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The Valuation of Callable Russian Options for Double Exponential Jump Diffusion Processes

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1. Introduction

Russian option was introduced by Shepp and Shiryaev [6], [7] and is one of perpetual American lookback options. In Russian option the buyer has the right to exercise it at any time. On the other hand, in callable Russian option not only the buyer but also the seller has the right to cancel it at any time. This option is formulated as coupled optimal stopping problem. See Cvitanic and Karatzas [1] Kifer [2].

Kyprianou [5] derived the closed-form solution in the case where the dividend rate is zero. Suzuki and Sawaki [8] gave the pricing formula with positive dividend. Kou and Wang [4] gave the closed form for the value function of perpetual American put options without dividend and so on. Suzuki and Sawaki [9] derived the pricing formula of non-callable Russian option for double exponential jump diffusion processes.

In this paper, we deal with callable Russian options. A callable Russian option is a contact that the seller and the buyer have the rights to cancel and to exercise at any time, respectively. We present the pricing formula of callable Russian options for double exponential jump diffusion processes. The pricing of such an option can be formulated as a coupled optimal stopping problem which is analyzed as Dynkin game. We derive the value function of a callable Russian option and its optimal boundaries. Also some numerical results are presented to demonstrate analytical sensitives of the value function with respect to parameters.

This paper is organized as follows. In section 2 we introduce a pricing model of callable Russian options by means of a coupled optimal stopping problem given by Kifer [2]. Section 3 presents the value function of callable Russian options for double exponential jump diffusion processes. Section 4 presents numerical examples to verify analytical results. We end the paper with some concluding remarks and future work.

2. Pricing model

In this section we consider the pricing model for the callable Russian option. Let $B(t)$ be the process of the riskless asset price at time t defined by $B(t) = B(0)e^{rt}$, where r is positive interest rate. Let $W(t)$ be a standard Brownian motion and $N(t)$ be a Poisson process with the

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intensity λ. Let J_i denote i.i.d. positive random variables. $Y_i \equiv \log J_i$ has a double exponential distribution and its the density function is given by

$$f(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y<0\}},$$

where $\eta_1 > 1, \eta_2 > 0$ and $0 \leq p, q \leq 1$ such that $p + q = 1$. Under a risk-neutral probability, the process of the risky asset price $S(t)$ at time t satisfies the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu dt + \kappa dW(t) + d(\sum_{i=1}^{N(t)} (J_i - 1)), \tag{2.1}$$

where μ and $\kappa > 0$ are constants. Define another probability measure $\tilde{P}$ as

$$\frac{d\tilde{P}}{dP}\Big|_{\mathcal{F}_t} = \exp\left\{-bW(t) - \frac{1}{2}b^2 t\right\}, \; b = \frac{\mu - r + d + \lambda\zeta}{\kappa},$$

where d is the positive continuous dividend rate of the risky asset, $\mathcal{F}_t = \sigma(W(s), N(s), \{J_i\})$ and

$$\zeta = E[J_i] - 1 = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

By Girsanov's theorem, $\tilde{W}(t) = W(t) - bt$ is a Brownian motion with respect to $\tilde{P}$.

We can rewrite (2.1) as

$$\frac{dS(t)}{S(t-)} = (r - d - \lambda\zeta)dt + \kappa d\tilde{W}(t) + d(\sum_{i=1}^{N(t)} (J_i - 1)). \tag{2.2}$$

Solving (2.2) gives $S(t) = S(0) \exp X(t)$, where

$$X(t) = \left(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta\right) t + \kappa\tilde{W}(t) + \sum_{i=1}^{N(t)} Y_i.$$

Let $V(v)$ be a function of class C^2. Then the infinitesimal generator $\mathcal{L}$ of the process $S(t)$ is given by

$$\mathcal{L}V(v) = \frac{1}{2}\kappa^2 v^2 V''(v) + (r - d - \lambda\zeta)vV'(v) + \lambda \int_{-\infty}^{\infty} (V(ve^y) - V(v))f(y)dy$$

for all $v > 0$.

Next we introduce the four real numbers $\beta_1, \beta_2, \beta_3, \beta_4$. Kou and Wang (2003) showed that the equation $G(\theta) = \alpha$ for all $\alpha > 0$ has the solutions β_1, $\beta_2, -\beta_3, -\beta_4$, where

$$G(\theta) = \theta\left(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta\right) + \frac{1}{2}\theta^2\kappa^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \theta} + \frac{q\eta_2}{\eta_2 + \theta} - 1\right)$$

And the four solutions satisfy

$$0 < \beta_1 < \eta_1 < \beta_2 < \infty, \quad 0 < \beta_3 < \eta_2 < \beta_4 < \infty.$$

Remark 2.1. *When the dividend rate $d = 0$, $\beta_1 = 1$.*

Define the process

$$\Psi(t) \equiv \max(vs, \sup_{0\le u\le t} S(u))/S(t), \quad S(0)=s, v\ge 1.$$

Then the value function of non-callable Russian options is given by

$$V_R(v) = \sup_{\tau} \tilde{E}[e^{-r\tau}\Psi(\tau) \mid \Psi(0)=v],$$

where the supremum is taken for all stopping times τ.

Theorem 2.1. *(Suzuki and Sawaki [9]) The value function $V(v)$ of Russian option is given by*

$$V_R(v) = \begin{cases} A(v_1)v^{\beta_1} + B(v_1)v^{\beta_2} + C(v_1)v^{-\beta_3} + D(v_1)v^{-\beta_4}, & 1\le v\le v_1 \\ v, & v\ge v_1. \end{cases}$$

The coefficients are determined by

$$\begin{aligned} A(v_1) &= \frac{(\eta_1-\beta_1)v_1^{-\beta_1}}{(\beta_1+\beta_3)(\beta_2-\beta_1)}\left\{\frac{(\beta_2-1)(\beta_3+1)}{\eta_1-1}v_1 - \frac{(\beta_2+\beta_4)(\beta_4-\beta_3)}{\eta_1+\beta_4}Dv_1^{-\beta_4}\right\} \\ B(v_1) &= \frac{(\beta_2-\eta_1)v_1^{-\beta_2}}{(\beta_2-\beta_1)(\beta_2+\beta_3)}\left\{\frac{(\beta_1-1)(\beta_3+1)}{\eta_1-1}v_1 - \frac{(\beta_1+\beta_4)(\beta_4-\beta_3)}{\eta_1+\beta_4}Dv_1^{-\beta_4}\right\} \\ C(v_1) &= \frac{(\eta_1+\beta_3)v_1^{\beta_3}}{(\beta_1+\beta_3)(\beta_2+\beta_3)}\left\{\frac{(\beta_1-1)(\beta_2-1)}{\eta_1-1}v_1 - \frac{(\beta_1+\beta_4)(\beta_2+\beta_4)}{\eta_1+\beta_4}Dv_1^{-\beta_4}\right\} \end{aligned}$$

and

$$\frac{A(v_1)}{\eta_2+\beta_1} + \frac{B(v_1)}{\eta_2+\beta_2} + \frac{C(v_1)}{\eta_2-\beta_3} + \frac{D(v_1)}{\eta_2-\beta_4} = 0.$$

Moreover, the optimal boundary v_1 is the solution in $(1,\infty)$ to the equation

$$A(v)\beta_1 + B(v)\beta_2 - C(v)\beta_3 - D(v)\beta_4 = 0$$

and the optimal stopping time is given by

$$\hat{\tau} = \inf\{t>0 \mid \Psi(t)\ge v_1\}.$$

3. Callable Russian options

We assume that $p=1$ and $q=0$. It means that the jump is down only. Then we can express $G(\theta)$ as

$$G(\theta) = \theta\left(r-d-\frac{1}{2}\kappa^2 - \lambda\zeta\right) + \frac{1}{2}\theta^2\kappa^2 + \lambda\left(\frac{\eta_1}{\eta_1-\theta}-1\right)$$

and the equation $G(\theta)=r$ has three solutions $\beta_1, \beta_2, -\beta_4$, which satisfy

$$1\le\beta_1<\eta_1<\beta_2<\infty, \quad 0<\beta_4<\infty.$$

Let σ denote a cancel time for the seller and τ an exercise time for the buyer. If the seller cancels the contract, the buyer receives $\Psi(\sigma)+\delta$ from the seller. We can think of $\delta>0$ as the penalty cost for the cancel. On the other hand, if the buyer exercises it, (s)he receives $\Psi(\tau)$ from the seller. Therefore, the payoff function is given by

$$(\Psi(\sigma)+\delta)1_{\{\sigma<\tau\}} + \Psi(\tau)1_{\{\tau\le\sigma\}}.$$

Let $\mathcal{T}_{0,\infty}$ denote the set of all stopping times with values in the interval $[0,\infty]$. Then the value function $V^*(v)$ of the callable Russian option is defined by

$$V^*(v) = \inf_{\sigma\in\mathcal{T}_{0,\infty}} \sup_{\tau\in\mathcal{T}_{0,\infty}} J(\sigma,\tau,v), \tag{3.1}$$

where

$$J(\sigma,\tau,v) = \tilde{E}[e^{-\alpha(\sigma\wedge\tau)}\{(\Psi(\sigma)+\delta)1_{\{\sigma<\tau\}} + \Psi(\tau)1_{\{\tau\le\sigma\}}\} \mid \Psi(0)=v].$$

And the function $V^*(v)$ satisfies the inequalities

$$v \le V^*(v) \le v+\delta,$$

which provides the lower and the upper bounds for the value function of the callable Russian option.

We define two sets A and B as

$$\begin{aligned} A &= \{v\in\mathbf{R}^+ \mid V(v) = v+\delta\} \\ B &= \{v\in\mathbf{R}^+ \mid V(v) = v\}. \end{aligned}$$

A and B are called the seller's cancel region and the buyer's exercise region, respectively. Then the two optimal stopping times are given by

$$\begin{aligned} \sigma_A &= \inf\{t>0 \mid \Psi(t)\in A\}, \\ \tau_B &= \inf\{t>0 \mid \Psi(t)\in B\}. \end{aligned}$$

Then for any v, $\hat\sigma \equiv \sigma_A$ and $\hat\tau \equiv \tau_B$ attain the infimum and supremum in (3.1), *i.e.*, we have

$$V^*(v) = J(\hat\sigma,\hat\tau,v).$$

The pair $(\hat\sigma,\hat\tau)$ is the saddle point of $J(\sigma,\tau,v)$.

Remark 3.1. *The seller minimizes the payoff function and $\Psi(t)\ge\Psi(0)=v\ge 1$. From this, it follows that the seller's optimal cancel region is $\{1\}$.*

Lemma 3.1. *Suppose that $r-d-\frac{1}{2}\kappa^2-\lambda\zeta>0$. Then the function $V(v)$ is Lipschitz continuous and its Radon-Nikodym derivative satisfies*

$$0\le V'(v)\le 1, \quad a.e.\ v. \tag{3.2}$$

Proof. Since $\hat\sigma,\hat\tau$ and $\Psi(t)$ depends on the initial value v, we write them as $\hat\sigma^v,\hat\tau^v$ and $\Psi(t,v)$. Replacing the optimal stopping times $\hat\tau^v$ by another stopping time $\hat\tau^u$, we get the inequalities

$$V(v)\le J(\hat\sigma^u,\hat\tau^v,v), \quad V(u)\ge J(\hat\sigma^u,\hat\tau^v,u).$$

Note that $z_1^+ - z_2^+ \le (z_1-z_2)^+$ for any $z_1,z_2\in\mathbf{R}$. For any $v\ge u$, we have

$$\begin{aligned} 0 &\le V(v)-V(u) \\ &= J(\hat\sigma^u,\hat\tau^v,v) - J(\hat\sigma^u,\hat\tau^v,u) \\ &= \tilde{E}[e^{-\alpha(\hat\sigma^u\wedge\hat\tau^v)}(\Psi(\hat\sigma^u\wedge\hat\tau^v,v) - \Psi(\hat\sigma^u\wedge\hat\tau^v,u))] \\ &= \tilde{E}[e^{-\alpha(\hat\sigma^u\wedge\hat\tau^v)}H^{-1}(\hat\sigma^u\wedge\hat\tau^v)\{(v-\sup H_u)^+ - (u-\sup H_u)^+\}] \\ &\le (v-u)\tilde{E}[e^{-\alpha(\hat\sigma^u\wedge\hat\tau^v)}H^{-1}(\hat\sigma^u\wedge\hat\tau^v)] \\ &\le v-u, \end{aligned}$$

where $H(t)=\exp X(t)$. Therefore, we obtain

$$0\le \frac{V(v)-V(u)}{v-u}\le 1.$$

This means that $V(v)$ is Lipschitz continuous and satisfies (3.2). $\square$

If the penalty δ is too large, the seller never cancels. How large δ is it?

Lemma 3.2. *Set $\delta^* = V_R(1) - 1$. If the penalty $\delta > \delta^*$, the seller never cancels. In other words, the callable Russian option is reduced to Russian option.*

Proof. Consider the function $U(v) = V_R(v) - v - \delta$. Since it holds $U'(v) \leq 0$ by Lemma 3.1 and $U(1) = \delta^* - \delta < 0$, we have $V_R(v) < v + \delta$. Hence, it follows that $V^*(v) < v + \delta$ because it holds that $V^*(v) \leq V_R(v)$. □

We introduce the function for $v_0 = e^{x_0} > 1$

$$V(v) = \begin{cases} Av^{\beta_1} + Bv^{\beta_2} + Cv^{-\beta_4}, & 1 \leq v \leq v_0 \\ v, & v \geq v_0. \end{cases} \tag{3.3}$$

We set $v = e^x$ and $V(v) = V(e^x) \equiv \hat{V}(x)$. In what follows, we determine the coefficients A, B, C and e^{x_0}. In order to determine the coefficients, we prepare the conditions. By value matching condition, we have

$$Ae^{\beta_1 x_0} + Be^{\beta_2 x_0} + Ce^{-\beta_4 x_0} = e^{x_0}$$

and by smooth pasting condition, we have

$$A\beta_1 e^{\beta_1 x_0} + B\beta_2 e^{\beta_2 x_0} - C\beta_4 e^{-\beta_4 x_0} = e^{x_0}.$$

We can get the last condition by using the infinitesimal generator $\hat{\mathcal{L}}$ of the process $X(t)$ given by

$$\hat{\mathcal{L}}\hat{V}(x) = \frac{1}{2}\kappa^2 \hat{V}''(x) + (r - d - \frac{1}{2}\kappa^2 - \lambda\zeta)\hat{V}'(x) + \lambda \int_{-\infty}^{\infty} (\hat{V}(x+y) - \hat{V}(x)) f(y) dy$$

for all $v > 0$. For $x < x_0$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \hat{V}(x+y) f(y) dy \\
&= \int_0^{x_0 - x} (Ae^{\beta_1(x+y)} + Be^{\beta_2(x+y)} + Ce^{-\beta_4(x+y)})\eta_1 e^{-\eta_1 y} dy + \int_{x_0 - x}^{\infty} e^{x+y} \eta_1 e^{-\eta_1 y} dy \\
&= \eta_1 \left(\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x} + \frac{C}{\eta_1 + \beta_4} e^{-\beta_4 x} \right) \\
&\quad - \eta_1 e^{-\eta_1 (x_0 - x)} \left(\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} + \frac{C}{\eta_1 + \beta_4} e^{-\beta_4 x_0} - \frac{e^{x_0}}{\eta_1 - 1} \right).
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
&(\hat{L} - r)\hat{V}(x) \\
&= Ae^{\beta_1 x} \left(\frac{1}{2}\beta_1^2 + \beta_1(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta) \right) + Be^{\beta_2 x} \left(\frac{1}{2}\beta_2^2 + \beta_2(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta) \right) \\
&\quad + Ce^{-\beta_4 x} \left(\frac{1}{2}(-\beta_4)^2 - \beta_4(r - d - \frac{1}{2}\kappa^2 - \lambda\zeta) \right) \\
&\quad + \lambda \int_{-\infty}^{\infty} \hat{V}(x+y) f(y) dy - (\lambda + r)\hat{V}(x) \\
&= Ae^{\beta_1 x} g(\beta_1) + Be^{\beta_2 x} g(\beta_2) + Ce^{-\beta_4 x} g(-\beta_4) \\
&\quad - \lambda p \eta_1 e^{-\eta_1 (x_0 - x)} \left(\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} + \frac{C}{\eta_1 + \beta_4} e^{-\beta_4 x_0} - \frac{e^{x_0}}{\eta_1 - 1} \right),
\end{aligned}$$

where $g(x) = G(-x) - r$. By Lemma 2.1 in Kou and Wang [3], we have $g(\beta_1) = g(\beta_2) = g(\beta_4) = 0$. Since $(\hat{\mathcal{L}} - r)\hat{V}(x) = 0$ holds, we get the condition

$$\frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} + \frac{C}{\eta_1 + \beta_4} e^{-\beta_4 x_0} - \frac{e^{x_0}}{\eta_1 - 1} = 0. \tag{3.4}$$

Lemma 3.3. *Solving the following equations*

$$\begin{aligned} Ae^{\beta_1 x_0} + Be^{\beta_2 x_0} + Ce^{-\beta_4 x_0} &= e^{x_0} \\ A\beta_1 e^{\beta_1 x_0} + B\beta_2 e^{\beta_2 x_0} - C\beta_4 e^{-\beta_4 x_0} &= e^{x_0} \\ \frac{A}{\eta_1 - \beta_1} e^{\beta_1 x_0} + \frac{B}{\eta_1 - \beta_2} e^{\beta_2 x_0} + \frac{C}{\eta_1 + \beta_4} e^{-\beta_4 x_0} &= \frac{e^{x_0}}{\eta_1 - 1} \end{aligned}$$

gives the solutions

$$\begin{aligned} A &= \frac{(\eta_1 - \beta_1)(\beta_2 - 1)(\beta_4 + 1)}{(\eta_1 - 1)(\beta_1 + \beta_4)(\beta_2 - \beta_1)} e^{(1-\beta_1)x_0} \\ B &= \frac{(\beta_2 - \eta_1)(\beta_1 - 1)(\beta_4 + 1)}{(\eta_1 - 1)(\beta_2 + \beta_4)(\beta_2 - \beta_1)} e^{(1-\beta_2)x_0} \\ C &= \frac{(\eta_1 + \beta_4)(\beta_2 - 1)(\beta_1 - 1)}{(\eta_1 - 1)(\beta_1 + \beta_4)(\beta_2 + \beta_4)} e^{(1+\beta_4)x_0}. \end{aligned}$$

Since the coefficients A, B, C depend on x_0, we denote them as $A(x_0), B(x_0)$ and $C(x_0)$. The number $v_0 = e^{x_0}$ given by (3.3) satisfies the equation

$$A(x_0)e^{\beta_1 x_0} + B(x_0)e^{\beta_2 x_0} + C(x_0)e^{-\beta_4 x_0} = \delta + 1.$$

In the remainder of this section, we discuss the case where $p, q > 0$. We set a function $V(v)$

$$V(v) = \begin{cases} Av^{\beta_1} + Bv^{\beta_2} + Cv^{-\beta_3} + Dv^{-\beta_4}, & 1 \le v \le v_0 \\ v, & v \ge v_0. \end{cases}$$

By value matching condition and smooth pasting condition, we have

$$\begin{aligned} Ae^{\beta_1 x_0} + Be^{\beta_2 x_0} + Ce^{-\beta_3 x_0} + De^{-\beta_4 x_0} &= e^{x_0} \\ A\beta_1 e^{\beta_1 x_0} + B\beta_2 e^{\beta_2 x_0} - C\beta_3 e^{-\beta_3 x_0} - D\beta_4 e^{-\beta_4 x_0} &= e^{x_0}, \end{aligned}$$

respectively. For $x_1 < x < x_0$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \hat{V}(x+y) f(y) dy \\ &= \int_{x_1 - x}^{0} (Ae^{\beta_1(x+y)} + Be^{\beta_2(x+y)} + Ce^{-\beta_3(x+y)} + De^{-\beta_4(x+y)}) q\eta_2 e^{\eta_2 y} dy \\ &\quad + \int_{0}^{x_0 - x} (Ae^{\beta_1(x+y)} + Be^{\beta_2(x+y)} + Ce^{-\beta_3(x+y)} + De^{-\beta_4(x+y)}) p\eta_1 e^{-\eta_1 y} dy \\ &\quad + \int_{x_0 - x}^{\infty} e^{x+y} p\eta_1 e^{-\eta_1 y} dy \\ &= q\eta_2 \left(\frac{A}{\eta_2 + \beta_1} e^{\beta_1 x} + \frac{B}{\eta_2 + \beta_2} e^{\beta_2 x} + \frac{C}{\eta_2 - \beta_3} e^{-\beta_3 x} + \frac{D}{\eta_2 - \beta_4} e^{-\beta_4 x} \right) \\ &\quad - q\eta_2 e^{\eta_2(x_1 - x)} \left(\frac{A}{\eta_2 + \beta_1} e^{\beta_1 x_1} + \frac{B}{\eta_2 + \beta_2} e^{\beta_2 x_1} + \frac{C}{\eta_2 - \beta_3} e^{-\beta_3 x_1} + \frac{D}{\eta_2 - \beta_4} e^{-\beta_4 x_1} \right) \end{aligned}$$

$$
\begin{aligned}
&+p\eta_1\left(\frac{A}{\eta_1-\beta_1}e^{\beta_1 x}+\frac{B}{\eta_1-\beta_2}e^{\beta_2 x}+\frac{C}{\eta_1+\beta_3}e^{-\beta_3 x}+\frac{D}{\eta_1+\beta_4}e^{-\beta_4 x}\right)\\
&-p\eta_1 e^{-\eta_1(x_0-x)}\left(\frac{A}{\eta_1-\beta_1}e^{\beta_1 x_0}+\frac{B}{\eta_1-\beta_2}e^{\beta_2 x_0}+\frac{C}{\eta_1+\beta_3}e^{-\beta_3 x_0}+\frac{D}{\eta_1+\beta_4}e^{-\beta_4 x_0}\right)\\
&+p\eta_1 e^{-\eta_1(x_0-x)}\frac{e^{x_0}}{\eta_1-1}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&(\hat{L}-r)\hat{V}(x)\\
=\ & Ae^{\beta_1 x}\left(\frac{1}{2}\beta_1^2+(r-d-\frac{1}{2}\kappa^2-\lambda\zeta)\beta_1\right)\\
&+Be^{\beta_2 x}\left(\frac{1}{2}\beta_2^2+(r-d-\frac{1}{2}\kappa^2-\lambda\zeta)\beta_2\right)\\
&+Ce^{-\beta_3 x}\left(\frac{1}{2}(-\beta_3)^2+(r-d-\frac{1}{2}\kappa^2-\lambda\zeta)(-\beta_3)\right)\\
&+De^{-\beta_4 x}\left(\frac{1}{2}(-\beta_4)^2+(r-d-\frac{1}{2}\kappa^2-\lambda\zeta)(-\beta_4)\right)\\
&+\lambda\int_{-\infty}^{\infty}\hat{V}(x+y)f(y)dy-(\lambda+r)\hat{V}(x)\\
=\ & Ae^{\beta_1 x}g(\beta_1)+Be^{\beta_2 x}g(\beta_2)+Ce^{-\beta_3 x}g(-\beta_3)+De^{-\beta_4 x}g(-\beta_4)\\
&-\lambda p\eta_1 e^{-\eta_1(x_0-x)}\left(\frac{Ae^{\beta_1 x_0}}{\eta_1-\beta_1}+\frac{Be^{\beta_2 x_0}}{\eta_1-\beta_2}+\frac{Ce^{-\beta_3 x_0}}{\eta_1+\beta_3}+\frac{De^{-\beta_4 x_0}}{\eta_1+\beta_4}-\frac{e^{x_0}}{\eta_1-1}\right)\\
&-\lambda q\eta_2 e^{\eta_2(x_1-x)}\left(\frac{Ae^{\beta_1 x_1}}{\eta_2+\beta_1}+\frac{Be^{\beta_2 x_1}}{\eta_2+\beta_2}+\frac{Ce^{-\beta_3 x_1}}{\eta_2-\beta_3}+\frac{De^{-\beta_4 x_1}}{\eta_2-\beta_4}\right).
\end{aligned}
$$

Since $(\hat{\mathcal{L}}-r)\hat{V}(x)=0$ for $x_1<x<x_0$, we can get

$$
\begin{aligned}
\frac{A}{\eta_1-\beta_1}e^{\beta_1 x_0}+\frac{B}{\eta_1-\beta_2}e^{\beta_2 x_0}+\frac{C}{\eta_1+\beta_3}e^{-\beta_3 x_0}+\frac{D}{\eta_1+\beta_4}e^{-\beta_4 x_0}-\frac{e^{x_0}}{\eta_1-1} &= 0\\
\frac{A}{\eta_2+\beta_1}e^{\beta_1 x_1}+\frac{B}{\eta_2+\beta_2}e^{\beta_2 x_1}+\frac{C}{\eta_2-\beta_3}e^{-\beta_3 x_1}+\frac{D}{\eta_2-\beta_4}e^{-\beta_4 x_1} &= 0.
\end{aligned}
$$

Lemma 3.4. *Solving the equations yields*

$$
\begin{aligned}
Ae^{\beta_1 x_0}+Be^{\beta_2 x_0}+Ce^{-\beta_3 x_0}+De^{-\beta_4 x_0} &= e^{x_0}\\
A\beta_1 e^{\beta_1 x_0}+B\beta_2 e^{\beta_2 x_0}-C\beta_3 e^{-\beta_3 x_0}-D\beta_4 e^{-\beta_4 x_0} &= e^{x_0}\\
\frac{A}{\eta_1-\beta_1}e^{\beta_1 x_0}+\frac{B}{\eta_1-\beta_2}e^{\beta_2 x_0}+\frac{C}{\eta_1+\beta_3}e^{-\beta_3 x_0}+\frac{D}{\eta_1+\beta_4}e^{-\beta_4 x_0} &= \frac{e^{x_0}}{\eta_1-1}
\end{aligned}
$$

the solutions

$$
\begin{aligned}
A &= \frac{(\eta_1-\beta_1)e^{-\beta_1 x_0}}{(\beta_1+\beta_3)(\beta_2-\beta_1)}\left\{\frac{(\beta_2-1)(\beta_3+1)}{\eta_1-1}e^{x_0}-\frac{(\beta_2+\beta_4)(\beta_4-\beta_3)}{\eta_1+\beta_4}De^{-\beta_4 x_0}\right\}\\
B &= \frac{(\beta_2-\eta_1)e^{-\beta_2 x_0}}{(\beta_2-\beta_1)(\beta_2+\beta_3)}\left\{\frac{(\beta_1-1)(\beta_3+1)}{\eta_1-1}e^{x_0}-\frac{(\beta_1+\beta_4)(\beta_4-\beta_3)}{\eta_1+\beta_4}De^{-\beta_4 x_0}\right\}\\
C &= \frac{(\eta_1+\beta_3)e^{\beta_3 x_0}}{(\beta_1+\beta_3)(\beta_2+\beta_3)}\left\{\frac{(\beta_1-1)(\beta_2-1)}{\eta_1-1}e^{x_0}-\frac{(\beta_1+\beta_4)(\beta_2+\beta_4)}{\eta_1+\beta_4}De^{-\beta_4 x_0}\right\}.
\end{aligned}
$$

And the solutions of the equations

$$
\begin{aligned}
\frac{A}{\eta_2+\beta_1}e^{\beta_1 x_1}+\frac{B}{\eta_2+\beta_2}e^{\beta_2 x_1}+\frac{C}{\eta_2-\beta_3}e^{-\beta_3 x_1}+\frac{D}{\eta_2-\beta_4}e^{-\beta_4 x_1} &= 0\\
A+B+C+D-\delta-1 &= 0
\end{aligned}
$$

are given by

$$\begin{aligned} A &= -\frac{\eta_2+\beta_1}{\beta_2-\beta_1}\left\{\frac{\beta_2+\beta_3}{\eta_2-\beta_3}C+\frac{\beta_2+\beta_4}{\eta_2-\beta_4}D+\delta+1\right\} \\ B &= \frac{\eta_2+\beta_2}{\beta_2-\beta_1}\left\{\frac{\beta_1+\beta_3}{\eta_2-\beta_3}C+\frac{\beta_1+\beta_4}{\eta_2-\beta_4}D+\delta+1\right\}. \end{aligned}$$

By the above lemma, we can determine the coefficients A, B, C, D and v_0.

4. Main Theorem

In this section we give the main theorem. In order to prove it, we needs the following lemmas.

Lemma 4.1. *Assume that a function $V(v)$ has the following properties.*
1. $(\mathcal{L}-r)V(v) \le 0$, *for* $v > v_0$.
2. *It holds* $(\mathcal{L}-r)V(v) = 0$ *and* $V(x)$ *satisfies* $v < V(v) < v+\delta$ *for* $1 < v < v_0$.
3. *At* $v = v_0$ *we have* $V'(v_0-) = V'(v_0+)$.

Then, V is the value function of callable Russian options with dividend, i.e., *$V^* = V$ holds. The optimal exercise region is the interval $[v_0, \infty)$ and the optimal cancel region is $\{1\}$.*

In what follows we will explore the properties of the function $V(v)$ in Lemma 4.1.

Lemma 4.2. *For $v > v_0$ the function $V(v)$ satisfies*

$$(\mathcal{L}-r)V(v) \le 0.$$

Proof. Since $\hat{V}(x) = e^x$ for $x > x_0$, we have

$$\int_0^\infty \hat{V}(x+y)f(y)dy = \int_0^\infty \eta_1 e^{x+(1-\eta_1)y}dy = \frac{\eta_1 e^x}{\eta_1-1}.$$

Hence, we obtain

$$\begin{aligned} (\hat{\mathcal{L}}-r)\hat{V}(x) &= \frac{1}{2}\kappa^2 e^x + (r-d-\frac{1}{2}\kappa^2-\lambda\zeta)e^x + \frac{\lambda\eta_1}{\eta_1-1}e^x - (\lambda+r)e^x \\ &= -de^x < 0. \end{aligned}$$

That is, it holds $(\mathcal{L}-r)V(v) \le 0$. □

Lemma 4.3. *For $1 < v < v_0$ the function $V(v)$ satisfies $(\mathcal{L}-r)V(v) = 0$ and*

$$v < V(v) < v+\delta.$$

Proof. The former assertion is known. We will show the latter one. The second derivative of $V(v)$ is nonnegative because $\beta_1, \beta_2 > 1$ and $A, B, C > 0$. It follows that V is a convex function. Since $V(v)$ is a convex function, $V'(v)$ is increasing. From this, we can see that $V'(v) < 1$ for $1 < v < v_0$. By the boundary conditions $V(1) = \delta+1$ and $V(v_0) = v_0$, we have $v < V(v) < v+\delta$. □

Lemma 4.4. *Set*

$$h(v) = \delta + 1 - A(v)v^{\beta_1} - B(v)v^{\beta_2} - C(v)v^{-\beta_4}. \tag{4.1}$$

Then the equation $h(v) = 0$ has the unique solution in the interval $(1, \infty)$.

Proof. By (4.1), a direct computation yields

$$\begin{aligned} h(1) &= \delta+1 \\ &- \frac{(\eta_1-\beta_1)(\beta_2-1)(\beta_4+1)}{(\eta_1-1)(\beta_1+\beta_4)(\beta_2-\beta_1)} - \frac{(\beta_2-\eta_1)(\beta_1-1)(\beta_4+1)}{(\eta_1-1)(\beta_2+\beta_4)(\beta_2-\beta_1)} - \frac{(\eta_1+\beta_4)(\beta_2-1)(\beta_1-1)}{(\eta_1-1)(\beta_1+\beta_4)(\beta_2+\beta_4)} \\ &= \delta > 0. \end{aligned}$$

Furthermore, Since $h(\infty) = -\infty, h''(v) < 0$ and $h'(1) = 0$, the equation $h(v) = 0$ has the unique solution in $(1, \infty)$. □

Theorem 4.1. *Let $V^*(v)$ denote the value function of the callable Russian option. If $\delta \geq \delta^*$, the value function is equal to non-callable Russian option,* i.e. $V^*(v) = V_R(v)$. *If $\delta < \delta_*$, then $V^*(v)$ is given by*

$$V(v) = \begin{cases} A(v_0)v^{\beta_1} + B(v_0)v^{\beta_2} + C(v_0)v^{-\beta_4}, & 1 \leq v \leq v_0 \\ v, & v \geq v_0 \end{cases} \tag{4.2}$$

and the optimal stopping times are given by

$$\begin{aligned} \hat{\sigma} &= \inf\{t > 0 \mid \Psi(t) = 1\}, \\ \hat{\tau} &= \inf\{t > 0 \mid \Psi(t) \geq v_0\}. \end{aligned}$$

The optimal boundary v_0 for the buyer is the unique solution to the equation

$$A(v)v^{\beta_1} + B(v)v^{\beta_2} + C(x_0)v^{-\beta_4} = \delta + 1.$$

Moreover, the function $V(v)$ is also represented by

$$V(v) = \tilde{E}[\int_0^\infty e^{-\alpha t}(r - \mathcal{L})V(\Psi(t))dt].$$

5. Numerical example

In this section we present some numerical examples which show that theoretical results are varied and that some effects of the parameters on the price of callable Russian option. We set $r = 0.1,\ d = 0.09,\ \kappa = 0.3,\ p = 1,\ q = 0,\ \eta_1 = 50,\ \lambda = 3$. Using these parameter, δ^* is 0.248.

Figure 1 shows that the optimal exercise boundary as the penalty δ increases from 0.1 up to δ^*. From the figure, we can see that the optimal boundary v_0 is increasing in the penalty δ.

Figure 2 demonstrates the value function of the callable Russian option with jumps. Dashed lines represent $\delta = 0.1, 0.15$ from the bottom. Real line represents $\delta = 0.2$. From this figure, we can recognize that $V(v)$ is convex and increasing in v.

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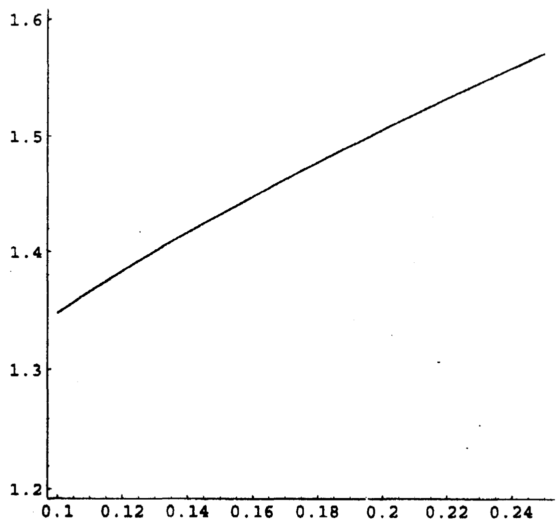

Figure 1: Optimal boundary for the buyer

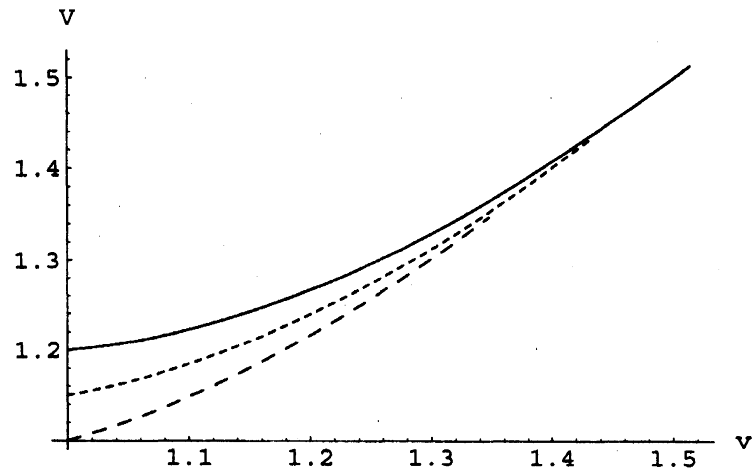

Figure 2: The value function

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