Abundance of $\sigma$-compact non-compactly generated groups witnessed by the Bohr topology of an abelian group *

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Abstract

Answering negatively a question of Fujita and Shakhmatov [2], Tkachenko and Torres Falcón [6] have constructed an example of a countable (and thus, $\sigma$-compact) totally bounded group that is not compactly generated. We observe that any countable non-finitely generated abelian group $G$ equipped with its Bohr topology fails to be compactly generated, thereby obtaining an abundant supply of totally bounded groups providing a counter-example to the question of Fujita and Shakhmatov [2].

A group $G$ is finitely generated if $G$ is algebraically generated by its finite subset.

A group $G$ is called $\sigma$-compact provided that $G$ can be represented as a union of a countable family of its compact subsets, and $G$ is called compactly

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generated if $G$ is algebraically generated by its compact subset. One can easily see that a compactly generated group is $\sigma$-compact.

Fujita and Shakhmatov [2] proved that a $\sigma$-compact metric group is compactly generated. The same result also holds for a wider class of groups that contains both metric and locally compact groups, see [3].

Answering a question of Fujita and Shakhmatov [2], Tkachenko and Torres Falcón [6] have constructed an example of a countable (thus, $\sigma$-compact) totally bounded group that is not compactly generated. (Recall that a group $G$ is totally bounded if it is (topologically and algebraically) isomorphic to a subgroup of some compact group.)

The main purpose of this note is to observe that any countable non-finitely generated abelian group $G$ equipped with its Bohr topology fails to be compactly generated, thereby obtaining an abundant supply of totally bounded groups providing a counter-example to the question of Fujita and Shakhmatov [2].

Let $G$ be an abelian group. The Bohr topology of $G$ is the weakest group topology on $G$ making all characters $\chi : G \to \mathbb{T}$ continuous, where $\mathbb{T}$ is the torus group. The Bohr topology of $G$ is totally bounded, and the group $G$ equipped with this topology is usually denoted by $G^\#$. According to the celebrated result of Glicksberg [4], $G^\#$ has no infinite compact subsets. Thus, $G^\#$ is compactly generated if and only if $G^\#$ is finitely generated. This immediately yields the following

**Theorem 1.** Let $G$ be a countable abelian group that is not finitely generated. Then $G^\#$ is a $\sigma$-compact totally bounded group that is not compactly generated.

It should be noted that there are only countably many finitely generated abelian groups, as every such group has the form

$$\mathbb{Z}^n \times \mathbb{Z}(m_1) \times \ldots \times \mathbb{Z}(m_k),$$

where $n, m_1, \ldots, m_k$ are integer numbers and $\mathbb{Z}(k)$ denotes the cyclic group of order $k$. On the other hand, there are continuum many pairwise non-isomorphic countable abelian groups. (In fact, even the group $\mathbb{Q}$ of rational numbers contains continuum many pairwise non-isomorphic subgroups.)

**Corollary 2.** Let $G$ be a countable abelian group that is not isomorphic to a group of the form (1). Then $G^\#$ is a $\sigma$-compact totally bounded group that is not compactly generated.

Let us finish with some concrete examples.

**Corollary 3.** $\mathbb{Q}^\#$ is a ($\sigma$-compact) divisible totally bounded group that is not compactly generated.
Corollary 4. If $G$ is a countably infinite torsion group, then $G^{\#}$ is a ($\sigma$-compact) totally bounded group that is not compactly generated.

It follows from Corollary 4 that $(\mathbb{Q}/\mathbb{Z})^{\#}$ is a ($\sigma$-compact) divisible totally bounded group that is not compactly generated.

Furthermore, from Corollary 4 one can easily obtain an infinite family of ($\sigma$-compact) non-compactly-generated totally bounded torsion groups that are pairwise non-homeomorphic even as topological spaces. Indeed, if $G$ and $H$ are countably infinite torsion groups of distinct prime exponent, then $G^{\#}$ and $H^{\#}$ are not homeomorphic as topological spaces ([5, 1]).

References


