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The Valuation of Callable Currency Linked Bonds

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1 Introduction

A currency linked bond (here after, abbreviated by CLB) is a type of structured bonds issued by a firm. Structured bonds are designed to meet needs of the issuer and the investor. The face value and the coupon payment of the structured bonds are linked to the stock price, the interest rate and the exchange rate. Structured bonds includes Nikkei Stock Average linked bonds, exchageable bonds and CLBs. In this paper we consider the callable CLB that has both the face value and the coupon payment linked to the exchange rate. Since the face value of the CLB is linked to the exchange rate, the original principal may not be guaranteed. However, the buyer of the CLB can receive higher coupon payment than that of straight bond. Also, the callable CLB provides the issuer with the right to call the CLB at any time.

In the early literature on the callable securities, Merton [3] and Brennan and Schwartz [1] are pioneer work. Merton [3] has provided the fundamental model for the valuation of callable warrant, and derived the closed form solution to the perpetual warrant. Brennan and Schwartz [1] has presented an algorithm to solve the differential equation which governs the value of the convertible bond with some boundary conditions. Recently, Kifer [2] has introduced a pricing model for the game option in the setting of a coupled stopping problem with the added feature that the firm can also call the option. Seko and Sawaki [6] has considered the pricing model of callable contingent claims rather in a general framework. Yagi and Sawaki [7] has studied on valuing callable convertible bonds and explored analytical properties for the optimal call and conversion strategies. These studies [6][7] have provided the decompositions for the value of callable securities using the framework in Myneni [4]. Realdon [5] has provided the valuation model of the exchangeable bonds which is a type of structured bonds. There are not many theoretical studies about structured bonds.

In the following section we present a pricing model for the callable CLB under a setting of the optimal stopping problem. In section 3 we explore analytical properties of the optimal call strategies of the issuer. In section 4, we show that the value of the callable CLB can be decomposed into the value of the European CLB and the early callable discounted value using the framework in Myneni [4]. In section 5 we provide some numerical results recognizing some analytical properties of the value of the CLB by using the numerical integration. Finally, in section 6 we summarize this paper with some concluding remarks.

2 The Pricing Model of Currency Linked Bonds

In this section we present a pricing model of CLB. We consider the dual CLB which has coupon payments in domestic currency linked to the exchange rate, and the terminal payoff in either domestic or foreign currency. Under the risk-neutral measure $\tilde{P}$ we suppose that the stochastic differential equation of the exchange rate X_t is given by the well known geometric Brownian motion

$$dX_t = (r_d - r_f)X_t dt + \sigma X_t d\tilde{Z}_t, \tag{2.1}$$

where r_d, r_f and σ are the constant domestic interest rate, the constant foreign interest rate and the constant volatility, respectively, and $\tilde{Z}_t$ is the standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \tilde{P})$.

Let $V(t, X_t)$ denote the value of the callable CLB with the face value F and the maturity T. The CLB holder can receive the continuous coupon payment with coupon rate

$$c(t, X_t) = \max\left(\alpha \frac{X_t}{g} - \beta, 0\right) F, \tag{2.2}$$

where α and β are the parameters for the coupon rate, $\beta < \alpha < 1$, and g is the criterial exchange rate on coupon. The terminal payoff at the maturity is paid in either domestic currency or foreign currency. That is, the terminal payoff is linked to the exchange rate. Let K be the criterial exchange rate on the terminal payoff. At the maturity T the value of the dual CLB is given by the minimum of the face value in either foreign or domestic currency,

$$V(T, X_T) = \min\left(\frac{X_T}{K} F, F\right). \tag{2.3}$$

The firm can call the CLB at the face value in domestic currency before the maturity. Then, the value of the CLB must satisfy

$$V(t, X_t) \leq F \tag{2.4}$$

for all t.

Let τ be a call time (a stopping time) by the issuer and $\mathcal{T}_{t,T}$ be the set of stopping times with respect to the filtration $\{\mathcal{F}_t$. Then, we can obtain the value of the callable CLB which is represented by

$$V(t, x) = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_{t,T}} J_t^x(\tau), \tag{2.5}$$

where $J_t^x(\tau)$ is

$$\begin{aligned} J_t^x(\tau) \quad = \quad & \tilde{E}\Bigg[e^{-r_d(\tau - t)} F 1_{\{\tau < T\}} + e^{-r(T-t)} \min\left(\frac{X_T}{K}, 1\right) F 1_{\{\tau = T\}} \\ & + \int_t^T e^{-r_d(u-t)} c(u, X_u) du \Big| X_t = x\Bigg]. \end{aligned} \tag{2.6}$$

Moreover, the optimal stopping times $\hat{\tau}_t$ for the issuer is given by

$$\hat{\tau}_t = \inf\{\tau \in [t, T) \mid V(\tau, S_\tau) = F\} \wedge T. \tag{2.7}$$

Hence, $V(t,x) = J_t^x(\hat{\tau}_t)$ is attained at the saddle point $\hat{\tau}_t$ for each t.

Let $\overline{V}(t, X_t)$ denote the value of the callable domestic CLB which has the terminal payoff in domestic currency. The value of the domestic CLB can be provided by

$$\overline{V}(t,x) = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_{t,T}} I_t^x(\tau), \tag{2.8}$$

where $I_t^x(\tau)$ is

$$I_t^x(\tau) = \tilde{E}\left[e^{-r_d(\tau-t)}F + \int_t^\tau e^{-r_d(u-t)} c(u, X_u) du \;\middle|\; X_t = x \right]. \tag{2.9}$$

Let $V_E(t, X_t)$ denote the value of the non-callable European CLB. This value is equal to the expected value of equation (2.3) plus total coupon payments,

$$\begin{aligned} V_E(t,x) &= \tilde{E}\left[e^{-r_d(T-t)} \min\left(\frac{X_T}{K}, 1\right) F + \int_t^T e^{-r_d(T-t)} c(u, X_u) du \middle| \; X_t = x \right] \\ &= e^{-r_f(T-t)} \frac{x}{K} F\Phi(-h^+(T-t, x, K)) + e^{-r_d(T-t)} F\Phi(h^-(T-t,x,K)) + q(t,x) \end{aligned} \tag{2.10}$$

where $q(t,x)$ is the expected total coupon payments between time t and maturity T,

$$\begin{aligned} q(t,x) &= \int_t^T e^{-r_f(u-t)} \alpha \frac{x}{g} F\Phi\left(h^+\left(u-t, x, \frac{\beta g}{\alpha}\right)\right) du \\ &\quad - \int_t^T e^{-r_d(u-t)} \beta F\Phi\left(h^-\left(u-t, x, \frac{\beta g}{\alpha}\right)\right) du, \end{aligned} \tag{2.11}$$

$\Phi(\cdot)$ is the cumulative standard normal distribution function and $h^{\pm}(\tau, x, y)$ are

$$h^{\pm}(\tau, x, y) = \frac{\log(x/y) + \left(r_d - r_f \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}.$$

Hence, the value of the European CLB satisfies $\mathcal{L}V_E(t,x) + c(t,x) = 0$, where the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + (r_d - r_f)x\frac{\partial}{\partial x} - r_d + \frac{\partial}{\partial t}.$$

3 Optimal Policies and Boundaries

In this section we explore analytical properties for the optimal call strategies for the issuer. We define $\mathcal{S}$ and $\mathcal{C}$ as the stopping and continuation regions of the dual CLB and $\overline{\mathcal{S}}$ and $\overline{\mathcal{C}}$ as the stopping and continuation regions of the domestic CLB as follows,

$$\begin{aligned} \mathcal{S} &= \{(t,x) \mid V(t,x) = F\}, \; \mathcal{C} = \{(t,x) \mid V(t,x) < F\}, \\ \overline{\mathcal{S}} &= \{(t,x) \mid \overline{V}(t,x) = F\}, \; \overline{\mathcal{C}} = \{(t,x) \mid \overline{V}(t,x) < F\}. \end{aligned}$$

For each t we define

$$\mathcal{S}_t = \{x \mid (t,x) \in \mathcal{S}\}, \quad \mathcal{C}_t = \{x \mid (t,x) \in \mathcal{C}\}, \quad \overline{\mathcal{S}}_t = \{x \mid (t,x) \in \overline{\mathcal{S}}\}, \quad \overline{\mathcal{C}}_t = \{x \mid (t,x) \in \overline{\mathcal{C}}\}.$$

Lemma 1 The value of the dual CLB is lower than or equal to ones of the domestic CLB and European CLB,

$$V(t,x) \le \overline{V}(t,x) \text{ and } V(t,x) \le V_E(t,x) \tag{3.1}$$

for all t and x, and moreover,

$$\mathcal{S}_t \subseteq \overline{\mathcal{S}}_t, \quad \mathcal{C}_t \supset \overline{\mathcal{C}}_t \tag{3.2}$$

for each t.

Proof. The terminal payoff of the dual CLB is paid in either domestic or foreign currency, while that of the domestic CLB is paid in domestic currency, that is, $V(T, X_T) \le \overline{V}(T, X_T)$. And both CLBs have the same coupon rate and the same call property. Hence, the value of the dual CLB is less than that of the domestic CLB. Also, the issuer can repurchase the dual CLB, minimizing the value of CLB at any time. Hence, the value of the callable dual CLB is less than that of the non-callable European dual CLB. Equation (3.2) follows from $V(t,x) \le \overline{V}(t,x)$. □

Lemma 2 The values of the dual CLB $V(t,x)$ and the domestic CLB $\overline{V}(t,x)$ are increasing in t and x.

Proof. Since $V(t,x)$ is given by equation (2.5), it follows that $V(t,x)$ is increasing in t. Next, we show that $V(t,x)$ is increasing in x. The solution of equation (2.1) starting from $X_u = x$, $x > 0$ is denoted by $X_{u,t}(x)$, for $u \le t$. We have $X_{u,t}(x_1) < X_{u,t}(x_2)$ if $x_1 < x_2$. Since $\min(x/K, 1)F$ and $\max(\alpha x/g - \beta, 0)$ are increasing in x, we have $I_t^{x_1}(\sigma) \le I_t^{y_2}(\sigma)$ for any $\sigma \in \mathcal{T}_{s,T}$. This means $V(t, x_1) \le V(t, x_2)$ if $x_1 \le x_2$, for any $0 \le t \le T$. Hence, $V(t,x)$ is increasing in x. In the same way, we can obtain the monotonicity for the domestic CLB. □

The values of the dual CLB $V(t,x)$ and the domestic CLB $\overline{V}(t,x)$ are continuously differentiable in t and twice continuously differentiable in x with $\mathcal{L}V = 0$ and $\mathcal{L}\overline{V} = 0$ on the continuation regions $\mathcal{C}$ and $\overline{\mathcal{C}}$, respectively.

Now, for the dual CLB the optimal call boundary at time t can be defined as the graph of $x_t^* \equiv \inf\{x |\ x \in \mathcal{S}_t\}$. Similarly, for the domestic the optimal call boundary at time t is the graph of $\overline{x}_t^* \equiv \sup\{x |\ x \in \mathcal{S}_t\}$.

Proposition 3 For the optimal call boundary, we have

1. $\mathcal{S}_t = [x_t^*, \infty)$, $\mathcal{C}_t = [0, x_t^*)$, $\overline{\mathcal{S}}_t = [\overline{x}_t^*, \infty)$, $\overline{\mathcal{C}}_t = [0, \overline{x}_t^*)$.
2. x_t^* and $\overline{x}_t^*$ is decreasing in t.
3. $x_t^* \ge \overline{x}_t^*$ for all t.

Proof. Property 1 follows from the definitions of the regions $\mathcal{S}_t$, $\mathcal{C}_t$, $\overline{\mathcal{S}}_t$ and $\overline{\mathcal{C}}_t$. Next, we show property 2. Since $V(t,x)$ is increasing in t, we have for any $a > 0$, $b > 0$ that

$$V(t-a, x_t^* - b) \le V(t, x_t^* - b) < F.$$

Thus, we have $x_{t-a}^* > x_t^* - b$. Since $b > 0$ is arbitrary, $x_{t-a}^* > x_t^*$, which shows that x_t^* is decreasing in t. In the same way, we can show the monotonicity of the optimal call boundary for the domestic CLB. Property 3 can be easily proved form Lemma 1. □

4 The Decomposition for the Value of Currency Linked Bonds

In this section we provide the decomposition of the values of the dual and domestic CLB. At the optimal call boundary the following lemma holds for the derivation of the value of CLB.

Lemma 4 The value of the dual CLB $V(t,x)$ satisfies

$$\frac{\partial V}{\partial x}(t, x_t^*) = 0 \tag{4.1}$$

at the optimal call boundary x_t^*. Similarly, the value of the domestic CLB $\overline{V}(t,x)$ satisfies

$$\frac{\partial \overline{V}}{\partial x}(t, \overline{x}_t^*) = 0 \tag{4.2}$$

at the optimal call boundary $\overline{x}_t^*$ for all t.

Theorem 5 1. The value of the dual CLB can be decomposed as follows;

$$V(t,x) = V_E(t,x) - d(t,x) \tag{4.3}$$

where $d(t,x)$ is the discounted value of the early call given by

$$\begin{aligned} d(t,x) &= \int_t^T e^{-r_f(u-t)}\alpha\frac{x}{g}F\Phi(h^+(u-t,x,x_u^*))du \\ &\quad - \int_t^T e^{-r_d(u-t)}(\beta + r_d)F\Phi(h^-(u-t,x,x_u^*))du. \end{aligned} \tag{4.4}$$

2. The value of the domestic CLB can be decomposed as follows;

$$\overline{V}(t,x) = e^{-r_d(T-t)}F + q(t,x) - \overline{d}(t,x) \tag{4.5}$$

where $\overline{d}(t,x)$ is the early discounted value given by

$$\begin{aligned} \overline{d}(t,x) &= \int_t^T e^{-r_f(u-t)}\alpha\frac{x}{g}F\Phi(h^+(u-t,x,\overline{x}_u^*))du \\ &\quad - \int_t^T e^{-r_d(u-t)}(\beta + r_d)F\Phi(h^-(u-t,x,\overline{x}_u^*))du. \end{aligned} \tag{4.6}$$

Proof. Putting $f(t,x) = e^{-r_d t}V(t,x)$ and applying the Ito's lemma, we obtain

$$f(T,X_T) = f(t,x) + \int_t^T \frac{\partial f}{\partial x}dX_u + \int_t^T \left(\frac{1}{2}\sigma^2 X_u^2 \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial u}\right) du.$$

Therefore,

$$\begin{aligned} &e^{-r_d T}V(T,X_T) \\ &= e^{-r_d t}V(t,x) + \int_t^T e^{-r_d u}\frac{\partial V}{\partial x}dX_u + \int_t^T e^{-r_d u}\left(\frac{1}{2}\sigma^2 X_u^2\frac{\partial^2 V}{\partial x^2} - rV + \frac{\partial V}{\partial u}\right)du \\ &= e^{-r_d t}V(t,x) + \int_t^T e^{-r_d u}\sigma X_u \frac{\partial V}{\partial x}d\tilde{Z}_u \\ &\quad + \int_t^T e^{-r_d u}\left(\frac{1}{2}\sigma^2 X_u^2\frac{\partial^2 V}{\partial x^2} + (r_d - r_f)X_u\frac{\partial V}{\partial x} - r_d V + \frac{\partial V}{\partial u}\right)du. \end{aligned} \tag{4.7}$$

Note that

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + (r_d - r_f)x\frac{\partial V}{\partial x} - r_d V + \frac{\partial V}{\partial t} = \begin{cases} -\max\left(\alpha\frac{x}{g} - \beta, 0\right) F, & (t,x) \in \mathcal{C} \\ -r_d F, & (t,x) \in \mathcal{S}. \end{cases}$$

Equation (4.7) can be rearranged as

$$\begin{aligned} e^{-r_d T} V(T, X_T) &= e^{-r_d t} V(t,x) + \int_t^T e^{-r_d u} \sigma X_u \frac{\partial V}{\partial x} d\tilde{Z}_u \\ &\quad - \int_t^T e^{-r_d u} \max\left(\alpha \frac{X_u}{g} - \beta, 0\right) F 1_{\{X_u < x_u^*\}} du - \int_t^T e^{-r_d u} r_d F 1_{\{X_u \geq x_u^*\}} du. \end{aligned}$$

From $1_{\{X_t < x_t^*\}} = 1 - 1_{\{X_t \geq x_t^*\}}$,

$$\begin{aligned} e^{-r_d T} V(T, X_T) &= e^{-r_d t} V(t,x) + \int_t^T e^{-r_d u} \sigma X_u \frac{\partial V}{\partial x} d\tilde{Z}_u - \int_t^T e^{-r_d u} \max\left(\alpha \frac{X_u}{g} - \beta, 0\right) F du \\ &\quad + \int_t^T e^{-r_d u} \left\{ \max\left(\alpha \frac{X_u}{g} - \beta, 0\right) - r_d \right\} F 1_{\{X_u \geq x_u^*\}} du. \end{aligned}$$

After multiplying by $e^{r_d t}$ and taking the expectation conditioning on $X_t = x$, we have

$$\begin{aligned} &\tilde{E}\left[e^{-r_d (T-t)} \min\left(\frac{X_T}{K}, 1\right) F \mid X_t = x \right] \\ &= V(t,x) - \tilde{E}\left[\int_t^T e^{-r_d(u-t)} \max\left(\alpha \frac{X_u}{g} - \beta, 0\right) F du \;\Big|\; X_t = x \right] \\ &\quad + \tilde{E}\left[\int_t^T e^{-r_d(u-t)} \left\{ \max\left(\alpha \frac{X_u}{g} - \beta, 0\right) - r_d \right\} F 1_{\{X_u \geq x_u^*\}} du \;\Big|\; X_t = x \right] \\ &= V(t,x) - q(t,x) + \int_t^T e^{-r_f(u-t)} \alpha \frac{x}{g} F \Phi(h^+(u-t, x, x_u^*)) du \\ &\quad - \int_t^T e^{-r_d(u-t)} (\beta + r_d) F \Phi(h^-(u-t, x, x_u^*)) du, \end{aligned}$$

which give us equations (4.3) and (4.4). In the same way as the dual CLB, we can obtain equations (4.5) and (4.6) for the domestic CLB. □

Corollary 6 The early callable discounted values for the dual CLB $d(t,x)$ and the domestic CLB $\overline{d}(t,x)$ satisfy

$$d(t,x) \leq \overline{d}(t,x). \tag{4.8}$$

Proof. Both the optimal call boundaries, x_t^* and $\overline{x}_t^*$ have the same structure from equations (4.4) and (4.6). From $x_t^* \leq \overline{x}_t^*$ in Proposition 3, the early callable discounted values satisfy inequality (4.8). □

5 Numerical Examples

In this section we present the numerical examples of the value of CLB, the early callable discounted value and the optimal call boundary. The integrations for both the values of CLB and the early callable discounted value calculate using the extended trapezoidal rule. The optimal boundary computes using the bisection method. Table 1 shows the data we use to evaluate the CLB.

Table 1: Parameters

Face value	F	100
Criterial exchange rate on terminal payoff	K	80
Domestic Interest rate	r_d	0.04
Foreign Interest rate	r_f	0.02
Volatility	σ	0.3
Maturity	T	3
Parameter for coupon rate (1)	α	0.12
Parameter for coupon rate (2)	β	0.10
Criterial exchange rate on coupon	g	120

In Figure 1 the value of the CLB is drawn as a function of the exchange rate. This figure numerically demonstrates that the value of the dual CLB is lower than the ones of the domestic and European CLB. as Lemma 1 insists. Also, we may recognize that the value of CLB is increasing in exchange rate x in Proposition 3 and the derivation of the value of CLB is equal to 0 at the optimal call boundary in Lemma 4. Figure 2 shows the early callable discounted value for the exchange rate. We can make sure that this value for the dual CLB is higher than the one for the domestic CLB in Corollary 6. Figure 3 shows the behavior of the optimal call boundaries for the dual CLB and the domestic CLB. The region above the the boundary is the stopping region, while the region below the boundary is the continuation region. We can observe that the both optimal call boundaries are decreasing in time t and the optimal boundary for the dual CLB is higher than the one for the domestic CLB in Proposition 3.

6 Conclusion

In this paper we have studied the valuation model of the callable CLB in the setting of the optimal stopping problem. The value of the dual CLB is lower than the ones of both the domestic CLB and the European CLB has been shown. Furthermore, we have shown some analytical properties of the optimal call boundaries for the dual and domestic CLB. The optimal call boundary for the dual CLB is higher than the one for the domestic CLB. This means that if the CLB has the risk of the exchange rate on the terminal payoff, then the issuer calls earlier. Furthermore, we have demonstrated that the value of the dual CLB can be decomposed into the value of the European CLB minus the early the early callable discounted value. We have also provided the numerical results; the value of the CLB and the early callable discounted value and the optimal call boundary.

In future work we will extend the callable clause to the model using the calla price less than the face vale. Furthermore, we'll investigate some properties using the stochastic interest rate. And we'll apply the model in this paper to other structured bonds.

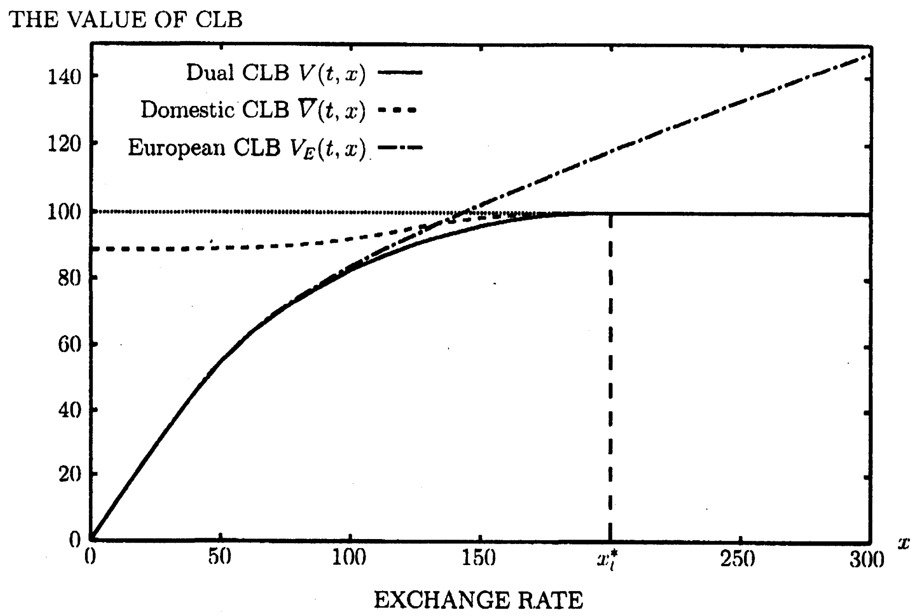

Figure 1: The value of CLB

$F = 100,\ K = 80,\ r_d = 0.04,\ r_f = 0.02,\ \sigma = 0.3,\ T = 3,\ \alpha = 0.12,\ \beta = 0.10,\ g = 120$

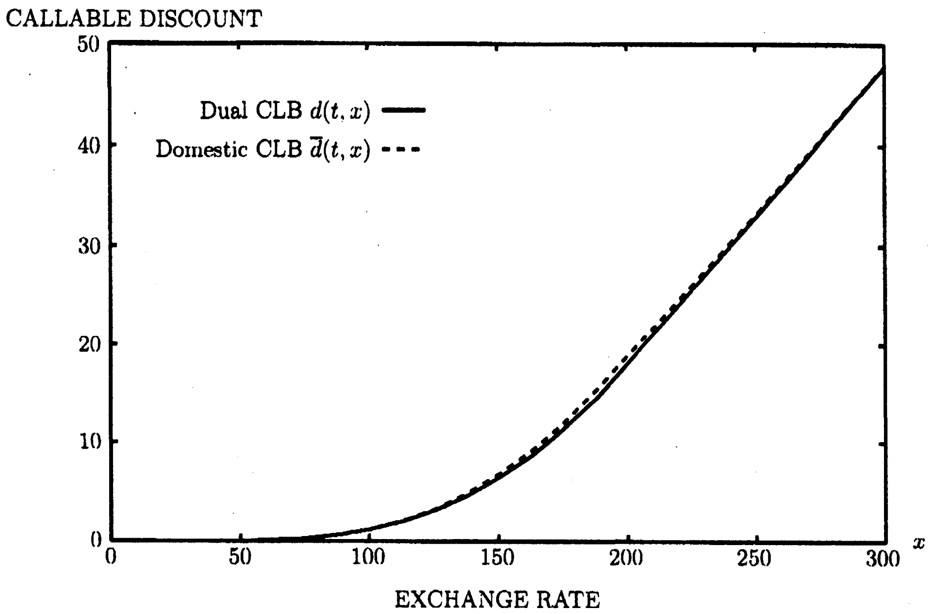

Figure 2: The early callable discounted value

$F = 100,\ K = 80,\ r_d = 0.04,\ r_f = 0.02,\ \sigma = 0.3,\ T = 3,\ \alpha = 0.12,\ \beta = 0.10,\ g = 120$

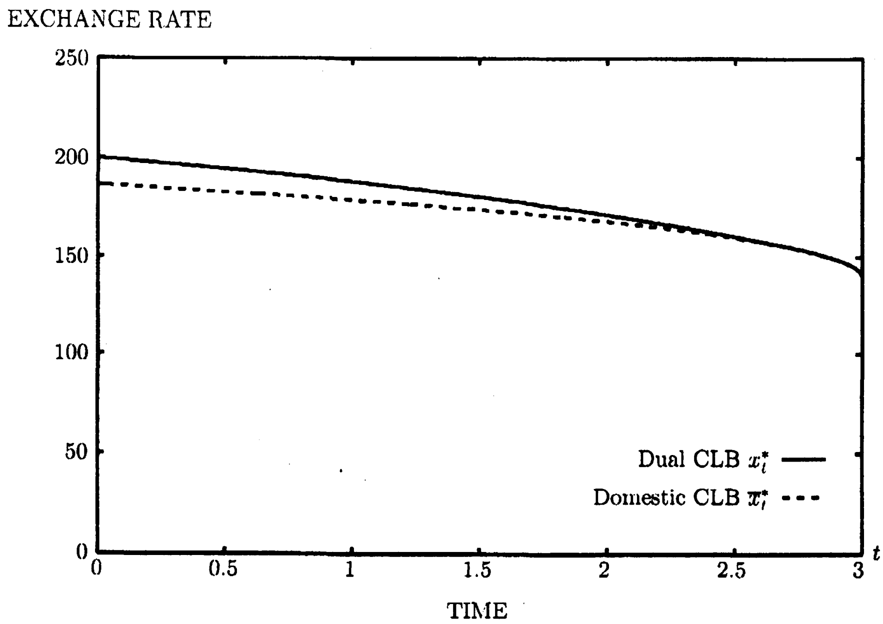

Figure 3: Optimal call boundary

$F = 100,\ n = 2,\ r = 0.04,\ \delta = 0.02,\ \kappa = 0.3,\ T = 5,\ X = 85,\ \alpha = 0.05$

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