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Abundance of σ-compact non-compactly generated groups witnessed by the Bohr topology of an abelian group *

Dikran Dikranjan [1]

Dipartimento di Matematica e Informatica
Università di Udine, Via delle Scienze 206, 33100 Udine, Italy

Dmitri Shakhmatov [2]

Graduate School of Science and Engineering
Division of Mathematics, Physics and Earth Sciences
Ehime University, Matsuyama 790-8577, Japan

Abstract

Answering negatively a question of Fujita and Shakhmatov [2], Tkachenko and Torres Falcón [6] have constructed an example of a countable (and thus, σ-compact) totally bounded group that is not compactly generated. We observe that *any* countable non-finitely generated abelian group G equipped with its Bohr topology fails to be compactly generated, thereby obtaining an abundant supply of totally bounded groups providing a counter-example to the question of Fujita and Shakhmatov [2].

A group G is *finitely generated* if G is algebraically generated by its finite subset.

A group G is called σ-*compact* provided that G can be represented as a union of a countable family of its compact subsets, and G is called *compactly*

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[2]The second author was partially supported by the Grant-in-Aid for Scientific Research no. 19540092 by the Japan Society for the Promotion of Science (JSPS). *e-mail*: dmitri@dpc.ehime-u.ac.jp

generated if G is algebraically generated by its compact subset. One can easily see that a compactly generated group is σ-compact.

Fujita and Shakhmatov [2] proved that a σ-compact metric group is compactly generated. The same result also holds for a wider class of groups that contains both metric and locally compact groups, see [3].

Answering a question of Fujita and Shakhmatov [2], Tkachenko and Torres Falcón [6] have constructed an example of a countable (thus, σ-compact) totally bounded group that is not compactly generated. (Recall that a group G is *totally bounded* if it is (topologically and algebraically) isomorphic to a subgroup of some compact group.)

The main purpose of this note is to observe that *any* countable non-finitely generated abelian group G equipped with its Bohr topology fails to be compactly generated, thereby obtaining an abundant supply of totally bounded groups providing a counter-example to the question of Fujita and Shakhmatov [2].

Let G be an abelian group. The *Bohr topology* of G is the weakest group topology on G making all characters $\chi : G \to \mathbb{T}$ continuous, where $\mathbb{T}$ is the torus group. The Bohr topology of G is totally bounded, and the group G equipped with this topology is usually denoted by $G^{\#}$.

According to the celebrated result of Glicksberg [4], $G^{\#}$ has no infinite compact subsets. Thus, $G^{\#}$ is compactly generated if and only if $G^{\#}$ is finitely generated. This immediately yields the following

Theorem 1. *Let G be a countable abelian group that is not finitely generated. Then $G^{\#}$ is a σ-compact totally bounded group that is not compactly generated.*

It should be noted that there are only countably many finitely generated abelian groups, as every such group has the form

$$\mathbb{Z}^n \times \mathbb{Z}(m_1) \times \ldots \times \mathbb{Z}(m_k), \tag{1}$$

where $n, m_1, \ldots, m_k$ are integer numbers and $\mathbb{Z}(k)$ denotes the cyclic group of order k. On the other hand, there are continuum many pairwise non-isomorphic countable abelian groups. (In fact, even the group $\mathbb{Q}$ of rational numbers contains continuum many pairwise non-isomorphic subgroups.)

Corollary 2. *Let G be a countable abelian group that is not isomorphic to a group of the form* (1). *Then $G^{\#}$ is a σ-compact totally bounded group that is not compactly generated.*

Let us finish with some concrete examples.

Corollary 3. *$\mathbb{Q}^{\#}$ is a (σ-compact) divisible totally bounded group that is not compactly generated.*

Corollary 4. *If G is a countably infinite torsion group, then $G^{\#}$ is a (σ-compact) totally bounded group that is not compactly generated.*

It follows from Corollary 4 that $(\mathbb{Q}/\mathbb{Z})^{\#}$ is a (σ-compact) divisible totally bounded group that is not compactly generated.

Furthermore, from Corollary 4 one can easily obtain an infinite family of (σ-compact) non-compactly-generated totally bounded torsion groups that are pairwise non-homeomorphic even as topological spaces. Indeed, if G and H are countably infinite torsion groups of distinct prime exponent, then $G^{\#}$ and $H^{\#}$ are not homeomorphic as topological spaces ([5, 1]).

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