Estimating Multi-dimensional Density Functions through the Malliavin-Thalmaier Formula and Its Application to Finance [1]

大阪大学・基礎工学研究科 コハツ-ヒガ アルトゥーロ (Arturo Kohatsu-Higa)
Graduate school of Engineering Science
Osaka University
大阪大学・基礎工学研究科 安田 和弘 (Kazuhiro Yasuda)
Graduate school of Engineering Science
Osaka University

Abstract

The Malliavin-Thalmaier formula was introduced in [5] for use in Monte-Carlo simulation. This is an integration by parts formula for high dimensional probability density functions. But when this formula is applied directly for computer simulation, we show that it is unstable. We propose an approximation to the Malliavin-Thalmaier formula. In the first part of this paper, we prove the central limit theorem to obtain the values of the parameters in Monte-Carlo simulations which achieves a prescribed error level. To prove it, we need the order of the bias and the variance of the approximation error, and we prove the central limit theorem by using these error estimation. And in the latter part, we obtain an explicit Malliavin-Thalmaier formula for the calculation of Greeks in finance. The weights obtained are free from the curse of dimensionality.

1 Introduction

The goal of the present article is to estimate through simulations the probability density function of multi-dimensional random variables using Malliavin Calculus and discuss some of its applications, particularly in Finance.

Usually, a result applied to estimate a multidimensional density is the usual integration by parts formula of Malliavin Calculus that is stated, for example, in Proposition 2.1.5 of Nualart [6].

Proposition 1.1 *Let $G \in \mathbb{D}^{\infty}$, $F = (F_1, ..., F_d)$ be a nondegenerate random vector. Then for $\mathbf{x} \in \mathbb{R}^d$,*

$$p_{F,G}(\mathbf{x}) = E\left[\prod_{i=1}^{d} \mathbf{1}_{[0,\infty)}(F_i - x_i) H_{(1,2,\dots,d)}(F;G)\right], \tag{1.1}$$

where $\mathbf{1}_{[0,\infty)}(x)$ denotes the indicator function and for $i = 2, ..., d$,

$$\begin{aligned} H_{(1)}(F;G) &:= \sum_{j=1}^{d} D^*\Big(G(\gamma_F^{-1})^{1j} DF_j\Big), \\ H_{(1,\dots,i)}(F;G) &:= \sum_{j=1}^{d} D^*\Big(H_{(1,\dots,i-1)}(F;G)(\gamma_F^{-1})^{ij} DF_j\Big). \end{aligned}$$

[1]This paper is an abbreviated version of Kohatsu-Higa, Yasuda [4]. All proofs are omitted due to the page restriction.

Here D^* denotes the adjoint operator of the Malliavin derivative operator D and γ_F the Malliavin covariance matrix of F.

Expression (1.1) has lead to various results concerning theoretical estimates of the density, its support etc. However that expression is not very efficient for computer simulation, that is, it has an iterated Skorohod integral. Recently, Malliavin and Thalmaier [5], Theorem 4.23, gave a new integration by parts formula that seems to alleviate the computational burden for simulation of densities in high dimension. We call this formula the Malliavin-Thalmaier formula. In this formula, one needs to simulate only one Skorohod integral instead of the previous multiple Skorohod integrals. But there is still a problem, that is, the variance of the estimator is infinite. Therefore we propose a slightly modified estimator that depends on a modification parameter h, which will converge to the Malliavin-Thalmaier formula as $h \to 0$. This will generate a small bias and a large variance which is not infinite.

First to obtain the sufficient number of Monte-Carlo simulation times, we prove the central limit theorem even though the variance of the density estimators explode. To prove it, we need some estimations of the order of the bias and the variance of the approximation error. Finally, this central limit theorem gives the corresponding optimal parameter h. Next we apply the Malliavin-Thalmaier formula to finance, especially to the calculation of Greeks. In the one dimensional case, a method to obtain Greeks by the integration by parts formula was introduced by Fournié *et al* [3]. Here we focus our attention to the high dimensional case. We give an expression of Greeks, which is derived using the Malliavin-Thalmaier formula. In particular, the weights are free from the curse of dimensionality. That is, the expression does not have a multiple Skorohod integral.

We have not tried to introduce approximations for F in the theoretical study of the error in order not to burden the reader with technical issues. A typical result incorporating these issues should be a combination with other known techniques (see e.g. Clement *et al.* [2]).

Also note that the expression in (1.1) corresponds to a density only in the case that $G = 1$. In general, it represents a conditional expectation multiplied by the density. To avoid introducing further terminology, we will keep referring to $p_{F,G}(\mathbf{x})$ as the "density".

2 Preliminaries

Let us introduce some notations. For a multi-index $\alpha = (\alpha_1, ..., \alpha_m) \in \{1, ..., d\}^m$, we denote by $|\alpha| = m$ the length of the multi-index.

2.1 Malliavin Calculus

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Suppose that H is a real separable Hilbert space whose norm and inner product are denoted by $\|\cdot\|_H$ and $< \cdot, \cdot >_H$ respectively. Let $W(h)$ denote a Wiener process on H.

We denote by $C_p^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that f and all of its partial derivatives have at most polynomial growth.

Let $\mathcal{S}$ denote the class of *smooth* random variables of the form

$$F = f(W(h_1), ..., W(h_n)), \tag{2.1}$$

where $f \in C_p^\infty(\mathbb{R}^n)$, $h_1, ..., h_n \in H$, and $n \geq 1$.

If F has the form (2.1) we define its derivative DF as the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i.$$

We will denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$. This space is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left\{E\left[|F|^p\right] + E\left[\|DF\|_H^p\right]\right\}^{\frac{1}{p}}.$$

We can define the iteration of the operator D in such a way that for a smooth random variable F, the derivative D^kF is a random variable with values on $H^{\otimes k}$. Then for every $p \geq 1$ and $k \in \mathbb{N}$ we introduce a seminorm on $\mathcal{S}$ defined by

$$\|F\|_{k,p}^p = E\left[|F|^p\right] + \sum_{j=1}^{k} E\left[\|D^jF\|_{H^{\otimes j}}^p\right].$$

For any real $p \geq 1$ and any natural number $k \geq 0$, we will denote by $\mathbb{D}^{k,p}$ the completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$. Note that $\mathbb{D}^{j,p} \subset \mathbb{D}^{k,q}$ if $j \geq k$ and $p \geq q$.

Consider the intersection

$$\mathbb{D}^\infty = \bigcap_{p\geq 1}\bigcap_{k\geq 1}\mathbb{D}^{k,p}.$$

Then $\mathbb{D}^\infty$ is a complete, countably normed, metric space.

We will denote by D^* *the adjoint of the operator* D as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega; H)$. That is, the domain of D^*, denoted by $\mathrm{Dom}(D^*)$, is the set of H-valued square integrable random variables u such that

$$|E[< DF, u >_H]| \leq c\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, where c is some positive constant depending on u. (here $\|\cdot\|_2$ denotes the $L^2(\Omega)$-norm.)

Suppose that $F = (F_1, ..., F_d)$ is a random vector whose components belong to the space $\mathbb{D}^{1,1}$. We associate with F the following random symmetric nonnegative definite matrix:

$$\gamma_F = \left(< DF_i, DF_j >_H \right)_{1\leq i,j\leq d}.$$

This matrix is called the *Malliavin covariance matrix* of the random vector F.

Definition 2.1 *We say that the random vector $F = (F_1, ..., F_d) \in (\mathbb{D}^\infty)^d$ is nondegenerate if the matrix γ_F is invertible a.s. and*

$$(\det\gamma_F)^{-1} \in \bigcap_{p\geq 1} L^p(\Omega).$$

2.2 Malliavin-Thalmaier Representation of Multi-Dimensional Density Functions

Assume that $d \geq 2$ is fixed through this paper.

Definition 2.2 *Given the $\mathbb{R}^d$-valued random vector F and the $\mathbb{R}$-valued random variable G, a multi-index α and a power $p \geq 1$ we say that there is an integration by parts formula in Malliavin sense if there exists a random variable $H_\alpha(F;G) \in L^p(\Omega)$ such that*

$$IP_{\alpha,p}(F,G): \quad E\left[\frac{\partial^{|\alpha|}}{\partial \mathbf{x}^\alpha} f(F)G\right] = E\left[f(F)H_\alpha(F;G)\right] \quad \text{for all } f \in C_0^{|\alpha|}(\mathbb{R}^d).$$

We represent the delta function by $\delta_0(\mathbf{x}) = \Delta Q_d(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^d$, where Δ means Laplacian. If $f \in C_0^2(\mathbb{R}^d)$, then the solution of the Poisson equation $\Delta u = f$ is given by the convolution $Q_d * f$ where the fundamental solution (also called Poisson kernel) Q_d has the following explicit form;

$$Q_2(\mathbf{x}) := a_2^{-1} \ln |\mathbf{x}| \quad \text{for } d = 2 \quad \text{and} \quad Q_d(\mathbf{x}) := -a_d^{-1} \frac{1}{|\mathbf{x}|^{d-2}} \quad \text{for } d \geq 3,$$

where a_d is the area of the unit sphere in $\mathbb{R}^d$. The derivative of the Poisson kernel is $\frac{\partial Q_d}{\partial x_i}(\mathbf{x}) = A_d \frac{x_i}{|\mathbf{x}|^d}$, where $i = 1, ..., d$, $A_2 := a_2^{-1}$ and for $d \geq 3$, $A_d := a_d^{-1}(d-2)$.

Related to the Malliavin-Thalmaier formula, Bally and Caramellino [1], have obtained the following result.

Proposition 2.3 *(Bally, Caramellino [1]) Suppose that for some $p > 1$, $\sup_{|\mathbf{a}| \leq R} E[|\frac{\partial}{\partial x_i} Q_d(F - \mathbf{a})|^{\frac{p}{p-1}} + |Q_d(F - \mathbf{a})|^{\frac{p}{p-1}}] < \infty$ for all $R > 0$, $\mathbf{a} \in \mathbb{R}^d$. If $IP_{i,p}(F;G)$, $i = 1, ..., d$, holds then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and the density $p_{F,G}$ is represented as, for $\mathbf{x} \in \mathbb{R}^d$,*

$$p_{F,G}(\mathbf{x}) = E\left[\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d(F - \mathbf{x}) H_{(i)}(F;G)\right]. \tag{2.2}$$

Corollary 2.4 *If $F = (F_1, ..., F_d)$ is a nondegenerate random vector and $G \in \mathbb{D}^\infty$, then (2.2) holds for the probability density function of the random vector F at $\mathbf{x} \in \mathbb{R}^d$.*

3 Error Estimation

In this section, we introduce an approximation of the Malliavin-Thalmaier formula to avoid the singularity of $\frac{\partial}{\partial x_i} Q_d(\mathbf{x})$. And we give the rate of convergence of the modified estimator of the density at $\mathbf{x} \in \mathbb{R}^d$.

Definitions and Notations

1. For $h > 0$ and $\mathbf{x} \in \mathbb{R}^d$, define $|\cdot|_h$ by $|\mathbf{x}|_h := \sqrt{\sum_{i=1}^d x_i^2 + h}$. Without loss of generality, we assume $0 < h < 1$.
2. For $i = 1, ..., d$, define the following approximation to $\frac{\partial}{\partial x_i} Q_d$, for $\mathbf{x} \in \mathbb{R}^d$, $\frac{\partial}{\partial x_i} Q_d^h(\mathbf{x}) := A_d \frac{x_i}{|\mathbf{x}|_h^d}$.
3. Then we define the approximation to the density function of F as; for $\mathbf{x} \in \mathbb{R}^d$,

$$p_{F,G}^h(\mathbf{x}) := E\left[\sum_{i=1}^d \frac{\partial}{\partial x_i} Q_d^h(F - \mathbf{x}) H_{(i)}(F;G)\right]. \tag{3.1}$$

Now we estimate errors between the Malliavin-Thalmaier formula (2.2) and the approximation (3.1). Here we consider the bias and the variance.

First we give the bias.

Theorem 3.1 *Let F be a nondegenerate random vector and $G \in \mathbb{D}^\infty$, then for $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$,*

$$p_{F,G}(\mathbf{x}) - p^h_{F,G}(\mathbf{x}) = C_1^{\mathbf{x}} h \ln \frac{1}{h} + C_2^{\mathbf{x}} h + o(h),$$

where $C_1^{\mathbf{x}}$ and $C_2^{\mathbf{x}}$ are constants which depend on $\mathbf{x}$, but are independent of h. The constants can be written explicitly.

Next we compute the rate at which the variance of the estimator using Q_d^h diverges.

Theorem 3.2 *Let F be a nondegenerate random vector and $G \in \mathbb{D}^\infty$.*
(i). *For $d = 2$ and $\mathbf{x} \in \mathbb{R}^d$,*

$$E\left[\left(\sum_{i=1}^{2} \frac{\partial}{\partial x_i} Q_2^h(F - \mathbf{x}) H_{(i)}(F;G) - p_{F,G}(\mathbf{x})\right)^2\right] = C_3^{\mathbf{x}} \ln \frac{1}{h} + O(1),$$

where $C_3^{\mathbf{x}}$ is a constant which depends on $\mathbf{x}$, but is independent of h. The constants can be written explicitly.
(ii). *For $d \geq 3$ and $\mathbf{x} \in \mathbb{R}^d$,*

$$E\left[\left(\sum_{i=1}^{d} \frac{\partial}{\partial x_i} Q_d^h(F - \mathbf{x}) H_{(i)}(F;G) - p_{F,G}(\mathbf{x})\right)^2\right] = C_4^{\mathbf{x}} \frac{1}{h^{\frac{d}{2}-1}} + o\left(\frac{1}{h^{\frac{d}{2}-1}}\right),$$

where $C_4^{\mathbf{x}}$ is a constant which depends on $\mathbf{x}$, but is independent of h. The constants can be written explicitly.

Remark 3.3 *In particular, for $h = 0$ one obtains that the variance of the Malliavin-Thalmaier estimator is infinite.*

4 Central Limit Theorem

Obviously when performing simulations, one is also interested in obtaining confidence intervals and therefore the Central Limit Theorem is useful in such a situation. Here we give the central limit theorem to the approximation (3.1). In what follows $\Rightarrow$ denotes weak convergence and the index $j = 1, \dots, N$ denote N independent copies of the respective random variables.

Theorem 4.1 *Let Z be a random variable with standard normal distribution. And $F^{(j)} \in (\mathbb{D}^\infty)^d$ and $G^{(j)} \in \mathbb{D}^\infty$ are respectively a random vector and a random variable which have independent identical distribution.*
(i). *When $d = 2$, set $n = \frac{C}{h \ln \frac{1}{h}}$ and $N = \frac{C^2}{h^2 \ln \frac{1}{h}}$ for some positive constant C fixed throughout. Then*

$$n\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{2} \frac{\partial}{\partial x_i} Q_2^h(F^{(j)} - \mathbf{x}) H_{(i)}(F;G)^{(j)} - p_{F,G}(\mathbf{x})\right) \quad \Rightarrow \quad \sqrt{C_3^{\mathbf{x}}} Z - C_1^{\mathbf{x}} C,$$

where $H_{(i)}(F;G)^{(j)}$, $i = 1, \ldots, d$, $j = 1, \ldots, N$, denotes the weight obtained in the j-th independent simulation (the same that generates $F^{(j)}$ and $G^{(j)}$) and $C_1^{\mathbf{x}}$, $C_3^{\mathbf{x}}$ are some constants.
(ii). *When $d \geq 3$, set $n = \frac{C}{h \ln \frac{1}{h}}$ and $N = \frac{C^2}{h^{\frac{d}{2}+1}(\ln \frac{1}{h})^2}$ for some positive constant C fixed throughout. Then*

$$n\left(\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h(F^{(j)}-\mathbf{x})H_{(i)}(F;G)^{(j)} - p_{F,G}(\mathbf{x})\right) \implies \sqrt{C_4^{\mathbf{x}}}Z - C_1^{\mathbf{x}}C,$$

where $C_1^{\mathbf{x}}$, $C_4^{\mathbf{x}}$ are some constants.

Remark 4.2

(i). *In the assertion of Theorem 4.1, we can freely choose the constant C. Therefore we have that if C is small (wrt $C_1^{\mathbf{x}}$), then the bias becomes small.*

(ii). *This theorem also gives an idea on how to choose h once n or N is fixed.*

To prove Theorem 4.1, we need Theorem 3.1, Theorem 3.2 and the following estimations.

Lemma 4.3 *Under the same assumptions of Theorem 4.1, the followings hold.*
(i).

$$E\left[\left(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h(F^{(1)}-\mathbf{x})H_{(i)}(F;G)^{(1)} - p_{F,G}^h(\mathbf{x})\right)^2\right] = \begin{cases} C_3^{\mathbf{x}}\ln\frac{1}{h} + O(1) & \text{if } d = 2 \\ C_4^{\mathbf{x}}\frac{1}{h^{\frac{d}{2}-1}} + o\left(\frac{1}{h^{\frac{d}{2}-1}}\right) & \text{if } d \geq 3, \end{cases}$$

where $C_3^{\mathbf{x}}$, $C_4^{\mathbf{x}}$ are the same constants of Proposition 3.2.
(ii). *For any $d \geq 2$ and $0 < p < \frac{1}{2}$,*

$$N \times \left|E\left[\exp\left(\frac{\sqrt{-1}un}{N}\zeta_1^{n,N,h}\right) - \left\{1 - \frac{1}{2}\frac{u^2n^2}{N^2}\left(\zeta_1^{n,N,h}\right)^2\right\}\right]\right| \leq o(h^p),$$

where

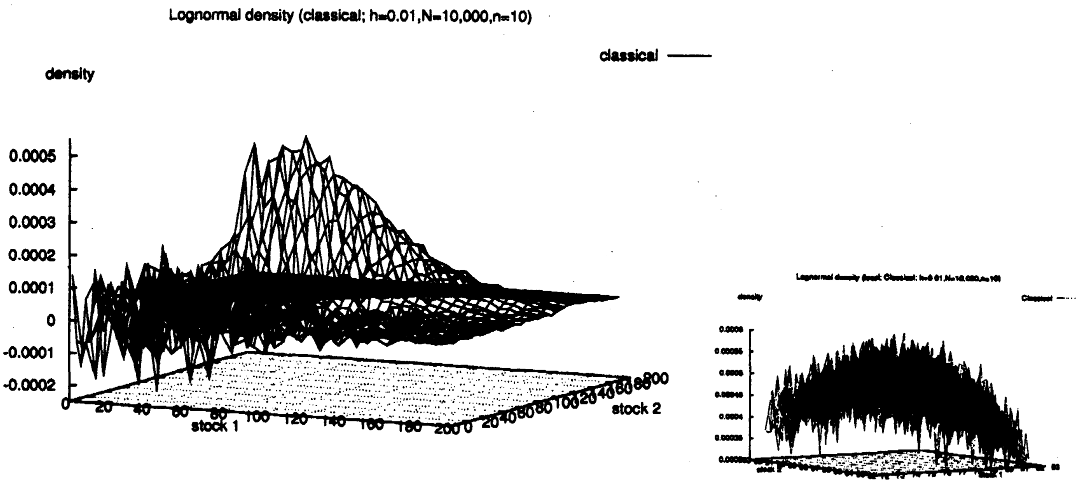
$$\zeta_1^{n,N,h} := \sum_{i=1}^{d}\frac{\partial}{\partial x_i}Q_d^h\left(F^{(j)}-\mathbf{x}\right)H_{(i)}\left(F^{(j)};G\right)^{(j)} - p_{F,G}^h(\mathbf{x}).$$

5 Simulation 1 : Density of 2-dim. Black-Scholes Model

Here we give a simulation result in the case of 2-dimensional log-normal density. In Figures 2 and 3, we show the result of the simulation of (2.2) and (3.1) at time 1. That is,

$$dX_t^1 = X_t^1\left\{0.01dt + 0.1dW_t^1 + 0.2dW_t^2\right\} \quad \text{and} \quad dX_t^2 = X_t^2\left\{0.02dt + 0.3dW_t^1 + 0.2dW_t^2\right\}.$$

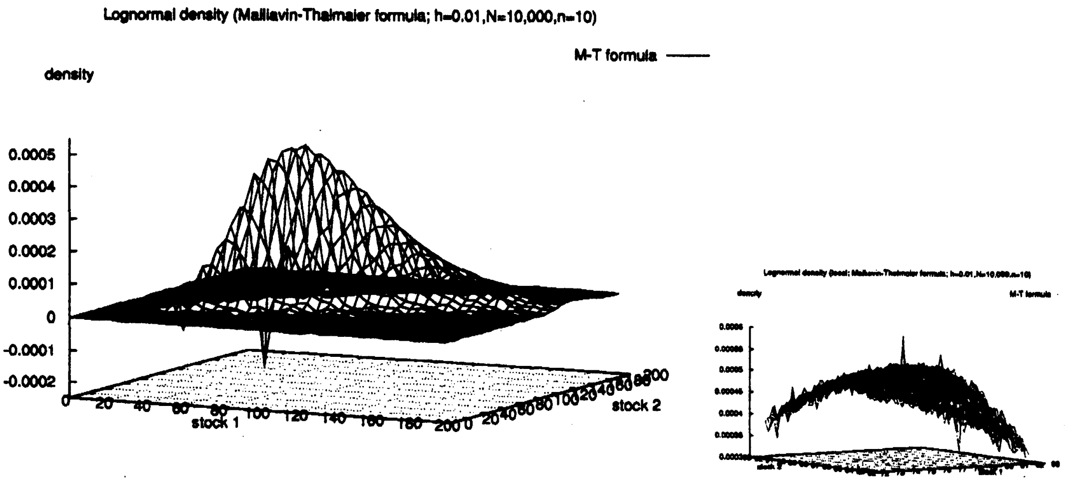
We have used the Euler-Maruyama approximation with 10 time steps and $N = 10^4$ Monte Carlo simulations at each point. The range of global views is $[0, 200] \times [0, 200]$ for the initial values of X_t^1 and X_t^2, and the range of local views is $[72.5, 82.5] \times [82.5, 92.5]$. From Figure 1, the usual method does not work well for both cases. As it can be seen from Figure 2, there are some points where the estimate is unstable. This is clearly due to the infinite variance of the Malliavin-Thalmaier estimator. Finally from Figure 3, we find that we can erase their singular points in Figure 2.



(a) Global view

(b) Local view

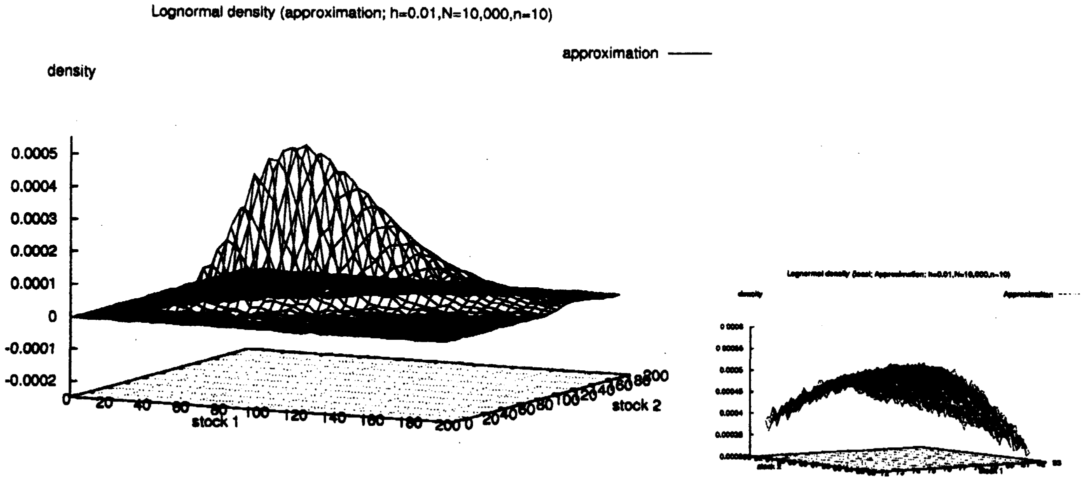
Figure 1: The usual formula (1.1)



(a) Global view

(b) Local view

Figure 2: The Malliavin-Thalmaier formula (2.2)



(a) Global view

(b) Local view

Figure 3: The approximation of the Malliavin-Thalmaier formula (3.1) ($h = 0.01$)

6 Application to Finance

In this section, we compute Greeks using the Malliavin-Thalmaier Formula. We consider a random vector $F^\mu = (F_1^\mu, ..., F_d^\mu)$, $\mu \in \mathbb{R}^m$, $m \in \mathbb{N}$ which depends on a parameter μ. Suppose that $F^\mu \in (\mathbb{D}^\infty)^d$ is a nondegenerate random vector. And let $f(x_1, ..., x_d)$ be a payoff function in a class [2]

$$\mathcal{A} := \left\{ f : \mathbb{R}^d \to \mathbb{R} : \begin{array}{l} \text{continuous a.e. wrt Lebesgue measure, and} \\ \text{there exist constants } c, a \text{ such that } |f(\mathbf{x})| \le \frac{c}{(1+|\mathbf{x}|)^{ad}} \ (a > 1) \end{array} \right\}.$$

A greek is defined for $f \in \mathcal{A}$, as the following quantity for some $j \in \{1, ...,, m\}$;

$$\frac{\partial}{\partial \mu_j} E\left[f\left(F_1^\mu, ..., F_d^\mu\right)\right].$$

We denote the integration with respect to $p_{F^\mu,G}^h(\mathbf{x})$ by $E^h[\cdot]$. That is,

$$E^h\left[f\left(F^\mu\right)\right] := \int \cdots \int_{\mathbb{R}^d} f(\mathbf{x}) p_{F^\mu,1}^h(\mathbf{x}) d\mathbf{x}.$$

[2] Note that in the case of a put option, if we define the payoff function $(K - x)_+ = (K - x)\mathbf{1}_{[0,K]}(x)$ then $(K - x)_+ \in \mathcal{A}$.

In a digital put option case, the payoff function is $\mathbf{1}_{[0,K]}(x)$. Therefore it is in $\mathcal{A}$.

Next in a digital call option case, the payoff function $\mathbf{1}_{[K,\infty)}(x)$ does not go to 0 as $x \to \infty$. But since stocks do not take negative value, then we can transform as it follows,

$$\mathbf{1}_{[K,\infty)}(x) = 1 - \mathbf{1}_{[0,K)}(x).$$

And now we want to calculate Greeks, that is, derivation of the term 1 is 0. It is enough to calculate the term $\mathbf{1}_{[0,K)}(x)$, which has a compact support.

Finally if we want to compute a Greeks for call option case $(x - K)_+$, then one uses directly the Malliavin-Thalmaier formula after taking the derivative. Although it is known that then a localization is needed.

And for $i, j = 1, \ldots, d$, set

$$g^h_{i,j}(\mathbf{y}) := \frac{\partial}{\partial y_j} \int \cdots \int_{\mathbb{R}^d} f(\mathbf{x}) \frac{\partial}{\partial x_i} Q^h_d(\mathbf{y} - \mathbf{x}) d\mathbf{x}, \ \mathbf{y} \in \mathbb{R}^d.$$

Theorem 6.1 *Let $k \in \{1, \ldots, m\}$ be fixed. Let $f \in \mathcal{A}$. Let F^μ be a nondegenerate random vector, which is differentiable with respect to μ_k. Suppose that for $j = 1, \ldots, d$, $\frac{\partial F^\mu_j}{\partial \mu_k} \in \mathbb{D}^\infty$.*

$$\frac{\partial}{\partial \mu_k} \sum_{i=1}^d E\left[\int \cdots \int_{\mathbb{R}^d} f(\mathbf{x}) \frac{\partial}{\partial x_i} Q^h_d(\mathbf{y} - \mathbf{x}) d\mathbf{x} H_{(i)}\left(F^\mu; 1\right)\right] = \sum_{i,j=1}^d E\left[g^h_{i,j}(F^\mu) H_{(i)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right]. \tag{6.1}$$

Moreover if we assume that for all $i = 1, \ldots, d$, there exists some $g_{i,j}$ such that $g^h_{i,j} \to g_{i,j}$ a.e. as $h \to 0$, then

$$\begin{aligned} \frac{\partial}{\partial \mu_k} E^h\left[f(F^\mu)\right] &= \sum_{i,j=1}^d E\left[g^h_{i,j}(F^\mu) H_{(i)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right] \\ &\longrightarrow \sum_{i,j=1}^d E\left[g_{i,j}(F^\mu) H_{(i)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right] = \frac{\partial}{\partial \mu_k} E\left[f(F^\mu)\right] \quad \text{as } h \to 0. \end{aligned} \tag{6.2}$$

Remark 6.2

(i). *The expression in Theorem 6.1 is obviously not unique; e.g.*

$$(6.1) = \sum_{i,j=1}^d E\left[g^h_{i,i}(F^\mu) H_{(j)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k}\right)\right].$$

(ii). *If $g^h_{i,j}$, $i, j = 1, \ldots, d$ has an explicit representation, then one can calculate Greeks easily. If we do not have an explicit expression for the multiple integral then one can use any approximation for multiple Lebesgue integrals. An example of the case that $g_{i,i}$ has an explicit expression. In the digital put case, let $d = 2$ and $f(x_1, x_2) = \mathbf{1}(0 \le x_1 \le K_1)\mathbf{1}(0 \le x_2 \le K_2) \in \mathcal{A}$ where K_1 and K_2 are positive constants. Then*

$$\begin{aligned} g_{1,1}(\mathbf{y}) &= A_2 \left\{\arctan \frac{y_2}{y_1} - \arctan \frac{y_2 - K_2}{y_1} - \arctan \frac{y_2}{y_1 - K_1} + \arctan \frac{y_2 - K_2}{y_1 - K_1}\right\}, \\ g_{2,1}(\mathbf{y}) &= \frac{A_2}{2} \ln \frac{(y_1^2 + y_2^2)((y_1 - K_1)^2 + (y_2 - K_2)^2)}{((y_1 - K_1)^2 + y_2^2)(y_1^2 + (y_2 - K_2)^2)}. \end{aligned} \tag{6.3}$$

These expressions are obtained after taking limits of $g^h_{i,j}(\mathbf{y})$ as $h \to 0$ for $i = 1, 2$.

(iii). *If we use the usual expression of the density, for example Proposition 2.1.5 in Nualart [6], then we need a multi dimensional Skorohod integral to write Greeks explicitly; under some assumptions,*

$$\begin{aligned} &\frac{\partial}{\partial \mu_k} E\left[f(F^\mu)\right] \\ &= E\left[\int_{-\infty}^{F^\mu_d} \cdots \int_{-\infty}^{F^\mu_1} f(\mathbf{x}) d\mathbf{x} \left\{\sum_{j=1}^d H_{(j)}\left(F^\mu; \frac{\partial F^\mu_j}{\partial \mu_k} H_{(1,\ldots,d)}(F^\mu; 1)\right) + \frac{\partial}{\partial \mu_k} H_{(1,\ldots,d)}(F^\mu; 1)\right\}\right]. \end{aligned}$$

(iv). *We remark that in Theorem 6.1, $H_{(i)}$ requires only one Skorohod integral. Even if higher derivatives with respect to μ are considered this fact remains unchanged.*

7 Simulation 2 : Delta of 2 assets digital put option

Now we consider an example. We calculate Delta in a digital put option and the asset is characterized by 2-dimensional Black-Scholes model. First we define the model as follows;

$$\begin{aligned} dS_t^{(1)} &= \mu_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} dW_t^{(1)}, \\ dS_t^{(2)} &= \mu_2 S_t^{(2)} dt + \rho\sigma_2 S_t^{(2)} dW_t^{(1)} + \sqrt{1-\rho^2}\sigma_2 S_t^{(2)} dW_t^{(2)}, \end{aligned}$$

where their initial values for the stock price process are $s_0^{(1)}$ and $s_0^{(2)}$, respectively. μ_1, μ_2 are constants, σ_1, σ_2 are positive constants and $\rho \in [-1, 1]$ is a constant. $W_t^{(1)}$ and $W_t^{(2)}$ are Brownian motion and independent of each other.

We study the Delta of the following option

$$E^Q\left[e^{-rT}\mathbf{1}\left(S_T^{(1)} \le K_1\right)\mathbf{1}\left(S_T^{(2)} \le K_2\right)\right],$$

where r expresses a constant interest rate. Without loss of generality, we assume that $r = 0$. K_1 and K_2 are strike prices of stocks and E^Q is an expectation with respect to the risk neutral measure.

Then the Delta of above option with respect to the first stock $S_t^{(1)}$ is;

$$\begin{aligned} &\frac{\partial}{\partial s_0^{(1)}} E^Q\left[e^{-rT}\mathbf{1}\left(S_T^{(1)} \le K_1\right)\mathbf{1}\left(S_T^{(2)} \le K_2\right)\right] \\ &= E^Q\left[g_{1,1}\left(S_T^{(1)}, S_T^{(2)}\right) H_{(1)}\left(S_T^{(1)}, S_T^{(2)}; \frac{\partial S_T^{(1)}}{\partial s_0^{(1)}}\right)\right] + E^Q\left[g_{2,1}\left(S_T^{(1)}, S_T^{(2)}\right) H_{(2)}\left(S_T^{(1)}, S_T^{(2)}; \frac{\partial S_T^{(1)}}{\partial s_0^{(1)}}\right)\right]. \end{aligned} \tag{7.1}$$

We simulate above Delta by using the following parameters; $s_0^{(1)} = s_0^{(2)} = 100$, $\mu_1 = \mu_2 = 0$, $\sigma_1 = 0.25$, $\sigma_2 = 0.2$, $\rho = 0.2$, $K_1 = K_2 = 100$, $T = 1$. For the simulation of $S_t^{(1)}$ and $S_t^{(2)}$, we use the Euler-Maruyama approximation with 8 time steps. The Delta by equation (7.1) is Figure 4. And we compare their results to the two-sided finite difference method with the bumping size 1 (Figure 5). Here the "average" means arithmetic average of the last 20 values respectively. Delta by the Malliavin-Thalmaier formula (Figure 4) is mostly between "average+0.5%" and "average-0.5%". But delta by finite difference method (Figure 5) still moves between "average+1%" and "average-1%".

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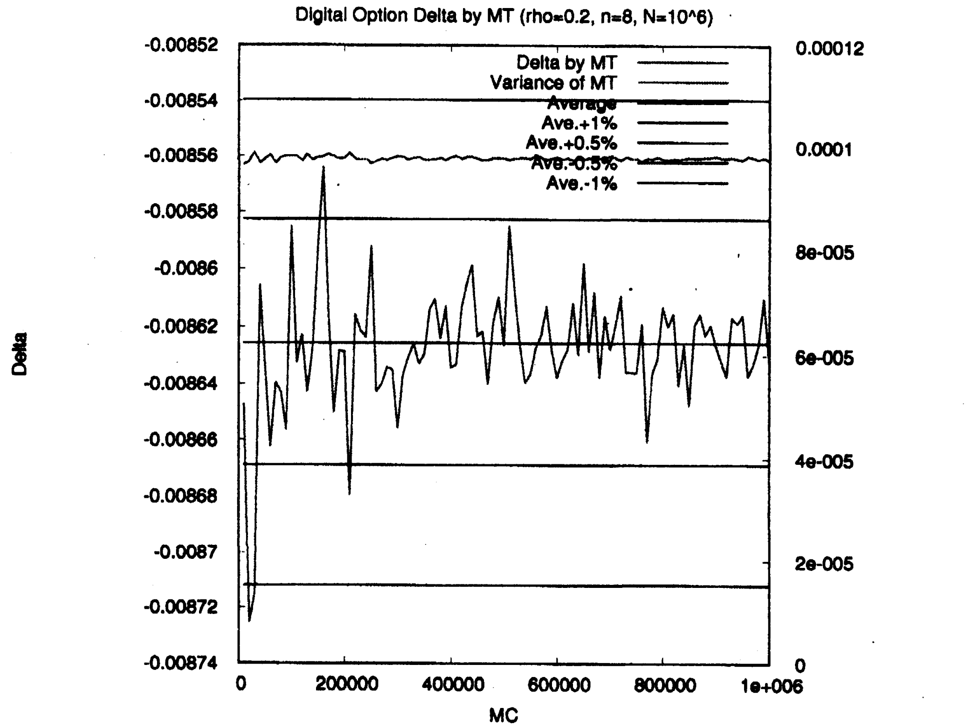

Figure 4: MC - Delta by (7.1)

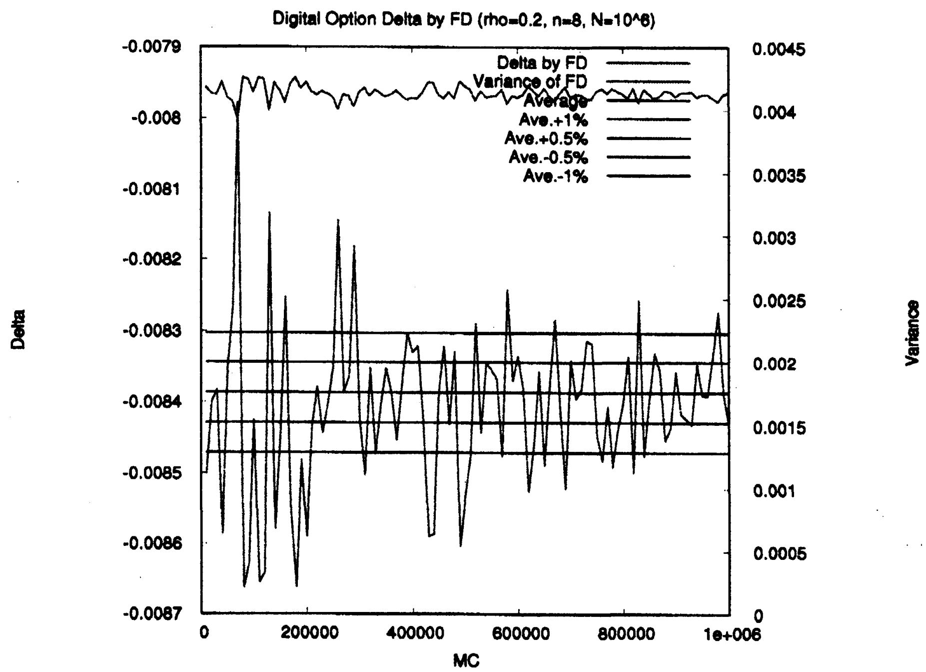

Figure 5: MC - Delta by FD

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