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Expanding Ratios, Box counting Dimension and Hausdorff Dimension

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1 Introduction

All spaces in this note are metric spaces and maps are continuous functions. Let $f : X \to X$ be a map of a compactum X. We say that f is *positively expansive* ([12]) if there is an admissible metric d for X and a positive number $c > 0$ such that if $x, y \in X$ and $x \neq y$, then there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) > c$. Note that this property is independent of the choice of metrics for X. We say that f is a *Ruelle expanding map* ([14]) if f is positively expansive and an open onto map. Note that by invariance of domain in $n-$manifolds, if $f : M \to M$ is a positively expansive map, then f is a Ruelle expanding map. We say that f *expands small distances* if there is an admissible metric d for X and $\epsilon > 0$ and $\lambda > 1$ such that if $0 < d(x, y) \leq \epsilon$, then $d(f(x), f(y)) > \lambda d(x, y)$. In this case, we say that $f : (X, d) \to (X, d)$ expands small distances. A map $f : X \to X$ *increases small distances* if there is an admissble metric d for X and $\epsilon > 0$ such that if $0 < d(x, y) \leq \epsilon$, then $d(f(x), f(y)) > d(x, y)$. The above two notions are dependent of the choice of metrics for X.

In [12], by use of the Frink's metrization theorem ([5]), Reddy proved that the following notions are equivalent:

1. $f : X \to X$ is positively expansive.

2. f expands small distances.

3. f increases small distances.

Hence for any onto open map $f : X \to X$, the following notions are equivalent:

1. f is a Ruelle expanding map.

2. f expands small distances.

3. f increases small distances.

In this note, we are interested in "metrics" related to expandability of maps and we investigate more precise expandability of maps as follows. We say that f *expands strictly small distances with an expanding ratio* $\lambda > 1$ if there is an admissible metric d for X and a positive number $\epsilon > 0$ such that if $x, y \in X$ and $d(x, y) \leq \epsilon$, then $d(f(x), f(y)) = \lambda d(x, y)$. In this case, we say that $f : (X, d) \to (X, d)$ expands strictly small distances with an expanding ratio $\lambda > 1$. Let $\mathbb{R}$ denote the real line, and let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{Z}$ the set of all integers.

Example 1.1. *Let* $L : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear map such that* $L(\mathbb{Z}^n) \subset \mathbb{Z}^n$ *and* $|\lambda_i| > 1$, $|\lambda_i| \neq |\lambda_j|$ $(i \neq j)$ *for eigenvalues* λ_i $(i = 1, 2, ..., n)$ *of* L. *If* $f : T^n \to T^n$ *is the map of the* n*-dimensional torus* T^n *induced by* L, *then for the Euclidean metric* ρ *for* T^n, $f : (T^n, \rho) \to (T^n, \rho)$ *expands small distances, but it does not expand strictly small distances with a common expanding ratio.*

In this note, by use of the Alexandroff-Urysohn's metrization theorem we obtain the following theorem which is a more precise result in case of Ruelle expanding maps: If $f : X \to X$ is a Ruelle expanding map of a compactum X and any positive number $s > 1$, then there exists an admissible metric d for X and positive numbers $\epsilon > 0$, λ $(1 < \lambda < s)$ such that if $x, y \in X$ and $d(x, y) \leq \epsilon$, then $d(f(x), f(y)) = \lambda d(x, y)$. For a case of graphs, we obtain that if $f : X \to X$ is a positively expansive map of a graph X (=1-dimensional compact polyhedron), then the same conclusion holds. In these cases, the metrics d satisfy the following equality:

$$\dim_H(X, d) = \underline{D}_d(X) = D_d(X) = \frac{h(f)}{\log \lambda},$$

where $\dim_H(X, d)$, $\underline{D}_d(X)$ and $D_d(X)$ denote the Hausdorff dimension, the lower box-counting dimension and the upper box-counting dimension of the compact metric space (X, d) and $h(f)$ is the topological entropy of f. This implies that such a metric d is a "fractal" metric for X. In fact, we can consider that the compact metric space (X, d) has some sort of local self-similarity with respect to the inverse f^{-1} of f and the similarity ratio $1/\lambda$. Also, we obtain that if $f : X \to X$ is an expanding homeomorphism of a noncompact metric space X, then there exist an admissible metric d for X and a positive number $\lambda > 1$ such that if $x, y \in X$, then $d(f(x), f(y)) = \lambda d(x, y)$.

2 Metrics of Ruelle expanding maps

In this section, we need the following terminology and concepts. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a space X. We assume that each element of any open cover of a space is not an empty set. If $\mathcal{V}$ refines $\mathcal{U}$, then we denote $\mathcal{V} \leq \mathcal{U}$ (e.g. see [9] and [10]). Suppose that $x \in X$ and $\mathcal{U}$ is an open cover of X. Then we denote

$$St(x, \mathcal{U}) = \bigcup \{U \in \mathcal{U} |\ x \in U\}.$$

We put

$$\mathcal{U}^\Delta = \{St(x, \mathcal{U}) \mid x \in X\}.$$

An open cover $\mathcal{V}$ of X is a *delta-refinement* of an open cover $\mathcal{U}$ of X if $\mathcal{V}^\Delta \leq \mathcal{U}$. Let $\{\mathcal{U}_i\}_{i=1}^\infty$ be a sequence of open covers of X. Then $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *normal delta-sequence* (e.g. see [9] and [10]) if $\mathcal{U}_{i+1}$ is a delta-refinement of $\mathcal{U}_i$ $(i = 1, 2, ...,)$. Also, $\{\mathcal{U}_i\}_{i=1}^\infty$ is called a *development* of X if $\{St(x, \mathcal{U}_i) |\ i = 1, 2, ..., \}$ is a neighborhood base for each point x of X. The following theorem is well known as the Alexandroff-Urysohn's metrization theorem (e.g. see [2], [9] and [10]).

Theorem 2.1. (the Alexandroff-Urysohn's metrization theorem [2]) *A T_1-space X is metrizable if and only if there exists a sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of open covers of X such that $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a normal delta-sequence and a development of X.*

In this section, by use of the construction of the Alexandroff-Urysohn's metrics we obtain the theorem which is a more precise result in case of Ruelle expanding maps. For the proof of Theorem 2.5, we need the following propositions.

Proposition 2.2. *Let X be a compactum and let $f : X \to X$ be a local embedding. Then there exists $k \in \mathbb{N}$ such that f is at most k-to-1 map.*

Let (X, d) be a metric space and $x \in X$. Also, let $U_\epsilon(x)$ be the ϵ neighborhood of x in X, i.e., $U_\epsilon(x) = \{y \in X |\ d(y, x) < \epsilon\}$.

Proposition 2.3. *Let $f : X \to X$ be a map of a compactum (X, d). Suppose that $\mathcal{W}$ is an open cover of X such that for each $x \in X$, there exists $W \in \mathcal{W}$ such that $f^{-1}(x) \subset W$. Then there is a positive number $r > 0$ such that if A is a subset of X with* $\mathrm{diam}(A) \le r$, *then there exists $W \in \mathcal{W}$ with $f^{-1}(A) \subset W$.*

Proposition 2.4. (Reddy [10, p.330, Construction Lemma]) *Let (X, d) be a compact metric space and $f : X \to X$ a positively expansive map with an expansive constant $c > 0$. Then for each positive number $r < c$, there exists a natural number $N(r) \in \mathbb{N}$ such that*

$$r \le d(x, y) \le c \ (x, y \in X) \Rightarrow \max\{d(f^i(x), f^i(y)) \mid 0 \le i \le N(r) - 1\} > c.$$

Theorem 2.5. *Let $f : X \to X$ be a Ruelle expanding map of a compactum X. For any $s > 1$, there exist an admissible metric $\tilde{d}$ for X and a positive number λ $(s > \lambda > 1)$ such that $f : (X, \tilde{d}) \to (X, \tilde{d})$ expands strictly small distances with the expanding ratio λ, that is, for some $\epsilon > 0$,*

$$\tilde{d}(x, y) \le \epsilon \ (x, y \in X) \Rightarrow \tilde{d}(f(x), f(y)) = \lambda \tilde{d}(x, y).$$

Generally, we have the following problem.

Problem 2.6. *Does Positively expansive maps expand strictly small distances?*

In a case of graphs, we obtain the following partial answer to Problem 2.6.

Theorem 2.7. *Let $f : X \to X$ be a positively expansive map of a compact connected graph $X = G$ (=1-dimensional compact polyhedron). Then for any $s > 1$, there exist an admissible metric $\tilde{d}$ for X and positive numbers $\epsilon > 0$, $s > \lambda > 1$ such that*

$$\tilde{d}(x, y) \le \epsilon \ (x, y \in X) \Rightarrow \tilde{d}(f(x), f(y)) = \lambda \tilde{d}(x, y).$$

3 Expanding homeomorphisms of noncompact metric spaces

In this section, we deal with the case of noncompact metric spaces. We obtain the following theorem (cf. Example 1.1).

Theorem 3.1. *Let (X, d) be a (noncompact) metric space. If $f : (X, d) \to (X, d)$ is an expanding homeomorphism, that is, there is $\lambda > 1$ such that $d(f(x), f(y)) \geq \lambda d(x, y)$ for $x, y \in X$, then for any $s > 1$ there is an admissible metric $\tilde{d}$ for X and a positive number r $(s > r > 1)$ such that $f : (X, \tilde{d}) \to (X, \tilde{d})$ expands strictly distances with the expanding ratio r, that is, for any $x, y \in X$,*

$$\tilde{d}(f(x), f(y)) = r\tilde{d}(x, y).$$

Remark 3.2. (Alexandroff-Urysohn's metrization theorem [7, Theorem 2.16]) It follows that D and d' in the proof of Theorem 3.1 satisfy the following condition: For any $x, y \in X$,

$$\frac{1}{4}D(x, y) \leq d'(x, y) \leq D(x, y).$$

Remark 3.3. There is the following relations between the given metric d of Theorem 3.1 and the metric d' in the proof of Theorem 3.1:

(a) There are $A > 0$ and $\alpha > 0$ such that if $d(x, y) \geq 1/2$ then

$$d'(x, y) \leq Ad(x, y)^{\alpha}.$$

(b) There are $B > 0$ and $\beta > 0$ such that if $d(x, y) < 1/2$ then

$$d'(x, y) \geq Bd(x, y)^{\beta}.$$

4 Topological entropy of Ruelle expanding maps and upper box-counting dimension

In this section, we study the dynamical property which is related to Ruelle expanding map, positively expansive map, topological entropy and box-counting dimension. For a map $f : X \to X$ of a compactum X, we define the topological entropy $h(f)$ of f as follows (see [1] and [6]): Let n be a natural number and $\epsilon > 0$. A subset F of X is an (n, ϵ)-*spanning set* for f if for each $x \in X$, there is $y \in F$ such that

$$\max\{d(f^i(x), f^i(y)) \mid 0 \leq i \leq n-1\} \leq \epsilon.$$

Let $r_n(f, \epsilon)$ be the smallest cardinality of all (n, ϵ)-spanning sets for f. A subset E of X is an (n, ϵ)-*separated set* for f if for each $x, y \in E$ with $x \neq y$, there is $0 \leq j \leq n-1$ such that

$$d(f^j(x), f^j(y)) > \epsilon.$$

Let $s_n(f,\epsilon)$ be the maximal cardinality of all (n,ϵ)-separated sets for f. Put

$$r(f,\epsilon) = \limsup_{n\to\infty}\ (1/n)\log r_n(f,\epsilon)$$

and

$$s(f,\epsilon) = \limsup_{n\to\infty}\ (1/n)\log s_n(f,\epsilon).$$

Also, put

$$h(f) = \lim_{\epsilon\to 0} r(f,\epsilon).$$

It is well known that $h(f) = \lim_{\epsilon\to 0} s(f,\epsilon)$ and $h(f)$ is equal to the topological entropy of f which was defined by Adler, Konheim and McAndrew (see [1]).

Let (X,d) be a compact metric space and $b(\epsilon)$ the minimum cardinality of a covering of X by ϵ-balls. Put

$$D_d(X) = \limsup_{\epsilon\to 0} \frac{\log b(\epsilon)}{|\log \epsilon|} \in \mathbb{R}\cup\{\infty\}.$$

Similarly, put

$$\underline{D}_d(X) = \liminf_{\epsilon\to 0} \frac{\log b(\epsilon)}{|\log \epsilon|} \in \mathbb{R}\cup\{\infty\}.$$

$D_d(X)$ is called the upper box-counting dimension of (X,d), and $\underline{D}_d(X)$ is called the lower box-counting dimension of (X,d).

Let $p \geq 0$ be any real number. Given $\epsilon > 0$, let

$$m_p^\epsilon(X,d) = \inf \Sigma_{i=1}^\infty [\operatorname{diam}(A_i)]^p$$

where $X = \cup_{i=1}^\infty A_i$ is any decomposition of X in a countable number of subset of diameter less than ϵ. Let

$$m_p(X,d) = \sup_{\epsilon>0} m_p^\epsilon(X,d).$$

Finally, we denote by the Hausdorff dimension $\dim_H(X,d)$ of (X,d) the supremum of all real numbers p such that $m_p(X,d) > 0$. It is well known that $\dim X \leq \dim_H(X,d) \leq \underline{D}_d(X) \leq D_d(X)$.

Proposition 4.1. (cf. [7], Theorem 3.2.9) *Let $f : X \to X$ be a map of a compactum X with a metric d. Suppose that there exist positive numbers $\epsilon > 0$ and $1 < \lambda_2 \leq \lambda_1$ such that if $x, y \in X$ and $0 < d(x,y) \leq \epsilon$, then $\lambda_2 d(x,y) \leq d(f(x), f(y)) \leq \lambda_1 d(x,y)$. Then the following inequalities hold*

$$D_d(X)\log\lambda_2 \leq h(f) \leq D_d(X)\log\lambda_1.$$

Dai-Zhou-Geng [4] and Misiurewicz [8] proved that the following interesting result.

Theorem 4.2. (Dai-Zhou-Geng [4] and Misiurewicz [8]) *If $f : X \to X$ is a Lipshitz continuous map of a compactum (X,d) with Lipshitz constant λ, then the following equality holds*

$$\frac{h(f)}{\log\lambda} \leq \dim_H(X,d).$$

Now, we obtain the following result.

Theorem 4.3. *Let $f : X \to X$ be a map of a compactum X with a metric d. Suppose that there exist positive numbers $\epsilon > 0$ and $\lambda > 1$ such that if $x, y \in X$ and $d(x,y) \le \epsilon$, then $d(f(x), f(y)) = \lambda d(x,y)$. Then the following equality holds*

$$h(f) = D_d(X) \log \lambda.$$

In particular, the followings hold.

1. *If $f : X \to X$ is a Ruelle expanding map of a compactum X and $s > 1$, then there exist an admissible metric d for X and a positive number $1 < \lambda \le s$ such that $f : (X,d) \to (X,d)$ expands strictly small distances with the expanding ratio λ, and hence*

$$\dim_H(X,d) = \underline{D}_d(X) = D_d(X) = \frac{h(f)}{\log \lambda}.$$

2. *If $f : G \to G$ is a positively expansive map of a gragh G and $s > 1$, then there exist an admissible metric d for G and a positive number $1 < \lambda \le s$ such that $f : (G,d) \to (G,d)$ expands strictly small distances with the expanding ratio λ, and hence*

$$\dim_H(G,d) = \underline{D}_d(G) = D_d(G) = \frac{h(f)}{\log \lambda}.$$

Remark 4.4. In [9], Pontrjagin and Schnirelmann proved that for any compactum X,

$$\dim X = \min\{\underline{D}_d(X) \mid d \text{ is a metric for } X\},$$

where $\dim X$ denotes the topological dimension of X. Suppose that $\dim X \ge 1$ and a map $f : (X,d) \to (X,d)$ expands strictly small distances with an expanding ratio $\lambda > 1$. Then $0 < \log \lambda \le h(f)/\dim X$, which implies that the set of expanding ratios of f are bounded. Note that there exist a sequence $\{d_i\}_{i=1}^{\infty}$ of metrics for X such that $f : (X, d_i) \to (X, d_i)$ expands strictly small distances with an expanding ratio λ_i satisfying $\lambda_i > \lambda_{i+1}$ and $\lim_{i\to\infty} \lambda_i = 1$. Then $\lim_{i\to\infty} D_{d_i}(X) = \infty$, which implies that d_i is a "fractal" metric on X. In fact, we can consider that the space (X, d_i) has some sort of local self-similarity with respect to the inverse f^{-1} of f and the similarity ratio $1/\lambda_i$. In [5], we investigated the relation between metrics d, box-counting dimensions $\underline{D}_d(X)$ and $D_d(X)$ of a separable metric space (X,d).

The topological entropy of endmorphisms of the n-dimensional torus T^n is well known and hence we have the following.

Corollary 4.5. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $L(\mathbb{Z}^n) \subset \mathbb{Z}^n$ and $|\lambda_i| > 1$ for each eigenvalue λ_i $(i = 1, 2, ..., n)$ of L. Then the followings hold.*

1. *For any $s > 1$, there exists an admissible metric d for $\mathbb{R}^n$ and a positive number λ with $s > \lambda > 1$ such that if $x, y \in \mathbb{R}^n$, then $d(L(x), L(y)) = \lambda d(x,y)$.*

2. Let T^n be the n-dimensional torus and let $f : T^n \to T^n$ be the map induced by the linear map L. Then for any $s > 1$, there exists an admissible metric d for T^n and positive numbers $\epsilon > 0$ and $1 < \lambda < s$ such that if $x, y \in T^n$ and $d(x,y) \leq \epsilon$, then $d(f(x), f(y)) = \lambda d(x,y)$. Also,

$$\Sigma_{i=1}^{n} \log|\lambda_i| = \Sigma_{|\lambda_i|>1} \log|\lambda_i| = h(f) = D_d(X) \log\lambda$$

and hence

$$\dim_H(X,d) = \underline{D}_d(X) = D_d(X) = \frac{\Sigma_{i=1}^{n} \log|\lambda_i|}{\log\lambda}.$$

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