SOME COMPLETE-TYPE MAPS

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1. INTRODUCTION

As well known, in the topological category $TOP$ uniform spaces are studied as the generalization of metric spaces, compact spaces and topological groups. In the fibrewise category $TOP_B$ with the base space $B$, the study of fibrewise uniform space in $TOP_B$ is found in James [5] Ch.3 and Konami-Miwa [6], [7]. Especially in [6] and [7], they studied the fibrewise uniform spaces by using coverings, and proved in [7] the equivalence of fibrewise uniform spaces by using entourages (in [5]) and their one (in [7]). The study of metrizable maps in $TOP_B$ is found in [11], [9], [2], [8] and [3]. But for a metrizable map $p : X \to B$, the study of fibrewise uniformity on $X$ has not been done.

In this paper, we announce the existence of fibrewise uniformities on some metrizable maps, and study the relations between the completeness induced by a trivial metric and the one defined by fibrewise uniformities. Further, we discuss the relations between completely metrizable maps and Čech-complete maps.

2. PRELIMINARIES

In this section, we refer to the notions and notations in Fibrewise Topology. For the definitions of undefined terms and notations, see [4], [3], [7] and [5].

Throughout this paper, we will use the abbreviation $nbd(s)$ for neighborhood(s). Let $B$ be a topological space with a fixed topology $\tau$. For each $b \in B$, $N(b)$ is the family of all open nbds of $b$, and $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ and $I$ are the sets of all natural numbers, all rational numbers, all real numbers and the unit interval, respectively. In this paper, we assume that $(B, \tau)$ is a regular space, all spaces are topological spaces and all maps are continuous.

For a map $p : X \to B$ and each $b \in B$, the fibre over $b$ is the subset $X_b = p^{-1}(b)$ of $X$. Also for each subset $B'$ of $B$, we denote $X_{B'} = p^{-1}B'$. For a filter $\mathcal{F}$ on $X$, by a $b$-filter on $X$ we mean a pair $(b, \mathcal{F})$ such that $b$ is a limit point of the filter $p_*(\mathcal{F})$ on $B$, where $p_*(\mathcal{F})$ is the filter generated by the family $\{p(F) | F \in \mathcal{F} \}$. By an adherence point of a $b$-filter $\mathcal{F}$ ($b \in B$) on $X$, we mean a point of the fibre $X_b$. 
which is an adherence point of \( \mathcal{F} \) as a filter on \( X \). For a projection \( p : X \to B \) and \( W \subset B \), we use the notation \( X_W \times X_W = X^2_W \) and \( X \times X = X^2 \). For \( D, E \subset X^2 \), \( D \circ E = \{(x, z) \mid \exists y \in X \text{ such that } (x, y) \in D, (y, z) \in E \} \) and \( D(x) = \{ y \mid (x, y) \in D \} \).

For a family \( \mathcal{U} \) of subsets of a set \( X \) and a subset \( A \) of \( X \), \( \mathcal{U}|_A = \{ U \cap A \mid U \in \mathcal{U} \} \).

Next, according to [11] let us refer to (completely) trivially metrizable maps. For a map \( p : X \to B \) with a pseudometric \( \rho \) on \( X \) is called a trivial metric (T-metric, for short) on \( p \) if the restriction of \( \rho \) to every fibre \( p^{-1}(b), b \in B \), is a metric and \( p^{-1} \tau \cup \tau_p \), where \( \tau_p \) is the topology on \( X \) generated by \( \rho \), is a subbase of the topology of \( X \). A map \( p : X \to B \) is called trivially metrizable (a TM-map, for short) if there exists a T-metric on \( p \). A T-metric on a map \( p : X \to B \) is called complete (a CT-metric, or short) if

\[(*) \text{ For any } b \text{-filter } \mathcal{F}, b \in B, \text{ on } X \text{ containing elements of arbitrary small diameter, } \mathcal{F} \text{ has adherence points.} \]

A map \( p : X \to B \) is called completely trivially metrizable (a complete TM-map, for short) if there exists a CT-metric on it.

A map \( p : X \to B \) is called (resp. closely) parallel to a space \( Z \) if there exists an embedding \( e : X \to B \times Z \) such that (resp. \( e(X) \) is closed in \( B \times Z \) and ) \( p = \pi \circ e \), where \( \pi : B \times Z \to B \) is the projection (see [10]).

The following are proved in [11]: A map \( p : X \to B \) is a TM-map if and only if \( p \) is parallel to a metrizable map, and \( p \) is a complete TM-map if and only if it is closely parallel to a completely metrizable (i.e., metrizable by complete metric) space.

Remark: By these, for a TM-map \( p : X \to B \) there exists a metric space \( (M, \rho) \) and an embedding \( e : X \to B \times M \) such that \( p = \pi \circ e \). Then it is easy to see that we can define a T-metric (pseudometric) \( \rho' \) on \( X \) by \( \rho'(x, y) = \rho(\pi \circ e(x), \pi \circ e(y)) \), and vice versa. So, we can identify \( \rho \) on \( M \) and \( \rho' \) on \( X \) in the above meaning. In latter sections, we use the same notation \( \rho \) on \( M \) and on \( X \).

We shall conclude this section by referring to fibrewise uniformities according to [7]. First, we recall the following definition.

**Definition 2.1.** Let \( p : X \to B \) be a projection, and \( \Delta \) be the diagonal of \( X \times X \). A *fibrewise entourage uniformity* on \( X \) is a filter \( \Omega \) on \( X \times X \) satisfying the following four conditions:

\[(J1) \Delta \subset D \text{ for every } D \in \Omega. \]

\[(J2) \text{ Let } D \in \Omega. \text{ Then for each } b \in B \text{ there exist } W \in N(b) \text{ and } E \in \Omega \text{ such that } E \cap X^2_W \subset D^{-1}. \]

\[(J3) \text{ Let } D \in \Omega. \text{ Then for each } b \in B \text{ there exist } W \in N(b) \text{ and } E \in \Omega \text{ such that } (E \cap X^2_W) \circ (E \cap X^2_W) \subset D \]

\[(J4) \text{ If } E \subset X \times X \text{ satisfies that for each } b \in B \text{ there exist } W \in N(b) \text{ and } D \in \Omega \text{ such that } D \cap X^2_W \subset E, \text{ then } E \in \Omega. \]
Note that in [5] Section 12, a filter \( \Omega \) on \( X \times X \) satisfying (J1),(J2) and (J3) is called a \textit{fibrewise uniform structure} on \( X \). So, the notion of a fibrewise entourage uniformity is slightly stronger than one of a fibrewise uniform structure.

For a projection \( p : X \to B \) and \( W \in \tau \), let \( \mu_{W} \) be a non-empty family of coverings of \( X_{W} \). We say that \( \{ \mu_{W} \}_{W \in \tau} \) is a \textit{system of coverings} of \( \{ X_{W} \}_{W \in \tau} \). (For this, we briefly use the notations \( \{ \mu_{W} \} \) and \( \{ X_{W} \} \).) Let \( \mathcal{U} \) and \( \mathcal{V} \) be families of subsets of a set \( X \). If \( \mathcal{V} \) refines \( \mathcal{U} \) in the usual sense, we denote \( \mathcal{V} < \mathcal{U} \). Let us define the notion of fibrewise covering uniformity.

**Definition 2.2.** Let \( p : X \to B \) be a projection, and \( \mu = \{ \mu_{W} \} \) be a system of coverings of \( \{ X_{W} \} \). We say that the system \( \{ \mu_{W} \} \) is a \textit{fibrewise covering uniformity} (and a pair \( (X, \mu) \) or \( (X, \{ \mu_{W} \}) \)) is a \textit{fibrewise covering uniform space} if the following conditions are satisfied:

(C1) Let \( \mathcal{U} \) be a covering of \( X_{W} \) and for each \( b \in W \) there exist \( W' \in N(b) \) and \( \mathcal{V} \in \mu_{W} \) such that \( W' \subset W \) and \( \mathcal{V} \subset \mathcal{U} \). Then \( \mathcal{U} \in \mu_{W} \).

(C2) For each \( U_{i} \in \mu_{W}, i = 1, 2 \), there exists \( U_{i} \in \mu_{W} \) such that \( U_{i} < U_{i}, i = 1, 2 \).

(C3) For each \( U \in \mu_{W} \) and \( b \in W \), there exist \( W' \in N(b) \) and \( \mathcal{V} \in \mu_{W} \) such that \( W' \subset W \) and \( \mathcal{V} \) is a star refinement of \( U \).

(C4) For \( W' \subset W, \mu_{W} \supset \mu_{W}|_{W} \), where

\[
\mu_{W}|_{W} = \{ U|_{W'} | U \in \mu_{W} \} \quad \text{and} \quad U|_{W'} = \{ U \cap X_{W} | U \in \mathcal{U} \}.
\]

For a fibrewise entourage uniformity \( \Omega \) on \( X, D \in \Omega \) and \( W \in \tau \), let \( \mathcal{U}(D, W) = \{ D(x) \cap X_{W} | x \in X_{W} \} \). Further let \( \mu_{W}(\Omega) \) be the family of coverings \( \mathcal{U} \) of \( X_{W} \) satisfying that for each \( b \in W \) there exist \( W' \in N(b) \) and \( D \in \Omega \) such that \( W' \subset W \) and \( \mathcal{U}(D, W) < \mathcal{U} \). Then the system \( \mu(\Omega) = \{ \mu_{W}(\Omega) \} \) is a fibrewise covering uniformity ([7] Proposition 3.7).

Conversely, for a fibrewise covering uniformity \( \mu = \{ \mu_{W} \} \), we can constructed a fibrewise entourage uniformity \( \Omega(\mu) \) as follows ([7] Construction 3.8): For \( U \in \mu_{W} \), \( D(U) = \{ U_{a} \times U_{a} | U_{a} \in \mathcal{U} \} \). Let \( \Omega(\mu) \) be the family of all subsets \( D \subset X \times X \) satisfying the following condition:

\[
\Delta \subset D, \text{ and for every } b \in B \text{ there exist } W \in N(b) \text{ and } U \in \mu_{W} \text{ such that } D(U) \subset D.
\]

Then \( \Omega(\mu) \) is a fibrewise entourage uniformity ([7] Proposition 3.10). Further, we proved the following:

**Theorem 2.3.** ([7] Theorem 3.11) For a projection \( p : X \to B \) and a fibrewise entourage uniformity \( \Omega \) on \( X \), we have \( \Omega = \Omega(\mu(\Omega)) \).

For a fibrewise entourage uniformity \( \Omega \) on \( X \) and a fibrewise covering uniformity \( \mu \) on \( X \), let \( \tau(\Omega) \) be the fibrewise topology induced by \( \Omega \) ([5] Section 13) and \( \tau(\mu) \) be the fibrewise topology induced by \( \mu \) ([7] Proposition 3.8). Then \( \tau(\Omega) = \tau(\mu(\Omega)) \) and \( \tau(\mu) = \tau(\Omega(\mu)) \) ([7] Proposition 3.12).
3. Fibrewise covering uniformities on TM-maps

For a TM-map $p : X \to B$ parallel to a metric space $(M, \rho)$, let $e : X \to B \times M$ be the embedding. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the family $\{U(x, \frac{1}{n}) | x \in M\}$, where $U(x, \frac{1}{n}) = \{y \in M | \rho(x, y) < \frac{1}{n}\}$ and $W_n = \{e^{-1}(B \times U) | U \in \mathcal{U}_n\}$. Then for each $W \in \tau$, let $\mu_W = \{U | \bigcup U = X_W\}$ and for each $b \in W$ there exists $n \in \mathbb{N}$ and $W' \in N(b)$ with $W' \subset W$ such that $\mathcal{W}_n|X_{W'} < U$.

Since $\mu_W$ and $\mu$ constructed above are induced by the metric $\rho$ on $M$ (on $X$), we call this $\mu = \{\mu_W\}$ a fibrewise covering uniformity on $X$ induced by the metric $\rho$, and denoted by $\mu = \{\mu_W\}_\rho$. Further, by the construction of $\{\mathcal{W}_n | n \in \mathbb{N}\}$ in the above, we say that the family $\{\mathcal{W}_n | n \in \mathbb{N}\}$ is the standard developable covering (sd-covering, for short) on $X$ induced by $\rho$. (Note that we exclusively use the notation $\{\mathcal{W}_n | n \in \mathbb{N}\}$ as sd-covering induced by $\rho$ in this paper.)

**Theorem 3.1.** For a TM-map $p : X \to B$ with a $T$-metric $\rho$, the system $\mu = \{\mu_W\}_\rho$ is a fibrewise covering uniformity on $X$ induced by $\rho$.

4. Equivalence of some completeness on TM-maps

**Definition 4.1.** ([5] Definition 14.1) For a map $p : X \to B$, let $\Omega$ be a fibrewise entourage uniformity on $X$.

1. A subset $D$ of $X$ is said to be $D$-small, where $D \subset X^2$, if $M^2$ is contained in $D$.
2. A $b$-filer $\mathcal{F}$, where $b \in B$, is Cauchy if $\mathcal{F}$ contains a $D$-small members for each $D \in \Omega$. (We call $\mathcal{F}$ $J$-Cauchy with respect to $\Omega$ (w.r.t. $\Omega$, for short), for convenience' sake.)

We shall define a new notion of Cauchy $b$-filter in fibrewise covering uniformity $\mu = \{\mu_W\}$ on $X$.

**Definition 4.2.** For a map $p : X \to B$, let $\mu = \{\mu_W\}$ be a fibrewise covering uniformity on $X$. A $b$-filer $\mathcal{F}$, where $b \in B$, is Cauchy if for each $W \in N(b)$ and $U \in \mu_W$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $F \subset U$. (We call $\mathcal{F}$ $CU$-Cauchy with respect to $\mu$ (w.r.t. $\mu$, for short), for convenience' sake.)

**Theorem 4.3.** For a map $p : X \to B$, let $\Omega$ be a fibrewise entourage uniformity on $X$. Then for each $b \in B$, a $b$-filer $\mathcal{F}$ is $J$-Cauchy w.r.t. $\Omega$ if and only if it is $CU$-Cauchy w.r.t. $\mu(\Omega)$.

For a space $X$, let $\Upsilon = \{\Phi_\alpha | \alpha \in \Lambda\}$ be a family of families of subsets of $X$. We say that a family $\Psi$ of subsets of $X$ is subordinated to the family $\Upsilon$ if for each $\alpha \in \Lambda$ there exists $U_\alpha \in \Phi_\alpha$ and $V \in \Psi$ such that $V \subset U_\alpha$.
Definition 4.4. Let \( p : X \rightarrow B \) be a \( TM \)-map with a \( T \)-metric \( \rho \).

1. ([11]) The map \( p \) is complete if for any \( b \)-filter \( \mathcal{F} \), \( b \in B \), on \( X \) subordinated to the \( sd \)-covering \( \{ \mathcal{W}_n | n \in \mathbb{N} \} \) induced by \( \rho \), it has adherence points. (We call this “complete” \( P \)-complete, and also call this \( b \)-filter satisfying this condition \( P \)-Cauchy w.r.t. \( \rho \).)

2. ([5] Definition 14.10) The map \( p \) is complete if for each \( b \in B \) any \( J \)-Cauchy \( b \)-filter \( \mathcal{F} \) w.r.t. \( \Omega(\mu_{\rho}) \) converges. (We call this “complete” \( J \)-complete.)

Theorem 4.5. For a \( TM \)-map \( p : X \rightarrow B \) with a \( T \)-metric \( \rho \) and each \( b \in B \), a \( b \)-filter \( \mathcal{F} \) is a \( P \)-Cauchy w.r.t. \( \rho \) if and only if it is a \( J \)-Cauchy w.r.t. \( \Omega_{\rho} \).

5. Complete \( TM \)-maps and \( \check{C} \)ech-complete maps

Definition 5.1. A \( T_2 \)-compactifiable map \( p : X \rightarrow B \) is \( \check{C} \)ech-complete if for each \( b \in B \), there exists a countable family \( \{ \mathcal{A}_n \}_{n \in \mathbb{N}} \) of open (in \( X \)) covers of \( X_b \) with the property that every \( b \)-filter \( \mathcal{F} \) which is subordinated to the family \( \{ \mathcal{A}_n \}_{n \in \mathbb{N}} \) has an adherence point.

Proposition 5.2. (1) ([1] Theorem 6.1) Every locally compact map is \( \check{C} \)ech-complete.

2. ([1] Theorem 4.1) For \( T_2 \)-compactifiable maps \( p : X \rightarrow B \), \( q : Y \rightarrow B \) and a perfect morphism \( f : p \rightarrow q \), \( p \) is \( \check{C} \)ech-complete if and only if \( q \) is \( \check{C} \)ech-complete.

Lemma 5.3. Every \( TM \)-map \( p : X \rightarrow B \) is a \( T_{3\frac{1}{2}} \)-map.

By this lemma, every \( TM \)-map is \( T_{3\frac{1}{2}} \)-compactifiable. For complete \( TM \)-maps, we can prove the following.

Theorem 5.4. If \( p : X \rightarrow B \) is a complete \( TM \)-map, then \( p \) is \( \check{C} \)ech-complete.

6. \( MT \)-maps and some problems

About the relations of \( TM \)-maps and \( MT \)-maps, we have the following.

(a) A closed \( TM \)-map is an \( MT \)-map.

(b) There exists a compact \( MT \)-map which is not a \( TM \)-map.

(c) There exists (complete) \( TM \)-maps which are not closed, so not \( MT \)-maps.

Theorem 6.1. If \( p : X \rightarrow B \) is a closed \( TM \)-map, then \( p \) is an \( MT \)-map.
As discussed in section 5, there seems to exist many problems about relations between metrizable maps and completeness. As an attempt to the problems, we define a new notion of $D$-complete $MT$-maps. For an $MT$-map $p:X \rightarrow B$, we use the following notation: $\{\{U_n(b)\}_{n \in \mathbb{N}}|b \in B\}$ is a $p$-development, where $\{U_n(b)\}_{n \in \mathbb{N}}$ is a $b$-development. First, we recall some definitions and theorems of $MT$-maps according to [3].

**Definition 6.2.** (1) ([3] Def. 2.8) For a map $p:X \rightarrow B$, a sequence $\{U_n\}_{n \in \mathbb{N}}$ of open (in $X$) covers of $X_b$, $b \in B$, is said to be a $b$-development if for every $x \in X_b$ and every $U \in N(x)$, there exists $n \in \mathbb{N}$ and $W \in N(b)$ such that $x \in st(x, U_n) \cap X_W \subset U$. The map $p$ is said to have a $p$-development if it has a $b$-development for every $b \in B$.

(2) ([3] Def. 2.9) A closed map $p:X \rightarrow B$ is said to be an $MT$-map if it is collectionwise normal and has a $p$-development.

**Definition 6.3.** For an $MT$-map $p:X \rightarrow B$ equipped with $p$-development $\{\{U_n(b)\}_{n \in \mathbb{N}}|b \in B\}$, we call $p$ $D$-complete with respect to the $p$-development if for each $b \in B$ every $b$-filter $\mathcal{F}$ subordinated to $\{U_n(b)\}_{n \in \mathbb{N}}$ has adherence points.

**Problem 6.4.** For an $MT$-map $p:X \rightarrow B$, let $\{\{U_n(b)\}_{n \in \mathbb{N}}|b \in B\}$ be a $p$-development.

1. Is there a fibrewise (covering) uniformity on $X$ related to the $p$-development?
2. If Problem (1) had an affirmative answer, then is the $J$-completion of $p$ w.r.t. the fibrewise (covering) uniformity on $X$ equivalent to $D$-completion?

**References**