{Z})$ that contains a principal congruence subgroup $\Gamma(n)$ of level n for some n where

$$\Gamma(n) = \mathrm{P}\{A \in \mathrm{SL}(2,\mathbf{Z}) : \mathrm{A} \equiv \mathrm{I} \bmod \mathrm{n}\}.$$

More generally, a non-cocompact arithmetic Fuchsian group Γ is congruence if some conjugate of Γ in $\mathrm{PGL}(2,\mathbf{R})$ contains some $\Gamma(n)$. Any non-cocompact arithmetic Fuchsian group is conjugate into a maximal group commensurable with $\mathrm{PSL}(2,\mathbf{Z})$, and these can be described as follows.

Any Eichler order of $M(2,\mathbf{Q})$ (an interesection of two maximal orders) is conjugate to

$$\mathcal{E}_0(n) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) \mid c \equiv 0 \bmod \mathrm{n}\}$$

and the conjugacy classes of the maximal Fuchsian groups are the groups $\Gamma^+_{\mathcal{E}_0(n)}$ for n square-free (the images in $\mathrm{PSL}(2,\mathbf{R})$ of the normalizers of the Eichler orders $\mathcal{E}_0(n)$). These groups have a more familiar description as the normalizers in $\mathrm{PSL}(2,\mathbf{R})$ of the groups

$$\Gamma_0(n) = \mathrm{P}\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ c \equiv 0 \bmod \mathrm{n}\}.$$

In particular these maximal arithmetic Fuchsian groups contain $\Gamma(n)$, and so are congruence subgroups.

In the context of cusped 2-orbifolds, the analogue of a rational homology 2-sphere is when $r(Q) = 0$ (recall §2.5). Thus before studying the cocompact setting, an easier question is to study non-cocompact arithmetic Fuchsian groups Γ with $r(\Gamma) = 0$ (there is after all only one commensurability class).

Now it is well known that $r(\Gamma_0(n)) = 0$ if and only if $n \in \{1,\ldots,10,12,13,16,18,25\}$, and the size of $r(\Gamma)$ carries important information regarding the theory of elliptic curves over $\mathbf{Q}$. Furthermore it is known when $r(\Gamma^+_{\mathcal{E}_0(n)}) = 0$ and that this is also related to interesting number theoretic phenomena; for example it is known that there are only finitely many n such that $r(\Gamma^+_{\mathcal{E}_0(n)}) = 0$, and that these values of n are related to the properties of the Monster simple group. Indeed, Ogg (cf. [43]) observed that that those primes p for which the groups $N(\Gamma_0(p))$ have genus 0, coincides with those primes p dividing the order of the Monster simple group (see [10], [36] and [43] for more details).

On the other hand, it is easy to construct infinitely many subgroups $\{\Gamma_i\}$ of finite index in $\mathrm{PSL}(2,\mathbf{Z})$ for which $r(\Gamma_i) = 0$; e.g. by constructing certain cyclic covers of the the thrice-punctured sphere.

5.2

The description of maximal arithmetic Fuchsian groups and congruence subgroups in terms of Eichler orders can be extended to the cocompact setting which we briefly recall (see [36]).

Let B be a quaternion algebra over a totally real field k and let $\mathcal{O}$ be a maximal order of B. Let I be an integral 2-sided $\mathcal{O}$-ideal in B which means that I is a complete R_k-lattice in B such that

$$\mathcal{O} = \{x \in B \mid xI \subset I\} = \{x \in B \mid Ix \subset I\}.$$

The principal congruence subgroup of $\Gamma^1_{\mathcal{O}}$ of level I, denoted $\Gamma^1_{\mathcal{O}}(I)$ is the image in $\mathrm{PSL}(2,\mathbf{R})$ of

$$\mathcal{O}^1(I) = \{\alpha \in \mathcal{O}^1 : \alpha - 1 \in I\}.$$

Since $\mathcal{O}/I$ is a finite ring, $\mathcal{O}^1(I)$ has finite index in $\mathcal{O}^1$.

Definition: *Let Γ be an arithmetic Fuchsian group. Γ is called a congruence subgroup if there is a quaternion algebra B defined over the totally real field k ramified at all real places except one, a maximal order $\mathcal{O}$ and an ideal I of $\mathcal{O}$ as described above such that Γ contains $\Gamma^1_{\mathcal{O}}(I)$.*

As in the case of the modular group, any maximal arithmetic Fuchsian group is a congruence subgroup (see [36]). In addition to proving Theorem 5.2, the methods also provide a generalization of the results when $r(\Gamma) = 0$ for Γ a maximal (or congruence) arithmetic Fuchsian group (see [36]).

Theorem 5.3. *There are finitely many conjugacy classes of congruence arithmetic Fuchsian groups of genus zero. There are thus finitely many conjugacy classes of maximal arithmetic Fuchsian groups of genus 0.*

5.3

The proof of Theorem 5.2 is completed in the following steps.

1. If a commensurability class contains an arithmetic Fuchsian group Γ of genus zero, then any Fuchsian group containing Γ will also have genus zero, so we can assume that we have an infinite sequence $\{\Gamma_i\}$ of non-conjugate maximal arithmetic Fuchsian groups of genus 0.

Note that $\mathrm{Area}(\mathbf{H}^2/\Gamma_i) \to \infty$, since as noted in §2.3, there are only finitely many conjugacy classes of arithmetic Fuchsian groups of bounded area.

2. We now exploit the following results that give contrasting behavior of the first non-zero eigenvalue of the Laplacian. The first is an important result about the first non-zero eigenvalue of the Laplacian for congruence arithmetic Fuchsian groups (see [18], [23] and [55]).

Theorem 5.4. *If Γ is a congruence arithmetic Fuchsian group then*

$$\lambda_1(\Gamma) \geq \frac{3}{16},$$

where $\lambda_1(\Gamma)$ is the first non-zero eigenvalue of the Laplacian of Γ.

Note by the discussion in §5.2, this applies to the maximal arithmetic Fuchsian groups Γ_i. On the other hand, we have the following result of P. Zograf [58].

Theorem 5.5. *Let Γ be a Fuchsian group of finite co-area and let the genus of $\mathbf{H}^2/\Gamma$ be denoted by $g(\Gamma)$. If $\mathrm{Area}(\mathbf{H}^2/\Gamma) \geq 32\pi(g(\Gamma)+1)$, then*

$$\lambda_1(\Gamma) < \frac{8\pi(g(\Gamma)+1)}{\mathrm{Area}(\mathbf{H}^2/\Gamma)}.$$

3. We now have a contradiction, for with $g(\Gamma_i) = 0$, and $\text{Area}(\mathbf{H}^2/\Gamma_i) \to \infty$, Theorem 5.5 implies that $\lambda_1 \to 0$. However this contradicts Theorem 5.4. □

Corollary 5.6. *There is an upper bound to the order of an elliptic element in an arithmetic Fuchsian group of genus 0.*

Proof: Since there are only finitely many commensurability classes of arithmetic Fuchsian groups of genus 0, there is a bound on the degree of the defining fields k of these groups. If an arithmetic Fuchsian group contains an element of order $n \geq 2$, then k contains $\mathbf{Q}(\cos 2\pi/n)$ and it is well-known that the degree of this field goes to infinity with n. □

Computations suggest (see [36]) the following conjecture.

Conjecture 5.7. *If Γ is an arithmetic Fuchsian group of genus 0, then the invariant trace-field has degree at most 7.*

Remarks:(1) The inequalities in Theorems 5.4 and 5.5 give an upper bound on the co-area of a maximal (resp. congruence) arithmetic Fuchsian group of genus zero as:

$$\text{Area}(\mathbf{H}^2/\Gamma) \leq \frac{128}{3}\pi$$

(2) The same arguments hold for arbitrary fixed g.
(3) Another feature of genus zero 2-orbifolds is that it is shown in [36] Theorem 1.5, that any Fuchsian group of genus 0 has a global upper bound to the length of the shortest translation length of a hyperbolic element.

6 Arithmetic rational homology 3-spheres

As discussed in §4, the fact that arithmetic hyperbolic 3-manifolds are commensurable with a 3-orbifold affords geometric and topological approaches to the Conjectures 1.1–1.4. The discussion of the case of 2-dimensional rational homology spheres in §5 motivates some interesting questions in dimension 3, that relate to Conjectures 1.1–1.4 but in the "opposite direction".

The analogue of Question 5.1 is:

Question 6.1. *Are there infinitely many commensurability classes of arithmetic rational homology 3-spheres (even allowing for orbifolds)?*

Continuing with the analogy, another consequence of Theorem 5.2 is that there is an upper bound on the degree of the invariant trace-field of an arithmetic Fuchsian group of genus 0 (from the area bound described in §5.3). Hence we pose:

Question 6.2. *Is there an upper bound to the degree of the invariant trace-field of an arithmetic rational homology 3-sphere?*

Note that a positive answer to Question 6.2 need not imply a negative answer to Question 6.1.

6.1

Here we provide a conjectural approach taken from [36], that provides a positive answer to Question 6.1.

As in the 2-dimensional case, we start with the non-cocompact setting which again is closely related to the theory of elliptic curves over quadratic imaginary number fields.

Let d be a square-free positive integer, and O_d be the ring of integers in $\mathbf{Q}(\sqrt{-d})$. The analogue of the modular group here is a family of groups, The Bianchi groups, $\mathrm{PSL}(2, \mathrm{O_d})$. In this setting, The Cuspidal Cohomology Problem posed in the 1980's asked which Bianchi groups have $r(PSL(2, O_d)) = 0$. In [56] it was shown that:

Theorem 6.3. $r(\mathrm{PSL}(2, \mathrm{O_d})) = 0$ *if and only if*

$$d \in \mathcal{D} = \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.$$

As in dimension 2, there is a class of congruence subgroups which play a key role in connecting with the arithmetic. For an ideal $\mathcal{A} \subset O_d$ we define the congruence subgroup:

$$\Gamma_0(\mathcal{A}) = \mathrm{P}\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ c \equiv 0 \bmod \mathcal{A}\}$$

Set $r(\mathcal{A}) = r(\Gamma_0(\mathcal{A}))$. Unlike the situation in dimension 2, $r(\mathcal{A})$ is far from understood at present. On the other hand, as in dimension 2, it is easy to construct infinitely many examples of non-cocompact arithmetic Kleinian groups Γ for which $r(\Gamma) = 0$; e.g. there are infinitely arithmetic links in S^3 whose complements are arithmetic.

In the context of the groups $\Gamma_0(\mathcal{A})$ work of Grunewald and Mennicke [19], and Cremona [14], suggest the following:

Conjecture 6.4. *For each $d \in \mathcal{D}$, there exist infinitely many ideals $\mathcal{A} \subset O_d$ such that $r(\mathcal{A}) = 0$.*

For the case of $\mathbf{Z}[i] = O_1$, more precise statements can be made. For example, the following seems reasonable.

Conjecture 6.5.

(a) There exist infinitely many pairs of prime ideals $\{\mathcal{P}_1, \mathcal{P}_2\} \subset \mathbf{Z}[i]$ such that $r(\mathcal{P}_1\mathcal{P}_2) = 0$.

(b) In fact, let $\mathcal{P} =< 1 + i >$, then there are infinitely many prime ideals $\mathcal{P}_j \subset \mathbf{Z}[i]$ with $N\mathcal{P}_j = 1 \bmod 12$ and $r(\mathcal{P}\mathcal{P}_j) = 0$.

As evidence of this, it is shown in [14] that of all integral ideals $\mathcal{A} \subset \mathbf{Z}[i]$ with norm ≤ 500 only 76 give rise to groups with $r(\mathcal{A}) > 0$. With reference to Conjecture 6.5(b) [14] shows that this holds for prime ideals of norms 13, 37, 61, and 73, but fails for a prime of norm 97.

The relevance of these conjectures to the cocompact setting follows from the Jacquet-Langlands correspondence (recall Theorem 4.15). A consequence of this is (cf. [6] Theorem 3.3).

Theorem 6.6. *Let B be a division algebra of quaternions over $\mathbf{Q}(\sqrt{-d})$ with the set of primes ramifying B being, $\{\mathcal{P}_1, \ldots, \mathcal{P}_{2r}\}$. Let $\mathcal{O}$ be a maximal order of B, and $\Gamma = \Gamma^1_{\mathcal{O}}$. If Γ^{ab} is infinite, then $r(\mathcal{P}_1 \ldots \mathcal{P}_{2r}) \neq 0$.*

A corollary of this result and Conjecture 6.5(b) is the following. We remark that the prime $\mathcal{P}$ in Conjecture 6.5(b) is chosen so that the resultant groups $\Gamma^1_{\mathcal{O}}$ are torsion-free.

Corollary 6.7. *If Conjecture 6.5(b) holds, then Question 6.1 has a positive answer.*

We finish this subsection with some further discussion motivated from the 2-dimensional setting. Corollary 5.6 provides an upper bound, say N, to the order of elliptic element in an arithmetic Fuchsian group of genus 0. Rephrasing this in a way that (although vacuous in dimension 2) is suggestive for dimension 3, Corollary 5.6 shows that if an arithmetic Fuchsian group Γ is commensurable with a group containing an element of order $> N$ then Γ is large. This prompts in the spirit of §4 a situation that may be more approachable than the general case—this is very much in the spirit of some of the results in [27].

Show that for all sufficiently large integers N, any (arithmetic) Kleinian group commensurable with one containing an element of order N is large.

Recall that if $M = \mathbf{H}^3/\Gamma$ is a closed orientable hyperbolic 3-manifold, the *injectivity radius* of M (denoted by injrad(M)) is defined as the largest number r such that for all $p \in M$, the ball $B(p,r)$ centered at p of radius r is isometric to the r-ball in $\mathbf{H}^3$. D. Cooper asked whether there is a $K \in \mathbf{R}$, independent of M, such that injrad$(M) \leq K$?

This question is of interest since, by residual finiteness, an affirmative answer to this question, implies every closed hyperbolic 3-manifold has a cover with first betti number at least 1. However, F. Calegari and N. Dunfield [6] answered this question in the negative; unlike the 2-dimensional case discussed in §5.3.

In [6], the authors construct arithmetic rational homology 3-spheres arising from the division algebra $\mathbf{Q}(\sqrt{-2})$ ramified at the two places lying above 3 in $\mathbf{Q}(\sqrt{-2})$ and with arbitrarily large injectivity radii. The construction of [6] assumes the existence of certain Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}(\sqrt{-2})}/\mathbf{Q}(\sqrt{-2}))$ that although predicted by the general framework of the Langlands correspondence are not yet known to exist. However an unconditional proof of the result of [6] was established in [3]. It is worth remarking in light of Question 6.1 that the construction of [6] provides commensurable rational homology 3-spheres. Also these examples show that there are infinitely many arithmetic rational homology 3-spheres in the same commensurability classes (as in the 2-dimensional setting with orbifolds).

Another interesting feature of the construction of [6] is that their manifolds are all Haken. Thus it seems interesting to refine Cooper's orginal question to.

Question 6.8. *Are there non-Haken hyperbolic 3-manifolds of arbitrarily large injectivity radius (even allowing for orbifolds)?*

6.2

There are many examples of arithmetic rational homology 3-spheres. These can be found efficiently using the census of hyperbolic 3-manifolds developed using the computer programs SnapPea, and the exact version Snap (see [13]). We list the 10 smallest known arithmetic hyperbolic 3-manifolds. The first two are known to be the smallest two by [7]. We list these using the notation of [13].

1: Manifold $m003(-3,1)$; Volume 0.94270736277692772092.
Minimum polynomial: $x^3 - x^2 + 1$; Root: -2.
Discriminant: -23; Signature $(1,1)$.
Real Ramification [1]; Finite Ramification ν_5.
Integral Traces; Arithmetic.

2: Manifold $m003(-2,3)$; Volume 0.9813688288922320880914.
Minimum polynomial: $x^4 - x - 1$; Root: 3.
Discriminant: -283; Signature $(2,1)$.

Real Ramification $[1,2]$; Finite Ramification $\emptyset$.
Integral Traces; Arithmetic.

3: Manifold $m010(-1,2)$; Volume 1.01494160640965362502120.
Minimum polynomial: $x^2 - x + 1$; Root: 1.
Discriminant: -3; Signature $(0,1)$.
Real Ramification $\emptyset$; Finite Ramification ν_2, ν_3.
Integral Traces; Arithmetic.

4: Manifold $m003(-4,3)$; Volume 1.26370923865804365584716.
Minimum polynomial: $x^4 - x^3 + x^2 + x - 1$; Root: -3.
Discriminant: -331; Signature $(2,1)$.
Real Ramification $[1,2]$; Finite Ramification $\emptyset$.
Integral Traces; Arithmetic.

5: Manifold $m004(6,1)$; Volume 1.28448530046835444246033370.
Minimum polynomial: $x^3 + 2x - 1$; Root: 2.
Discriminant: -59; Signature $(1,1)$.
Real Ramification $[1]$; Finite Ramification ν_2.
Integral Traces; Arithmetic.

6: Manifold $m003(-3,4)$; Volume 1.41406104416539158138194 9.
Minimum polynomial: $x^3 - x^2 + 1$; Root: 2.
Discriminant: -23; Signature $(1,1)$.
Real Ramification $[1]$; Finite Ramification ν_{19}.
Integral Traces; Arithmetic.

7: Manifold $m003(-5,3)$; Volume 1.54356891147185507432847.
Minimum polynomial: $x^5 - x^3 - 2x^2 + 1$; Root: 4.
Discriminant: -4511; Signature $(3,1)$.
Real Ramification $[1,2,3]$; Finite Ramification ν_{13}.
Integral Traces; Arithmetic.

8: Manifold $m007(4,1)$; Volume 1.58316666062481283616602 8.
Minimum polynomial: $x^3 + x - 1$; Root: 2.
Discriminant: -31; Signature $(1,1)$.
Real Ramification $[1]$; Finite Ramification ν_{13}.
Integral Traces; Arithmetic.

9: Manifold $m006(3,1)$; Volume 1.58864663930016298817691 3.
Minimum polynomial: $x^3 - x^2 + x + 1$; Root: -2.
Discriminant: -44; Signature $(1,1)$.
Real Ramification $[1]$; Finite Ramification ν_2.
Integral Traces; Arithmetic.

10: Manifold $m015(5,1)$; Volume 1.75712602918845136287474 65.
Minimum polynomial: $x^5 - x^4 - x^3 + 2x^2 - x - 1$; Root: -4.
Discriminant: -4903; Signature $(3,1)$.
Real Ramification $[1,2,3]$; Finite Ramification ν_{13}.

Integral Traces; Arithmetic.

We also give an example of an arithmetic rational homology 3-sphere that is of the type predicted by Theorem 6.6.

Example: Let $B/\mathbf{Q}(i)$ be ramified at $\{< 1+i >, < 2+3i >\}$. Let $\mathcal{O}$ be a maximal order in B, then $\Gamma^1_{\mathcal{O}}$ is a cocompact torsion-free Kleinian group of volume approximately 3.663862376708.... The quotient $M = \mathbf{H}^3/\Gamma^1_{\mathcal{O}}$ is the manifold $s705(1,2)$ in the SnapPea census of closed hyperbolic 3-manifolds. The first homology of this manifold is $\mathbf{Z}/21\mathbf{Z}$.

7 Geodesics in arithmetic hyperbolic 3-manifolds

Recall that if $M = \mathbf{H}^3/\Gamma$ is a closed orientable hyperbolic 3-manifold then the *length spectrum* $\mathcal{L}(M)$ of M is the set of all lengths of closed geodesics on M counted with multiplicities. The *length set* $L(M)$ of M is the set of lengths all closed geodesics counted without multiplicities. The *rational length spectrum* $\mathbf{Q}L(M)$ of M is the set of all rational multiples of lengths of closed geodesics of M. We shall denote the set of axes of hyperbolic elements in Γ by $\mathcal{A}(\Gamma)$.

This section is motivated by the following questions; in these, $M_j = \mathbf{H}^3/\Gamma_j$ $(j = 1,2)$ are closed orientable hyperbolic 3-manifolds.

Question 7.1. *If $\mathcal{L}(M_1) = \mathcal{L}(M_2)$, are M_1 and M_2 are commensurable?*

Question 7.2. *If $L(M_1) = L(M_2)$, are M_1 and M_2 are commensurable?*

Question 7.3. *If $\mathbf{Q}L(M_1) = \mathbf{Q}L(M_2)$, are M_1 and M_2 are commensurable?*

Note that if M_1 and M_2 are commensurable then $\mathbf{Q}L(M_1) = \mathbf{Q}L(M_2)$, and that a positive answer to Question 7.2 provides positive answers to Questions 7.1 and 7.3.

In addition, since it is known (see [16] pp. 415–417) that for closed hyperbolic manifolds, the spectrum of the Laplace-Beltrami operator action on $L^2(M)$, counting multiplicities, determines $L(M)$, a positive answer to Question 7.2 implies a positive answer to the next question. It should be noted that the currently known methods of producing isospectral (same spectrum of the Laplace-Beltrami operator counted with multiplcities) closed hyperbolic 3-manifolds produce commensurable ones (see [51] and [54]).

Question 7.4. *If M_1 and M_2 are isospectral, are they commensurable?*

"Dual" to these questions about lengths of geodesics is the following question about axes determining commensurability.

Question 7.5. *If $\mathcal{A}(\Gamma_1) = \mathcal{A}(\Gamma_2)$, are Γ_1 and Γ_2 commensurable?*

As above, it is easy to see that if Γ_1 and Γ_2 are commensurable, then $\mathcal{A}(\Gamma_1) = \mathcal{A}(\Gamma_2)$.

Addressing these questions for general closed hyperbolic 3-manifolds seems very hard at present, but arithmetic manifolds again provide an interesting class where these questions can be answered. Indeed, these questions can be asked for closed hyperbolic manifolds in arbitrary dimensisons, and considerable progress has been made recently in dimensions ≥ 4 in the arithmetic case by Prasad and Rapinchuk [44]. In particular, they show that there exist arithmetic hyperbolic 5-manifolds such that Question 7.3 has a negative answer.

In the remainder of this section we outline proofs of positive answers to all of the above questions for arithmetic hyperbolic 2 and 3-manifolds. This surveys work in [8], [33] and [46].

7.1

We begin by sketching the proof of the following result in dimension 2. This is a mild extension of the result in [46].

Theorem 7.6. *Let $M_1 = \mathbf{H}^2/\Gamma_1$ be an orientable arithmetic hyperbolic 2-manifold and assume that $M_2 = \mathbf{H}^2/\Gamma_2$ is a closed orientable hyperbolic 2-manifold with $L(\Gamma_1) = L(\Gamma_2)$. Then Γ_2 is arithmetic, and Γ_1 and Γ_2 are commensurable.*

Before commencing with the sketch of the proof, we discuss, what is perhaps the key feature that allows one to use arithmetic methods. We motivate this with an example

Example: Let $\gamma \in \mathrm{PSL}(2,\mathbf{Z})$ be a hyperbolic element of trace t. Then the eigenvalues of γ are $\frac{(-t\pm\sqrt{t^2-4})}{2}$, and these being roots of the polynomial $p(x) = x^2 - tx + 1$. Let $\lambda_\gamma = \frac{(-t+\sqrt{t^2-4})}{2}$ be the eigenvalue satisfying $|\lambda_\gamma| > 1$. The length of the closed geodesic on the modular surface $\mathbf{H}^2/\mathrm{PSL}(2,\mathbf{Z})$ determined by the projection of the axis of γ is $2\ln|\lambda_\gamma|$.

Note that λ_γ is a unit, and it is an elementary excercise to show that $p(x)$ is irreducible over $\mathbf{Z}$. Thus $\mathbf{Q}(\lambda_\gamma)$ determines a quadratic extension of $\mathbf{Q}$.

Now $\mathrm{PSL}(2,\mathbf{Z})$ coincides with the projection of the image of the elements of norm 1 in $M(2,\mathbf{Z})$. Thus we can find an element $u \in M(2,\mathbf{Q})$ such that $\mathrm{P}(u) = \gamma$, and we deduce from this that $\mathbf{Q}(\lambda_\gamma)$ embeds in $M(2,\mathbf{Q})$ as $\mathbf{Q}(u)$.

Conversely, given a real quadratic extension L of $\mathbf{Q}$ that embeds in $M(2,\mathbf{Q})$ we can constuct a hyperbolic element $\gamma \in \mathrm{PSL}(2,\mathbf{Z})$ for which $L \cong \mathbf{Q}(\lambda_\gamma)$ as follows. Since L is is a real quadratic extension of $\mathbf{Q}$, the unit group R_L^* of L has rank one. Choosing a fundamental unit u, then $u^n \notin \mathbf{Q}$ for all integers $n \neq 0$, and so $L = \mathbf{Q}(u^n)$ for all integers $n \neq 0$. Now L embeds in $M(2,\mathbf{Q})$, so R_L^* embeds in $M(2,\mathbf{Q})$ The key claim is that we can construct an order $\mathcal{O} \subset M(2,\mathbf{Q})$ for which $u \in \mathcal{O}$ (see [39] Chapter 12). It follows that since either u or u^2 has norm 1, we can arrange that u or u^2 is an element of $\mathcal{O}^1$. By commensurability, we deduce that there is an integer $m \neq 0$ such that $u^m \in SL(2,\mathbf{Z})$ and the image of this element in $\mathrm{PSL}(2,\mathbf{Z})$ is the required hyperbolic element.

That the quadratic extension is real is important, since otherwise, the unit group is finite, and we cannot construct a hyperbolic element as needed.

Extending this, the following facts are proved in [39, Chapter 12]. In the statement Γ is either an arithmetic Fuchsian or Kleinian group.

Theorem 7.7. *Suppose that Γ is derived from a quaternion algebra B/k.*

(i) Let γ be a hyperbolic element of Γ with eigenvalue λ_γ. The field $k(\lambda_\gamma)$ is a quadratic extension field of k which embeds into B.

(ii) Let L be a quadratic extension of k. If Γ is Fuchsian assume that L is not a totally imaginary quadratic extension of k.

Then L embeds in B/k if and only if $L = k(\lambda_\gamma)$ for some hyperbolic $\gamma \in \Gamma$. This will be true if and only if no place of k which splits in L is ramified in B.

Sketch proof of Theorem 7.6: We begin by showing that Γ_2 is arithmetic. Let the invariant quaternion algebra of Γ_1 be B_1/k. Standard considerations allow us to deduce that the sets of traces of elements in Γ_1 and Γ_2 are the same up to sign, and that the sets $\{\mathrm{tr}\ \gamma^2\}$ are the same. Since Γ_1 is arithmetic, $\mathrm{tr}\ \gamma^2$ is an algebraic integer in the totally real field k. Thus all elements of Γ_2 have algebraic integer trace and $k\Gamma_2 = k$. To complete the proof that Γ_2 is arithmetic we show that $B_2 = A\Gamma_2$ is ramified at all infinite places except one, for then the characterization theorem of Takeuchi [52] applies. To see this we argue as follows.

Let α_1 and α_2 be a pair non-commuting hyperbolic elements in $\Gamma_2^{(2)}$. Now a Hilbert Symbol for $A\Gamma_2^{(2)}$ can be computed as $\left(\frac{\mathrm{tr}^2(\alpha_1)-4, \mathrm{tr}[\alpha_1,\alpha_2]-2}{k}\right)$ (see [39] Chapter 3). Since Γ_1 and Γ_2 have the same sets of traces (up to sign), we can find $\beta_1, \beta_2 \in \Gamma_1$ such that $\mathrm{tr}(\beta_1) = \pm\mathrm{tr}(\alpha_1)$ and $\mathrm{tr}(\beta_2) = \pm\mathrm{tr}([\alpha_1, \alpha_2])$. It follows from [42] Theorem 2.2 that $\beta_1, \beta_2 \in \Gamma_1 \cap B_1^1$ since $\mathrm{tr}(\beta_1), \mathrm{tr}(\beta_2) \in \mathrm{k}$.

Let $\sigma : k \to \mathbf{R}$ be a non-identity embedding. Since B_1 is ramified at σ it follows that $|\sigma(\mathrm{tr}\ \mathrm{x})| < 2$ for all non-trivial elements $x \in B^1$. Hence we conclude that

$$\sigma(\mathrm{tr}^2(\beta_1) - 4) < 0 \text{ and } \sigma(\mathrm{tr}(\beta_2) - 2) < 0.$$

Hence it follows that

$$\sigma(\mathrm{tr}^2(\alpha_1) - 4) < 0 \text{ and } \sigma(\mathrm{tr}([\alpha_1, \alpha_2]) - 2) < 0,$$

from which it follows that B_2 is ramified at σ. Now σ was an arbitrary non-identity embedding, and so we have shown that B_2 is ramified at all such embeddings as required.

To establish commensurability, it suffices to show that $B_1 \cong B_2$ (see [39] Theorem 8.4.6). To this end, we make the following definition.

Definition: *For $j = 1, 2$ let*

$\mathcal{N}_j = \{L/k : [L : k] = 2,$ *L emebds in B_j and is not a totally imaginary quadratic extension of k*$\}$.

Clearly B_1 and B_2 are isomorphic if and only if the set of quadratic extensions that embed in one is precisely the set that embeds in the other. The key claim that needs to be shown is:

B_1 and B_2 are isomorphic if and only $\mathcal{N}_1 = \mathcal{N}_2$

This is established in [46]. Given this, the proof is completed as follows. Suppose $L \in \mathcal{N}_1$. Theorem 7.7 and commensurability shows that $L = k(\lambda_\gamma)$ for some hyperbolic $\gamma \in \Gamma_1$ with eigenvalue λ_γ. Since $L(M_1) = L(M_2)$, it follows that there exists an element $\gamma' \in \Gamma_2$ for which $\lambda_{\gamma'} = \lambda_\gamma$. Hence $L = k(\lambda_\gamma) = k(\lambda_{\gamma'})$ embeds in B_2 as required. □

In dimension 3, a similar result is established in [46] for the *complex length spectrum.* However, to deal with lengths only requires considerably more work. The following is proved in [8], and in particular gives an affirmative answer to Questions 7.1, 7.2 7.3 and 7.4 when M_1 and M_2 are both arithmetic.

Theorem 7.8. *If M is an arithmetic hyperbolic 3-manifold, then the rational length spectrum and the commensurability class of M determine one another.*

The proof of Theorem 7.8 is modelled on some of the ideas contained in the proof of Theorem 7.6, but more involved, since one cannot deal with traces directly. Briefly, the first step in proving that $\mathbf{Q}L(M)$ determines the commensurability class is to show that the invariant trace-field is determined by $\mathbf{Q}L(M)$. We then determine the commensurability class of M from $\mathbf{Q}L(M)$ following ideas similar to those used in the proof of Theorem 7.6. The arguments for the determination of $k\Gamma$ splits naturally into two cases; when $[k : k \cap \mathbf{R}] > 2$ and $[k : k \cap \mathbf{R}] = 2$. Both cases require a detailed understanding of the Galois theory of fields with one complex place and their quadratic extensions. We refer the reader to [8] for details.

7.2

If we consider only the "location" of the axes of hyperbolic elements in arithmetic Fuchsian and Kleinian groups, then again one can get extra information that allows one to prove (see [33]).

Theorem 7.9. *If Γ_1 and Γ_2 are arithmetic Fuchsian (resp. Kleinian) groups then Question 7.5 has a positive answer.*

For convenience, we shall call groups Γ_1 and Γ_2 *isoaxial* if $\mathcal{A}(\Gamma_1) = \mathcal{A}(\Gamma_2)$.

As with the proof of Theorem 7.6 the key issue is to show that the isoaxial condition forces the same invariant quaternion algebra. We sketch the proof for Kleinian groups that requires the following definitions. The obvious changes can be made for Fuchsian groups.

Let Γ be a Kleinian group and define

$$\Sigma(\Gamma) = \{\gamma \in \mathrm{PSL}(2, \mathbf{C}) \mid \gamma(\mathcal{A}(\Gamma)) = \mathcal{A}(\Gamma)\}.$$

It is easy to check that $\Sigma(\Gamma)$ is a subgroup of $\mathrm{PSL}(2, \mathbf{C})$, and that $\mathrm{Comm}(\Gamma) < \Sigma(\Gamma)$. The key point now is to show that when Γ is arithmetic, then $\mathrm{Comm}(\Gamma) = \Sigma(\Gamma)$. Given this, the proof is completed as follows.

If Γ_1 and Γ_2 are isoaxial Kleinian groups, then for any $\gamma \in \Gamma_2$, $\mathcal{A}(\Gamma_1) = \mathcal{A}(\gamma\Gamma_1\gamma^{-1})$, and therefore $\gamma \in \Sigma(\Gamma_1)$. Hence $\Gamma_2 < \Sigma(\Gamma_1)$. In addition, if Γ_1 is arithmetic, then the equality $\mathrm{Comm}(\Gamma_1) = \Sigma(\Gamma_1)$ implies that $\Gamma_2 < \mathrm{Comm}(\Gamma_1)$. It is now standard that, since Γ_2 is also arithmetic, Γ_1 and Γ_2 are commensurable. □

Remark: In [35] we construct examples of Fuchsian groups of finite co-area that have the same set of parabolic fixed points $\mathrm{PSL}(2, \mathbf{Z})$ but are non-arithmetic. The question of whether these Fuchsian groups share the same set of axes as hyperbolic elements of $\mathrm{PSL}(2, \mathbf{Z})$ remains open, although we suspect that this is not the case.

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