数理解析研究所講究録
第 1579 巻 2008 年 98-100

On two sufficient conditions for univalency of real coefficient functions

Mamoru Nunokawa

Abstract

It is well known that if the function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is analytic in $|z|<1$ and satisfies one of the following conditions

$$1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)>-\frac{1}{2}\qquad \mathrm{in}|z|<1$$

or

$$1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)<\frac{3}{2}\qquad \mathrm{in}|z|<1,$$

then $f(z)$ is univalent in $|z|<1$. In this paper, we improve the above conditions for the function $f(z)$ whose coefficients are all real.

1. Introduction

Let $\mathcal{A}$ be the set of analytic functions defined in the unit disk $\mathbf{E}=\{z\mid|z|<1\}$ normalized by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

and let

$$\mathcal{S}=\{f(z)\mid f(z)\in\mathcal{A}\text{ and }f(z)\text{ is univalent in }\mathbf{E}\}.$$

The late professor Ozaki [1] proved the following theorem.

Theorem A. *Let $f(z)\in\mathcal{A}$ and if $f(z)$ satisfies one of the following conditions*

(i) $1+\mathrm{Re}\left(\dfrac{zf''(z)}{f'(z)}\right)>-\dfrac{1}{2}\qquad$ in $|z|<1$

or

(ii) $1+\mathrm{Re}\left(\dfrac{zf''(z)}{f'(z)}\right)<\dfrac{3}{2}\qquad$ in $|z|<1,$

then we have $f(z)\in\mathcal{S}$.

2. Theorems

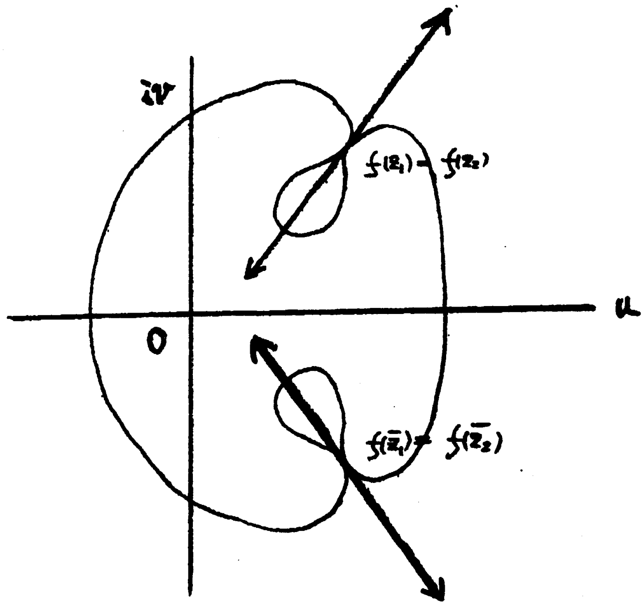
First our theorem is contained in

Theorem 1. *Let $f(z) \in \mathcal{A}$, all the coefficients a_n, $2 \leqq n \in \mathrm{N} = \{1, 2, 3, \cdots\}$ are real and suppose that*

$$(1) \qquad 1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > -1 \qquad (z \in \mathbf{E}).$$

Then we have $f(z) \in \mathcal{S}$.

Proof. Suppose that if there exists a positive real number $r, 0 < r < 1$ for which $f(z)$ is univalent in $|z| < r$ but $f(z)$ is not univalent in $|z| \leqq r$, then from the hypothesis, there exists two points $z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, \theta_1 < \theta_2$ and $\theta_2 - \theta_1 < \pi$ for which $f(z_1) = f(z_2)$.
From the hypothesis (1), we have $f'(z) \neq 0$ in $\mathbf{E}$, because if $f'(z)$ has a zero in $\mathbf{E}$, then it is impossible that $f(z)$ satisfies the condition (1).
Let us put
$C = \{z \mid z = re^{i\theta}, \theta_1 \leqq \theta \leqq \theta_2\}$
and
$C_{f(z)} = \{f(z) \mid z \in C\}$.
Then we have



$$\begin{aligned}
\int_{C_{f(z)}} d\arg df(z) &= -\pi \\
&= \int_C d\arg f'(z)dz \\
&= \int_{\theta_1}^{\theta_2} \left(1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta \\
&> \int_{\theta_1}^{\theta_2} (-1) d\theta = \theta_1 - \theta_2 > -\pi.
\end{aligned}$$

This is a contradiction and therefore it completes the proof.

Theorem 2. *Let $f(z) \in \mathcal{A}$, all the coefficients a_n, $2 \leqq n \in \mathrm{N} = \{1, 2, 3, \cdots\}$ are real and suppose that*

$$(2) \qquad 1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) < 2 \qquad (z \in \mathbf{E}).$$

Then we have $f(z) \in \mathcal{S}$.

Proof. Applying the same method as the proof of Theorem 1, if there exists a positive real number $r, 0 < r < 1$ for which $f(z)$ is univalent in $|z| < r$ but $f(z)$ is not univalent in $|z| \leqq r$, then there are four points such as the proof of Theorem 1, $z_1 = re^{i\theta_1}, z_2 = re^{i\theta_2}, z_3 = re^{i(2\pi - \theta_2)}$ and $z_4 = re^{i(2\pi - \theta_1)}, 0 < \theta_1 < \theta_2 < \pi$ for which we have $f(z_1) = f(z_2)$ and $f(z_3) = f(z_4)$. From the hypothesis, the tangent line at the point $f(z_1)$ and $f(z_2)$ is the common tangent and it is the same for the points $f(z_3)$ and $f(z_4)$.
Therefore, we have

$$\int_{\theta_1}^{\theta_2} \left(1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta = -\pi$$

and

$$\int_{2\pi-\theta_2}^{2\pi-\theta_1} \left(1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta = -\pi,$$

where $z = re^{i\theta}$.
From the hypothesis (2) and the same reason as the proof of Theorem 1, we have $f'(z) \neq 0$ in $\mathbf{E}$.
From the hypothesis (2), we have

$$\int_{|z|=r} \left(1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta = 2\pi$$

$$\begin{aligned}
&= \int_{\theta_1}^{\theta_2} \left(1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta + \int_{\theta_2}^{2\pi-\theta_2} \left(1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta \\
&+ \int_{2\pi-\theta_2}^{2\pi-\theta_1} \left(1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta + \int_{2\pi-\theta_1}^{2\pi+\theta_1} \left(1 + \mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right) d\theta \\
&\quad < -\pi + \int_{\theta_2}^{2\pi-\theta_2} 2d\theta - \pi + \int_{2\pi-\theta_1}^{2\pi+\theta_1} 2d\theta \\
&= \{4\pi - 2(\theta_2 - \theta_1)\} - 2\pi \\
&\quad < 4\pi - 2\pi = 2\pi.
\end{aligned}$$

This is a contradiction and so, it completes the proof.

Remark. A function $f(z) \in \mathcal{A}$ is typically real in $\mathbf{E}$ if $(\mathrm{Im} f(z))\,(\mathrm{Im} z) > 0$ for $\mathbf{E}/\mathbf{R} = \{z \mid z \in \mathbf{E} \cap z \notin \mathbf{R}\}$. In Theorem 1, if $f(z)$ is typically real and satisfies (1), then the conclusion continues to hold true.

References

[1] S. Ozaki, On the theory of multivalent functions II, Sci. Rep. Tokyo Bunrika Daigaku 4(1941), 45 - 86.

Mamoru Nunokawa
Emeritus Professor of University of Gunma
Hoshikuki-Cho 798 - 8
Chuou-Ward, Chiba City 260 - 0808, Japan
E-mail: mamoru_nuno@doctor.nifty.jp