Homotopy type of the box complexes of graphs without 4-cycles

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A graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a finite set and $E(G)$ is a family of 2-elements subsets of $V(G)$. We assume that graphs are connected. We follow [3] with respect to the standard notation in graph theory. For a graph $G$, an abstract simplicial complex $B(G)$ which is called the box complex of $G$ is introduced by J. Matoušek and G. M. Ziegler in [5]. We define the box complex of a graph following [5].

Let $G$ be a graph and $U$ a subset of $V(G)$. A vertex $v \in V(G)$ which is adjacent to each $u \in U$ is called a common neighbor of $U$ in $G$. The set of all common neighbors of $U$ in $G$ is denoted by $CN_{G}(U)$. For convenience, we define $CN_{G}(\phi) = V(G)$. For $U_{1}, U_{2} \subseteq V(G)$ such that $U_{1} \cap U_{2} = \phi$, we define $G[U_{1}, U_{2}]$ as the bipartite subgraph of $G$ with

$$V(G[U_{1}, U_{2}]) = U_{1} \cup U_{2} \text{ and } E(G[U_{1}, U_{2}]) = \{u_{1}u_{2} | u_{1} \in U_{1}, u_{2} \in U_{2}, u_{1}u_{2} \in E(G)\}.$$ 

The graph $G[U_{1}, U_{2}]$ is said to be complete if $u_{1}u_{2} \in E(G)$ for all $u_{1} \in U_{1}$ and $u_{2} \in U_{2}$. For convenience, $G[\phi, U_{2}]$ and $G[U_{1}, \phi]$ are also said to be complete.

Let $U_{1}, U_{2}$ be subsets of $V(G)$. The subset $U_{1} \uplus U_{2}$ of $V(G) \times \{1, 2\}$ is defined as

$$U_{1} \uplus U_{2} := (U_{1} \times \{1\}) \cup (U_{2} \times \{2\}).$$

For vertices $u_{1}, u_{2} \in V(G)$, $\{u_{1}\} \uplus \phi$, $\phi \uplus \{u_{2}\}$, and $\{u_{1}\} \uplus \{u_{2}\}$ are simply denoted by $u_{1} \uplus \phi$, $\phi \uplus u_{2}$ and $u_{1} \uplus u_{2}$ respectively.

The box complex of a graph $G$ is an abstract simplicial complex with the vertex set $V(G) \times \{1, 2\}$ and the family of simplices

$$B(G) = \{U_{1} \uplus U_{2} | U_{1}, U_{2} \subseteq V(G), U_{1} \cap U_{2} = \phi, G[U_{1}, U_{2}] \text{ is complete, } CN_{G}(U_{1}) \neq \phi \neq CN_{G}(U_{2})\}.$$ 

An abstract simplex $U_{1} \uplus U_{2}$ and its geometric simplex are denoted by the same symbol $U_{1} \uplus U_{2}$. The simplicial isomorphism $\nu : V(B(G)) \rightarrow V(B(G))$ is defined by

$$u \uplus \phi \mapsto \phi \uplus u \text{ and } \phi \uplus u \mapsto u \uplus \phi$$

for each $u \in V(G)$. This induces a homeomorphism on $\|B(G)\|$ satisfying $\nu \circ \nu = \text{id}_{\|B(G)\|}$. Moreover, we notice that this homeomorphism has no fixed point. In general, a homeomorphism $\nu$ on a topological space $X$ satisfying $\nu \circ \nu = \text{id}_{X}$ is called the $\mathbb{Z}_{2}$-action on $X$ and the pair $(X, \nu)$ is called the $\mathbb{Z}_{2}$-space. For two $\mathbb{Z}_{2}$-spaces $(X, \nu_{X})$ and $(Y, \nu_{Y})$, a continuous map $f : X \rightarrow Y$ satisfying $f \circ \nu_{X} = \nu_{Y} \circ f$ is called a $\mathbb{Z}_{2}$-map from $X$ to $Y$. We define the $\mathbb{Z}_{2}$-index of a $\mathbb{Z}_{2}$-space $(X, \nu)$ as

$$\text{ind}_{\mathbb{Z}_{2}}(X, \nu) := \min\{n | \text{there is a } \mathbb{Z}_{2}\text{-map } f : X \rightarrow S^{n}\},$$

where $S^{n} = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$ with the $\mathbb{Z}_{2}$-action on $S^{n}$ given by $x \mapsto -x$. If there exists a $\mathbb{Z}_{2}$-map from $X$ to $Y$, then we have $\text{ind}_{\mathbb{Z}_{2}}(X) \leq \text{ind}_{\mathbb{Z}_{2}}(Y)$.

In [5], J. Matoušek and G. M. Ziegler pointed out the following:

(1) For any graph $G$,
\[ \text{ind}_{\mathbb{Z}_2} (\|B(G)\|) \leq \chi(G) - 2, \]

where \( \chi(G) \) is the chromatic number of \( G \).

(2) If a graph \( G \) has no 4-cycle, there is a \( \mathbb{Z}_2 \)-retraction of \( \|\text{sd} \, B(G)\| \) onto a 1-dimensional subcomplex \( \|L\| \) of \( \|\text{sd} \, B(G)\| \) defined in [5], p.81, (H1). Then, we have \( \text{ind}_{\mathbb{Z}_2} (\|B(G)\|) \leq 1 \). This indicates that the difference between \( \text{ind}_{\mathbb{Z}_2} (\|B(G)\|) \) and \( \chi(G) - 2 \) can be arbitrarily large.

Let \( \overline{G} \) be the following 1-dimensional subcomplex of \( B(G) \):

\[ \overline{G} := \{ u \cup \phi, v \cup \phi, \phi \cup u, \phi \cup v, u \cup v, v \cup u \mid uv \in E(G) \}. \]

Then, \( \|\overline{G}\| \) is the \( \mathbb{Z}_2 \)-space with the restriction of the \( \mathbb{Z}_2 \)-action on \( \|B(G)\| \). This \( \mathbb{Z}_2 \)-action also has no fixed point. The preceding 1-dimensional subcomplex \( L \) of \( \text{sd} \, B(G) \) equals to \( \text{sd} \, \overline{G} \).

We are interested in the relation between the combinatorics of \( G \) and the topology of \( \|B(G)\| \). In what follows, we consider the topology of the box complex of a graph without 4-cycles. Such a box complex has the following two properties:

**Lemma 1** ([2], Lemma 4.1). A graph \( G \) contains no 4-cycle if and only if for any simplices \( U_1 \cup U_2 \in B(G) \), we have \( |U_1| \leq 1 \) or \( |U_2| \leq 1 \). For such a graph \( G \), each maximal simplex \( U_1 \cup U_2 \in B(G) \) satisfies \( |U_1| = 1 \) or \( |U_2| = 1 \).

**Lemma 2** ([2], Lemma 4.2). Let \( G \) be a graph without 4-cycles. For any two distinct maximal simplices of \( B(G) \) with nonempty intersection, the intersection is a simplex of \( \overline{G} \).

Let \( X \) be a \( \mathbb{Z}_2 \)-space and \( A \) a \( \mathbb{Z}_2 \)-subspace of \( X \). A strong deformation retraction \( \{ f_t \}_{t \in [0,1]} \) of \( X \) onto \( A \) such that each \( f_t : X \to X \) is a \( \mathbb{Z}_2 \)-map is called a strong \( \mathbb{Z}_2 \)-deformation retraction of \( X \) onto \( A \). Then, we notice that the retraction \( f_t \) of \( X \) onto \( A \) and the inclusion of \( A \) into \( X \) are \( \mathbb{Z}_2 \)-maps, so we have \( \text{ind}_{\mathbb{Z}_2} (X) = \text{ind}_{\mathbb{Z}_2} (A) \).

**Theorem 3** ([2], Theorem 4.3). A graph \( G \) contains no 4-cycle if and only if \( \|\overline{G}\| \) is a strong \( \mathbb{Z}_2 \)-deformation retract of \( \|B(G)\| \).

**Sketch of proof.** If a graph \( G \) contains a 4-cycle \( C_4 \), then \( \|B(C_4)\| (\subseteq \|B(G)\|) \) is the disjoint union of two 3-simplices and \( \|\overline{C_4}\| \) is the disjoint union of two circles, each of which is contractible in \( \|B(G)\| \).

\[ \|B(C_4)\| \]

( The polyhedron \( \|\overline{C_4}\| \) is illustrated with — — .)

**Figure 1. The box complex \( \|B(C_4)\| \)**

Suppose that there is a retraction \( r : \|B(G)\| \to \|\overline{G}\| \). We consider the nullhomotopic loop \( l \) in \( \|B(G)\| \) which goes around one of two circles of \( \|\overline{C_4}\| \). Then, we see that \( r \circ l \) is the circle in \( \|\overline{G}\| \) which must be nullhomotopic. This is impossible since \( \|\overline{G}\| \) is the 1-dimensional complex.

Conversely, we assume that a graph \( G \) has no 4-cycle. Then, by Lemma 1, we can divide all maximal simplices of \( B(G) \) into the two sets of simplices

\[ B_1 = \{ u \cup U \mid u \cup U \text{ is maximal} \} \quad \text{and} \quad B_2 = \{ U \cup v \mid U \cup v \text{ is maximal} \}. \]
The $\mathbb{Z}_2$-action $\nu$ on $\|B(G)\|$ induces a one-to-one correspondence between $B_1$ and $B_2$. For each simplex $v \cup U \in B_2$, we define a strong deformation retraction $\{f_t^v\}_{t \in [0,1]}$ of $v \cup U$ onto $K_v^- := \|\overline{G}\| \cap (v \cup U)$ starting with a collapsing from the free face $\phi \cup U$ of $v \cup U$ (see Figure 2):

![Diagram of the strong deformation retraction](image)

Figure 2. The strong deformation retraction $\{f_t^v\}_{t \in [0,1]}$ of $v \cup U$ onto $K_v^-$.  

For each simplex $U \cup v \in B_2$, a strong deformation retraction of $U \cup v$ onto $K_v^+ := \|\overline{G}\| \cap (U \cup v)$ is defined as $\{f_t^v \circ \nu\}_{t \in [0,1]}$. Let $X_v = (v \cup U) \cup (U \cup v)$, for any $v \in V(G)$. Then, a strong $\mathbb{Z}_2$-deformation retraction $F_v$ of $X_v$ onto $K_v^- \cup K_v^+$ is defined as

$$F_v(x,t) = \begin{cases} f_t^v(x) & \text{if } x \in v \cup U, \\ \nu \circ f_t^v \circ \nu(x) & \text{if } x \in U \cup v, \end{cases}$$

where $t \in [0,1]$. Since the homotopies $F_u$ and $F_v$ are stationary on $X_u \cap X_v$ for $u, v \in V(G)$ by Lemma 2, we see that the homotopies $\{F_v \mid v \in V(G)\}$ induce a strong $\mathbb{Z}_2$-deformation retraction of $\|B(G)\|$ onto $\|\overline{G}\|$.

For (2) above, this theorem shows that $\|L\|$ is indeed a strong $\mathbb{Z}_2$-deformation retract of $\|B(G)\|$ if $G$ contains no 4-cycle. The theorem also shows that the converse of this also holds and that we have $\text{ind}_{\mathbb{Z}_2}(\|B(G)\|) = \text{ind}_{\mathbb{Z}_2}(\|L\|) = \text{ind}_{\mathbb{Z}_2}(\|\overline{G}\|)$. On the other hand, $\|\overline{G}\|$ is the 1-dimensional complex with the $\mathbb{Z}_2$-action which has no fixed point, so that we have $\text{ind}_{\mathbb{Z}_2}(\|\overline{G}\|) \leq 1$. The homotopy type of $\|\overline{G}\|$ and the $\mathbb{Z}_2$-index of $\|\overline{G}\|$ are determined by the following theorem:

**Theorem 4 ([1], Theorem 4.4).** Let $G$ be a connected graph with $k$ induced cycles of $G$.

1. If $G$ has no cycle of odd length, we have $\|\overline{G}\| \simeq \bigvee_k S^1 \cup \bigvee_k S^1$ and $\text{ind}_{\mathbb{Z}_2}(\|\overline{G}\|) = 0$.
2. If $G$ has at least one cycle of odd length, we have $\|\overline{G}\| \simeq \bigvee_{2k-1} S^1$ and $\text{ind}_{\mathbb{Z}_2}(\|\overline{G}\|) = 1$.  

As a conclusion, if a graph $G$ contains no 4-cycle, the homotopy type of $\|B(G)\|$ and the $\mathbb{Z}_2$-index of $\|B(G)\|$ is determined by Theorem 3 and 4.

**Corollary 5 ([2], Corollary 4.5).** Let $G$ be a graph without 4-cycles and $k$ the number of induced cycles of $G$.

1Let $\|K\|$ be an $n$-dimensional simplicial complex with a $\mathbb{Z}_2$-action which has no fixed point, then we have $\text{ind}_{\mathbb{Z}_2}(\|K\|) \leq n$ (see [4], p.96).
(1) If $G$ has no cycle of odd length, we have $\|B(G)\| \simeq \bigvee_k S^1 \amalg \bigvee_k S^1$ and $\text{ind}_{\mathbb{Z}_2}(\|B(G)\|) = 0$.

(2) If $G$ has at least one cycle of odd length, we have $\|B(G)\| \simeq \bigvee_{2k-1} S^1$ and $\text{ind}_{\mathbb{Z}_2}(\|B(G)\|) = 1$.

References


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