数理解析研究所講究録
第 1580 巻 2008 年 234-244

Optimal Insurance Coverage for a Durable Consumption Good: The Second Best Solution

Teruyoshi Suzuki*
Graduate School of Economics, Hokkaido University

Nishi 7, Kita 9, Kita-Ku, Sapporo 060-0809, Japan
suzuki@econ.hokudai.ac.jp

Abstract

This article analyzes the optimal deductible level for a durable consumption good in a continuous-time economy with a risky asset, a riskless asset and a perishable consumption good. We first show a myopic strategy as a second best solution ignoring the fact that the insurance coverage must be positive.

Keywords: Insurance, deductible, durable consumption goods, optimal consumption and investment

1 Introduction

Arrow (1971) and Mossin (1968) were the first to examine the optimal insurance. Mossin (1968) showed that a full insurance is not optimal when premium includes a positive loading. Arrow (1971) showed that it is optimal to purchase a deductible insurance. The benefit of reducing coverage comes from the reduction of the positive insurance cost. These classic literatures were concerned with the case of a single insurable asset in a static model. Therefore the agent implicitly transforms the corresponding loss into the reduction of consumption or savings, and cannot hedge against the shock of loss by reducing his consumption over time. Then Moffet (1977) derived some propositions about the optimal deductible and consumption in a single period model. And then, Dionne and Eechkhoudt (1984) showed the interactions between consumption and saving decision in a two period model.

Since Merton(1969), a considerable number of studies have been conducted on the intertemporal consumption and investment strategy in a continuous time economy. The problem consists of maximizing total expected utility of consumption over trading interval and terminal wealth. In Merton (1969), the optimal portfolio is equal to the tangency portfolio given under the static model named CAPM (Capital Asset Pricing Model). Merton (1971,1973) provided a general framework for understanding the portfolio demands of long-term investors when investment opportunity varies over time. Merton (1973) showed that the optimal portfolio for long-term investors are affected by the possibility of uncertain changes in future investment opportunities and then differs from the tangency portfolio. The results have had a major influence in microeconomics as surveyed in Campbell and Viceira (2003).

To our knowledge, Briys(1986) first approached the optimal insurance with consumption and investment policy using the methods of Merton (1969). A sufficient condition for separability of the insurance and investment decisions was shown. Gollier (1994) extended Briys (1986) and showed that the demand for insurance vanishes in the long run if the loading factor exceeds a critical value. Moore and Young (2006) extended Gollier (1994) allowing the risk horizon to be random, and showed several examples using the Markov Chain approximation.

*I acknowledges financial support from the Ishii Memorial Securities Research Promotion Foundation.

Gollier (2003) examined the demand for insurance and showed that a liquidity constrained agent would demand cover for both low and high risk events as opposed to an agent without liqudity constraint who would have demand cover for high risk events.

We introduce durable consumption goods to investigate the effect of substituting the damaged durables with perishable consumption goods. Various literature has been published that study optimal consumption and investment including durable consumption goods. These include Hindy and Huang (1993), Detemple and Giannikos (1996), Cuoco and Liu (2000), Cocco (2004), Cauley, Pavlov and Schwarts (2005) and Grossman and Laroque (1990) .

We follow Damgaard, Fuglsbjerg and Munk (2003) that extend Grossman and Laroque (1990) to include a perishable consumption good and an indivisible durable consumption good in the model. The main scope of their models were transaction costs and indivisibility of durable goods. However we assume that the durable goods are divisible and can be traded with no transaction cost. We have therefore added new features to the model to take into account that durable goods can be damaged and the damaged goods can be insured.

Needless to say the insurance coverage must be positive. Without the positive coverage constraint, the optimal solution for insurance coverage can be negative. This constraint is in analogue to the leverage constraint studied by Grossman and Vila (1992). They examined the problem of the investor who has a limited ability to borrow for the purpose of investing in a risky asset. And they proved that in the presence of leverage constraint, the optimal solution for a risky asset when the constraint is binding was to invest a fixed proportion of his wealth. The strategy took the same form in the absence of constraint however the proportion level to invest in the risky asset was different because of the possibility that leverage constraint would become binding in the future. He and Pages (1992) and Zariphopoulou (1994) also examined the constrained optimal consumption and investment problem.

Our work is related to Gollier (1987) who dare to relax positive coverage constraint in simple settings. The solution has three domains: (1) short sale of an insurance policy, (2) no insurance, and (3) purchase an insurance policy.

Following on from the introduction, section II shows the strategy ignoring positive coverage constraint. Section III then examines the effect of the constraint. Section IV concludes this article.

2 A model

2.1 Set up

We consider an infinite-horizon, continuous-time stochastic economy with a perishable consumption good, a durable consumption good and two financial assets. One of the financial assets is a risk-free security paying a constant continuously compounded interest rate r. The other is a risky security whose price process follows a geometric Brownian motion

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_S dw_1(t), \quad t \geq 0 \tag{1}$$

where $(w_1(t), w_2(t))$ is uncorrelated two dimensional Wiener process and where μ and σ_S are constants.

We now make several assumptions about the market:

(a) Financial securities and durable goods can be bought in unlimited quantities and are infinitely divisible.

(b) Financial securities can be sold short but a durable good can not be sold short.
(c) There are no transaction costs.
The unit price of a durable good $P(t)$ also follows a geometric Brownian motion

$$\frac{dP(t)}{P(t)} = \mu_P dt + \sigma_{P1} dw_1(t) + \sigma_{P2} dw_2(t), \quad t \geq 0 \tag{2}$$

where μ_P, σ_{P1} and σ_{P2} are constants and where

$$\sigma_P^2 = \sigma_{P1}^2 + \sigma_{P2}^2.$$

We should note that the unit price of the durable good is partly correlated with the price of the financial risky asset.

We assume that the stock of the durable consumption good depreciates at a certain depreciation rate δ over time. We also assume that durable consumption goods can be damaged by an insured event represented by Poisson process $N(t)$ which is independent of $(w_1(t), w_2(t))$. We denote by λ the intensity of the events and by ℓ the constant loss rate of the durable good when the insured event occurs. Letting $K(t)$ be the number of units of the durable good held at time t, then $K(t)$ follows

$$\frac{dK(t)}{K(t)} = (-\delta + \lambda\ell)dt - \ell dN(t), \quad t \geq 0 \tag{3}$$

where δ, ℓ, λ are constants. We require $K(t) > 0$ from the assumption that the agent can not take a short position for durable goods.

The agent can purchase an insurance contract to cover the risk of loss. We denote the indemnity paid by the insurer at time by $q(t)$. The payment must be positive then the constraint is

$$q(t) \geq 0. \tag{4}$$

Assuming that insurance premium is payable continuously and include a positive loading which is represented by a factor ϕ, the premium to be paid and denoted by $p(t)$ is given by

$$p(t) = \lambda\phi q(t)$$

where $\phi \geq 1$. Assume that the premium loading is sufficient small to satisfy the solvency condition.

We denote by $\theta_0(t)$ and $\theta(t)$ the amount held in the risk-free and risky security at time t. We define the wealth of the agent as the sum of his investments in the risk-free and risky assets and the value of his current stock of durable goods $K(t)$ times the current price of durable goods $P(t)$. Therefore his wealth $X(t)$ is given as

$$X(t) = \theta_0(t) + \theta(t) + K(t)P(t), \quad t \geq 0. \tag{5}$$

Under the assumption that the agent follows a perishable consumption strategy $C(t)$ and self-financing strategy $(\theta_0(t), \theta(t), K(t))$, the wealth process $X(t)$ evolves as

$$\begin{aligned} dX(t) &= \Big(r(X(t) - K(t)P(t)) + \theta(t)(\mu - r) + (\mu_P - \delta + \lambda\ell)K(t)P(t) - C(t) - p(t)\Big)dt \\ &\quad + \Big(\theta(t)\sigma_S + K(t)P(t)\sigma_{P1}\Big)dw_1(t) + K(t)P(t)\sigma_{P2}dw_2(t) \\ &\quad + \Big(q(t-) - \ell P(t)K(t-)\Big)dN(t), \qquad t \geq 0. \end{aligned} \tag{6}$$

At the time η when an insured event occurs, there is a jump in his wealth due to the damage of his durable goods. We require that the consumption and trading strategies satisfy the solvency condition of the agent and that his total wealth is always positive although an insured event has occurred:

$$X(\eta) = X(\eta-) - \ell P(\eta)K(\eta-) + q(t) > 0, \quad t \geq 0. \tag{7}$$

A policy $\mathcal{S}_t = (\theta(t), K(t), C(t), q(t))$ is admissible if the policy satisfies (4), (7) and $K(t), C(t) > 0$. We denote by $\mathcal{A}(x, k, p)$ the set of admissible policies where $x = X(0),\ k = K(0),\ p = P(0)$. We assume $\mathcal{A}(x, k, p)$ is a non-empty set.

We assume that the utility function exhibits constant relative risk aversion, i.e.:

$$U(c, k) = \frac{1}{1-\gamma}\left(c^{\beta}k^{1-\beta}\right)^{1-\gamma}, \quad 0 < \beta < 1,\ 0 < \gamma < 1$$

where c denotes the perishable consumption rate and k denotes the stock of durable goods held. The agent's objective is to find the policy $\mathcal{S}_t \in \mathcal{A}$ that maximizes his time 0 expected utility:

$$J^{S}(x, p) = E\left[\int_0^{\infty} \mathrm{e}^{-\rho t} U(C(t), K(t))dt\right]$$

where ρ is time preference parameter. Therefore the value function of agents is given by

$$V(x, p) = \sup_{\mathcal{S}_t \in \mathcal{A},\ t>0} J^{S}(x, p). \tag{8}$$

From the dynamic programming principle, the value function satisfies

$$V(x, p) = \sup_{\mathcal{S} \in \mathcal{A},\ t>0} E\left[\int_0^{\eta} \mathrm{e}^{-\rho t} U(C(t), K(t))dt + \mathrm{e}^{-\rho\eta} V(X(\eta), P(\eta))\right]. \tag{9}$$

Then the Hamilton-Jacobi-Bellman (HJB) equation corresponding to this problem can be written as

$$\begin{aligned}
\rho V(x,p) \;=\; \sup_{\mathcal{S}\in\mathcal{A}} \Big\{ & \frac{1}{1-\gamma}(c^{\beta}k^{1-\beta})^{1-\gamma} + \Big(r(x - pk) + \theta(\mu - r) + (\mu_P - \delta)kp - c - \lambda\phi q\Big)\frac{\partial V}{\partial x}(x,p) \\
& + \frac{1}{2}\Big(\theta^2\sigma_S^2 + k^2p^2\sigma_P^2 + 2\theta\sigma_S\sigma_{P1}kp\Big)\frac{\partial^2 V}{\partial x^2}(x,p) + \mu_P p\frac{\partial V}{\partial p}(x,p) \\
& + \frac{1}{2}\sigma_P^2 p^2 \frac{\partial^2 V}{\partial p^2}(x,p) + \Big(\theta\sigma_S\sigma_{P1} + \sigma_P^2 kp\Big)p\frac{\partial^2 V}{\partial x \partial p}(x,p) \\
& + \lambda\Big(V(x - \ell kp + q, p) - V(x,p) + \ell kp\frac{\partial V}{\partial x}(x,p)\Big)\Big\}.
\end{aligned} \tag{10}$$

2.2 The Second Best Solution

In this section, we show a *myopic strategy* for problem (9). When the agent follows the myopic strategy, he is not aware of the positive coverage constrain $q(t) \geq 0$ until he meets it. More precisely, myopic insurance coverage is given by the maximum of two quantities: (a) no insurance, and (b) optimal insurance coverage ignoring the positive coverage constraint. The approach to get the myopic solution is simple. First, we find a solution ignoring the positive coverage constraint. Next we seek the cutoff level which the positive coverage constraint bind and we then set the *constraint* domain and *unconstraint* domain. Finally, we find the solution with no insurance in the constraint domain. We will later show that the myopic strategy is not optimal in general. While in the unconstraint domain, the optimal solution is affected by the fact that the positive coverage constraint may be binding in the future. However it is possible to obtain qualitative properties of the optimal controls.

Now we introduce some auxiliary parameters and give assumptions. We then show a solution under the assumptions. Constants are defined as follows:

$$\begin{aligned}
\Lambda_0 &= -\frac{\rho}{\gamma} + \frac{1-\gamma}{\gamma}\left\{r - (1-\beta)\mu_P + \frac{1}{2}(1-\beta)[1+(1-\beta)(1-\gamma)]\sigma_P^2\right\} \\
&\quad + \frac{1-\gamma}{2\gamma^2\sigma_S^2}(\mu - r - (1-\gamma)(1-\beta)\sigma_S\sigma_{P1})^2 && (11) \\
\Lambda_1 &= (1-\gamma)\sigma_{P2}^2 + \frac{1}{1-\beta}\left(r - \mu_P + \delta + (\mu - r)\frac{\sigma_{P1}}{\sigma_S}\right) && (12) \\
\Lambda_2 &= \left(\frac{\gamma}{1-\beta} + \frac{1-\gamma}{2}\right)\sigma_{P2}^2 && (13)
\end{aligned}$$

as in Damgaard, Fuglsbjerg and Munk (2003). If $\Lambda_0 < 0$, the nonlinear equation

$$F(\alpha_k) = 0 \tag{14}$$

where

$$F(\alpha_k) = \begin{cases} \Lambda_0 + \Lambda_1\alpha_k + \Lambda_2\alpha_k^2 + \dfrac{\lambda}{\gamma}\left\{(1-\ell\alpha_k)^{-\gamma}\left(1 + \dfrac{\beta\gamma\ell}{1-\beta}\alpha_k\right) - \left(1 + \dfrac{\gamma\ell}{1-\beta}\alpha_k\right)\right\}, & \alpha_k < \hat{\alpha}_k \\ \Lambda_0' + \left(\Lambda_1 + \dfrac{\lambda(\phi-1)\ell}{1-\beta}\right)\alpha_k + \Lambda_2\alpha_k^2, & \alpha_k \geq \hat{\alpha}_k, \end{cases}$$

and where

$$\hat{\alpha}_k = \frac{1-\phi^{-\frac{1}{\gamma}}}{\ell}, \quad \Lambda_0' = \Lambda_0 + \frac{\lambda(\phi-1)}{\gamma} + \lambda\phi(\phi^{-\frac{1}{\gamma}} - 1)$$

will have a single positive solution. It is noted that the argument α_k represents the optimal holding policy for durable consumption goods in the following.

The assumption below will give the transversarity condition.

Assumption 1 *If $\alpha_k < \hat{\alpha}_k$ then*

$$\Lambda_0 < -\frac{1}{2}(1-\gamma)\sigma_{P2}^2\alpha_k^2 + \frac{\lambda}{\gamma}\left\{1 + (1-\ell\alpha_k)^{-\gamma}(-1+\ell\gamma\alpha_k)\right\}.$$

If $\alpha_k \geq \hat{\alpha}_k$ then

$$\Lambda_0' < -\frac{1}{2}(1-\gamma)\sigma_{P2}^2\alpha_k^2.$$

The optimal solution for problem (9) is stated as follows. The proof is presented in Appendix.

Proposition 1 *Under Assumption 1, the value function for problem (9) is given by*

$$\bar{V}(x,p) = \frac{1}{1-\gamma}\alpha_v p^{-(1-\beta)(1-\gamma)}x^{1-\gamma} \tag{15}$$

and the controls are given in feedback form as

$$\bar{\theta}(t) = \alpha_\theta\bar{X}(t), \quad \bar{K}(t) = \alpha_k\bar{X}(t)/P(t), \quad \bar{C}(t) = \alpha_c\bar{X}(t), \quad \bar{q}(t) = \alpha_q\bar{X}(t) \tag{16}$$

where $\bar{X}(t)$ is the wealth process generated by these controls and where constants α_v, α_θ are written by

$$\alpha_v = {\alpha_c}^{\beta(1-\gamma)-1}{\alpha_k}^{(\beta-1)(\gamma-1)}\beta \tag{17}$$

$$\alpha_\theta = \frac{\mu - r}{\gamma\sigma_S^2} + \Big(\beta - (\alpha_k + \beta - 1)\gamma - 1\Big)\frac{\sigma_{P1}}{\gamma\sigma_S} \tag{18}$$

and where α_k is a root of the equation $F(\alpha_k) = 0$ and where constants α_q, α_c are given by as follows:
(i) *If $\alpha_k \geq \hat{\alpha}_k$ then insurance policy is given by deductible form as*

$$\alpha_q = \ell\alpha_k - \left(1 - \phi^{-\frac{1}{\gamma}}\right) \tag{19}$$

and α_c is given by

$$\alpha_c = -\beta\Lambda_0' - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2.$$

(ii) *If $\alpha_k < \hat{\alpha}_k$ then no insurance is optimal i.e. $\alpha_q = 0$ and α_c is given by*

$$\alpha_c = -\beta\Lambda_0 - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2 + \frac{\lambda\beta}{\gamma}\left\{1 + (1-\ell\alpha_k)^{-\gamma}(-1+\ell\gamma\alpha_k)\right\}.$$

The insurance coverage is given by (19). It follows that the state space can be divided by the ratio $z = x/(kp)$ at

$$z^* = \frac{\ell}{1 - \phi^{-\frac{1}{\gamma}}}$$

into the *constrained* domain

$$C = \{(x,k,p) \mid z > z^*\} \tag{20}$$

and into the *unconstrained* domain

$$U = \{(x,k,p) \mid z \leq z^*\}. \tag{21}$$

If $(x,k,p) \in C$, no insurance is optimal in the myopic sense. Therefore wealthier consumers will not insure. And if $(x,k,p) \in U$, positive coverage is needed. The consumer who would like to hold a large amount of durable goods as against his wealth has a demand for the insurance. The insurance policy is given by the deductible form and the deductible equals

$$\left(1 - \phi^{-\frac{1}{\gamma}}\right)\bar{X}(t)$$

which is proportional to wealth. Hence as in the no insurance domain, wealthier consumers can reduce the coverage of costly external insurance and partly follow self-insurance. Of course positive loading decreases the demand for the insurance.

Let us consider a special case where the premium loading goes to 0. In this case $\phi \to 1$ and then $z^* \to \infty$. Therefore there is only one domain U where insurance is demanded. Further the deductible goes to zero and then full insurance is optimal. Finally when the premium loading equals 0, myopic strategy is optimal.

It is noted that when durable goods are insured against damage the solvency condition (7) will be satisfied by the insurance payment and when no insurance is needed, the solvency condition (7) is satisfied even if an insured event occurs because the definition of constrained domain implies the equation

$$\frac{x}{kp} > \frac{\ell}{1 - \phi^{-\frac{1}{\gamma}}} > \ell$$

holds. We also note that the domains can be rewritten by

$$C = \{(x,k,p) \mid \alpha_k < \hat{\alpha}_k\}, \quad U = \{(x,k,p) \mid \alpha_k \geq \hat{\alpha}_k\}$$

and that the controlled consumption of perishable goods given by (??) differs as to the domains.

A Proof of Proposition 1

First we reduce the dimensionality of the problem. Second we show that the optimal strategy ignoring the positive coverage constraint in the constrained domain is equal to the myopic strategy. Third we also show that the optimal strategy constrained to be no insurance in the unconstraint domain is equal to the myopic strategy.

A.1 Reducing the dimensionality of the problem

As in Damgaard, Fuglsbjerg and Munk (2003), the dimensionality of problem (9) can be reduced as follows.

From (6), for all $\kappa > 0$, the strategy (Θ, K, C, Q) is admissible with initial wealth x and initial durable price p if and only if the strategy $(\kappa\Theta, K, \kappa C, \kappa Q)$ is admissible with initial wealth κx and initial durable price κp. Since $U(\kappa C, K) = \kappa^{\beta(1-\gamma)}U(C,K)$, it follows that

$$\bar{V}(\kappa x, \kappa p) = \kappa^{\beta(1-\gamma)}\bar{V}(x,p), \kappa > 0.$$

From the equation above, it follows that

$$\bar{V}(x,p) = p^{\beta(1-\gamma)}\bar{V}(x/p, 1).$$

Therefore, to set $y = x/p$ we can reduce the problem by

$$\bar{V}(x,p) = p^{\beta(1-\gamma)}\bar{v}(y).$$

Substitute this result to (10) and simplfy it, then we get the ordinary differential equation:

$$0 = \sup_{\hat{\theta}\in R, (\hat{c},k,\hat{q})\in R^3_+} J(v(y)) \tag{22}$$

where

$$\begin{aligned} J(v(y)) \;=\; & \frac{\left(\hat{c}^{\beta}k^{1-\beta}\right)^{1-\gamma}}{1-\gamma} + \frac{1}{2}\Big\{\beta(\gamma-1)\left((\beta(\gamma-1)+1)\sigma_P^2 - 2\mu_P\right) - 2\rho\Big\}v(y) \\ & + \Big\{ -\hat{c} + \left(-\delta + (-\gamma\beta+\beta-1)\sigma_P^2 + \mu_P - r\right)k - \phi\hat{q} \\ & + \left(\mu - r + (-\gamma\beta+\beta-1)\sigma_S\sigma_{P1}\right)\hat{\theta} + \left((\beta(\gamma-1)+1)\sigma_P^2 - \mu_P + r\right)y\Big\}v'(y) \\ & + \frac{1}{2}\Big\{k^2\sigma_P^2 + \sigma_S^2\hat{\theta}^2 + 2k\sigma_S\sigma_{P1}\hat{\theta} - 2\left(k\sigma_P^2 + \sigma_S\sigma_{P1}\hat{\theta}\right)y + \sigma_P^2 y^2\Big\}v''(y) \\ & + \lambda\Big\{v(y - \ell k + \hat{q}) - v(y) + \ell k v'(y)\Big\} \end{aligned} \tag{23}$$

and where we have set new control variables:

$$\hat{c} = c/p, \quad \hat{\theta} = \theta/p, \quad \hat{q} = q/p.$$

A.2 The solution in the constraint domain

We show that the optimal solution for the problem

$$0 = \sup_{\hat{\theta}\in R,(\hat{c},k)\in R_+^2,\hat{q}\in R} J(v(y)) \tag{24}$$

equals to the myopic solution given in Proposition 1 when α_k lies in constraint domain (20).

We suppose that the differential equation (24) has the solution

$$v(y) = \frac{1}{1-\gamma}\alpha_v y^{1-\gamma} \tag{25}$$

with the maximizing control values

$$\hat{c} = \alpha_c y, \quad \hat{\theta} = \alpha_\theta y, \quad k = \alpha_k y, \quad \hat{q} = \alpha_q y. \tag{26}$$

Ignoring the positive constraint $\hat{c}$ and k, the first order conditions for the maximizing control values $\hat{q}, \hat{c}, \hat{\theta}, k$ are:

$$v'(y - \ell k + \hat{q}) - \phi v(y) = 0, \tag{27}$$

$$U_c(c,k) - v'(y) = 0, \tag{28}$$

$$\Big(\mu - r + (-\gamma\beta + \beta - 1)\sigma_S\sigma_{P1}\Big)v'(y) + \Big(\sigma_S^2\theta - 2\sigma_S\sigma_{P1}(y-k)\Big)v''(y) = 0, \tag{29}$$

$$\begin{aligned} &U_k(c,k) + \Big(-\delta + (-\gamma\beta + \beta - 1)\sigma_P^2 + \mu_P - r\Big)v'(y) + \Big(\sigma_P^2(k-y) + \theta\sigma_S\sigma_{P1}\Big) \\ &\quad + \lambda\Big(- \ell v'(y - \ell k + \hat{q}) + \ell v'(y)\Big) = 0. \end{aligned} \tag{30}$$

Inserting the control values (26) and the supposed solution (25), we get from (27) that

$$\alpha_q = \ell\alpha_k - \Big(1 - \phi^{-\frac{1}{\gamma}}\Big) \tag{31}$$

and from (28) that

$$\alpha_v = \beta{\alpha_c}^{\beta(1-\gamma)-1}{\alpha_k}^{(\beta-1)(\gamma-1)} \tag{32}$$

and from (29) that

$$\alpha_\theta = \frac{\mu - r + (\beta - (\alpha_k + \beta - 1)\gamma - 1)\sigma_S\sigma_{P1}}{\gamma\sigma_S^2}. \tag{33}$$

Substituting (25) and (26) into (30) and applying

$$U(c,k) = \frac{c}{\beta(1-\gamma)}U_c(c,k), \quad U_k(c,k) = \frac{1-\beta}{\beta}\frac{c}{k}U_c(c,k) \tag{34}$$

and (28) yield

$$\alpha_C = \frac{\gamma\beta\alpha_k}{1-\beta}\left\{\frac{\lambda(\phi-1)\ell - (-\delta + (-\gamma\beta + \beta - 1)\sigma_P^2 + \mu_P - r)}{\gamma} + \Big(\sigma_P^2(\alpha_k - 1) + \alpha_\theta\sigma_S\sigma_{P1}\Big)\right\}. \tag{35}$$

Substituting (26) back into (24) and applying (34) and (28) to simplify, then inserting the candidate control values (31), (32), (33) and (35) yields the quadratic equation

$$\Lambda_0' + \left(\Lambda_1 + \frac{\lambda(\phi-1)\ell}{1-\beta}\right)\alpha_k + \Lambda_2\alpha_k^2 = 0 \tag{36}$$

which is equivalent to (14) when $\alpha_k \geq \hat{\alpha_k}$.

If (36) has a root that satisfies $\alpha_k \geq \hat{\alpha_k}$ then $q(t)$ can be positive from (31). Therefore the cutoff level from the right hand side is given by $\hat{\alpha}$. We will later show that the cutoff level from the left hand side is equal to $\hat{\alpha_k}$ to seek the optimal solution when we set $q(t) = 0$.

Supposing $\alpha_k \geq \hat{\alpha_k}$ we show α_c is positive in the following. We can show that (35) can be rewrttien by

$$\alpha_c = -\beta\Lambda_0' - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2$$

from (36). Then $\alpha_c > 0$ from Assumption ??. Supposing $\alpha_k > \hat{\alpha}_k$ the solvency condition $X(t) > 0$ is hold because the loss of durabel goods are insured.

Finally we can show the transversarity condition of problem (24) is equivalent to Assumption ?? after tedious manipulation. Then we conclude that the solution of HJB equation (24) above is the myopic solution of problem (9) supposing $\alpha_k > \hat{\alpha}_k$.

A.3 The solution in the unconstraint domain

We show that the optimal solution for the problem

$$0 = \sup_{\hat{\theta}\in R,(\hat{c},k)\in R_+^2} J(v(y)) \tag{37}$$

equals to the myopic solution given in Proposition 1 when α_k lies in unconstraint domain (21).

The optimal solution can be derived in the same manner as in the previous section except that the quadratic equation is replaced by the non-linear equation

$$\Lambda_0 + \Lambda_1\alpha_k + \Lambda_2\alpha_k^2 + \frac{\lambda}{\gamma}\left\{(1-\ell\alpha_k)^{-\gamma}\left(1+\frac{\beta\gamma\ell}{1-\beta}\alpha_k\right) - \left(1+\frac{\gamma\ell}{1-\beta}\alpha_k\right)\right\} = 0 \tag{38}$$

which is equivalent to (14) when $\alpha_k < \hat{\alpha_k}$ and that the optimal consumption is given by

$$\alpha_c = -\beta\Lambda_0 - \frac{1}{2}\beta(1-\gamma)\sigma_{P2}^2\alpha_k^2 + \frac{\lambda\beta}{\gamma}\left\{1 + (1-\ell\alpha_k)^{-\gamma}(-1+\ell\gamma\alpha_k)\right\}. \tag{39}$$

The solution α_v and α_θ can be given by substituting α_k as a root of (38) and applying (39) into (32) and (33).

We can see that the nonlinear equation (14) has at most one positive root. Then supposing $\alpha_k < \hat{\alpha}_k$, imply solvency condition is satisfied. Beside this α_c will be positive from (39) and Assumption 1. The transversarity condition is verified by the Assumption 1 after tedious manipulation we shall omit it.

References

[1] Arrow, L. J. (1971), *Essays in the Theory of Risk Bearing.* Chicago: Markham Publishing Co.

[2] Briys, E. (1986), "Insurance and Consumption: The continuous-time case," *Journal of Risk and Insurance*, **53**, 718–723.

[3] Campbell, J. Y. and L. M. Viceira (2002), *Strategic asset allocation: portfolio choice for long-term investors*, Oxford University Press.

[4] Caulay, S. D., A. D. Pavlov, and E. S. Schwartz (2005),"Homeownership as a constraint on asset allocation," working paper.

[5] Cocco, J. F. (2004), "Portfolio choice in the presence of housing," *Review of Financial Studies*, 18, 535–567.

[6] Cuoco, D. and H. Liu (2000), "Optimal Consumption of a Divisible Durable Dood," *Journal of Economic Dynamics and Control*, 24, 561–613.

[7] Damgaard, A., B. Fuglsbjerg and C. Munk (2003), "Optimal consumption and investment strategies with a perishable and an indivisible durable consumption good," *Journal of Economic Dynamics and Control*, 28, 209–253.

[8] Detemple, J. B. and C. I. Giannikos (1996),"Asset and commodity prices with multi-attribute durable good," *Journal of Economic Dynamics and Control*, 20, 1451–1504.

[9] Dionne, G. and E. Louis (1984), "Insurance and Saving: Some Further Results," *Insurance: Mathematics and Economics*, **3**, 101–110.

[10] Gollier, C. (1987), "The Design of Optimal Insurance Contracts Without the Nonnegativity Constraint on Claims," *Journal of Risk and Insurance*, **54**, 314–324.

[11] Gollier, C. (1994), "Insurance and Precautionary Capital Accumulation in a Continuous-Time Model," *Journal of Risk and Insurance*, **61**, 78–95.

[12] Gollier, C. (2003), "To Insure or Not to Insure?: An Insurance Puzzle," *The Geneva Papers on Risk and Insurance Theory*, **28**, 5–24.

[13] Grossman, S. J. and G. Laroque (1990), "Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods," *Econometorica*, 58, 25–51.

[14] Grossman, S. J. and J. Vila (1992), "Optimal Dynamic Trading with Leverage Constraints," *Journal of Financial and Quantitative Analysis*, **27**, 151–167.

[15] He, H. and F. F. Page (1993), "Labor Income, Borrowing Constraints, and Equilibrium Asset Prices," *Economic Theory*, **3**, 663–696.

[16] Hindy, A., and C. Huang (1993),"Optimal consumption and portfolio rules with durability and local substitution,"*Econometrica*, 61, 85–121.

[17] Merton, R. C. (1969), "Lifetime portfolio selection under uncertainty: the continuous time case," *Review of Economic Studies*, 32, 137 150.

[18] ———— (1971), "Optimal consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, 3, 373–413.

[19] ———— (1973), "An intertemporal capital asset pricing model," *Econometorica*, 41, 867–887.

[20] Moore, K. S. and V. R. Young (2006), "Optimal Insurance in a Continuous-Time Model," **39**, 47–68.

[21] Mossin, J. (1968), "Aspects of Rational Insurance Purchasing," *Journal of Political Economy*, **76**, 533–568.

[22] Moffet, D. (1977), "Optimal Deductible and Consumption Theory," *Journal of Risk and Insurance*, **44**, 669–683.

[23] Zariphopoulou, T. (1994), "Consumption-Investment Models with Constraints," *SIAM Journal on Control and Optimization*, **32**, 59–85.