On the transfer map for the Hochschild cohomology of Frobenius algebras

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1 Introduction

We describe a transfer map between the complete Hochschild cohomologies of Frobenius algebras $\Lambda$ and $\Gamma$. For a Frobenius algebra $\Lambda$ over a commutative ring $R$ which is finitely generated projective $R$-module, we can define a complete Hochschild cohomology $H^r(\Lambda, M)$ with a coefficient $\Lambda$-bimodule $M$ (see [Na]). If, in addition, we assume that $\Gamma$ is a Frobenius extension of $\Lambda$, then $\Gamma$ is a Frobenius $R$-algebra. Under this assumption, we can define $\text{Res} : H^r(\Gamma, \Gamma M_{\Gamma}) \to H^r(\Lambda, M)$ and $\text{Cor} : H^r(\Lambda, \Gamma M_{\Gamma}) \to H^r(\Gamma, M)$. $\text{Res}$ for $r \geq 0$ and $\text{Cor}$ for $r \leq -1$ are defined naturally, and, particularly for Frobenius algebras, $\text{Res}$ and $\text{Cor}$ can also be defined for other integers $r$, which we may call them the transfer maps (see [S3], [S4], and also [No1, No2]).

In this summary, we show the explicit description of $\text{Res}$ and $\text{Cor}$ by means of the standard resolutions of the Frobenius algebras above.

2 Complete Hochschild cohomology of Frobenius algebras

Let $R$ be a commutative ring with identity, $\Lambda$ an $R$-algebra which is finitely generated projective as $R$-module. $\Lambda^e = \Lambda \otimes_R \Lambda^{opp}$ denotes the enveloping algebra of $\Lambda$ and $Z\Lambda$ denotes the center of $\Lambda$.

If $M$ is a left $\Lambda^e$-module (i.e. $\Lambda$-bimodule), we define the Hochschild cohomology of $\Lambda$ with coefficient module $M$:

$$H^n(\Lambda, M) = \text{Ext}^n_{\Lambda^e}(\Lambda, M) \quad (n \geq 0).$$

It is easily verified that this is a $Z\Lambda$-module.

We denote $H^n(\Lambda, \Lambda)$ by $HH^n(\Lambda)$ in the following. By the definition, we see that

$$H^0(\Lambda, M) \cong M^\Lambda = \{x \in M \mid ax = xa \text{ for any } a \in \Lambda\}$$

and so we have $HH^0(\Lambda) = Z\Lambda$. 
2.1 Standard resolution, cup product

Let $n \geq 0$ be an integer, and we put $X_n = \Lambda \otimes \cdots \otimes \Lambda$ ($n + 2$-times tensor products over $R$). Then we have the following $\Lambda^e$-projective resolution of $\Lambda$ which is called the standard resolution of $\Lambda$:

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} \Lambda \longrightarrow 0,$$

$$d_n(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1},$$

$$d_1(x_0 \otimes x_1 \otimes x_2) = x_0 x_1 \otimes x_2 - x_0 \otimes x_1 x_2,$$

$$d_0(x_0 \otimes x_1) = x_0 x_1.$$

We can define the cup product $H^i(\Lambda, M) \otimes H^j(\Lambda, N) \xrightarrow{\cup_{ij}} H^{i+j}(\Lambda, M \otimes_{\Lambda} N)$, which satisfies the anti-commutativity:

$$\alpha \cup_{i,j} \beta = (-1)^{ij} \beta \cup_{j,i} \alpha \quad \text{for } \alpha \in H^i(\Lambda), \beta \in H^j(\Lambda, M).$$

and the cup product $Z\Lambda \otimes H^i(\Lambda, M) \xrightarrow{\cup_{0i}} H^i(\Lambda, M)$ gives the $Z\Lambda$-module structure for $H^i(\Lambda, M)$.

Furthermore, we have $HH^i(\Lambda) \otimes HH^j(\Lambda) \xrightarrow{\cup} HH^{i+j}(\Lambda)$, so this makes

$$HH^*(\Lambda) := \bigoplus_{k \geq 0} HH^k(\Lambda)$$

a ring containing $HH^0(\Lambda) = Z\Lambda$ as a subring, which is called the Hochschild cohomology ring of $\Lambda$.

2.2 Frobenius extensions and Frobenius algebras

Let $\Gamma/\Lambda$ be a Frobenius extension. That is,

$$\Gamma = a_1 \Lambda \oplus \cdots \oplus a_m \Lambda = \Lambda b_1 \oplus \cdots \oplus \Lambda b_m;$$

$$xa_i = \sum_{j=1}^{m} a_j \beta_{ji}(x), \quad b_j x = \sum_{i=1}^{m} \beta_{ji}(x) b_i \quad (x \in \Gamma, \beta_{ji}(x) \in \Lambda)$$

and there exist the following isomorphisms:

$$\phi_{\Gamma/\Lambda} : \Gamma \xrightarrow{\sim} Hom_{\Lambda,-}(\Gamma, \Lambda_\Lambda), \quad \phi_{\Gamma/\Lambda}(a_i)(b_j) = \delta_{ij},$$

$$\phi'_{\Gamma/\Lambda} : \Lambda \xrightarrow{\sim} Hom_{-,-}(\Gamma, \Lambda_\Lambda), \quad \phi'_{\Gamma/\Lambda}(b_j)(a_i) = \delta_{ij}.$$

We set

$$\mu_{\Gamma/\Lambda} = \phi_{\Gamma/\Lambda}(1), \quad N_{\Gamma/\Lambda}(x) = \sum_{i=1}^{m} a_i x b_i \quad (x \in \Gamma).$$
Then $\mu_{\Gamma/\Lambda} : \Gamma \to \Lambda$ is a two-sided $\Lambda$-module homomorphism and

$$x = \sum_{i=1}^{m} \mu_{\Gamma/\Lambda}(xa_{i})b_{i} = \sum_{j=1}^{m} a_{j}\mu_{\Gamma/\Lambda}(b_{j}x) \ (x \in \Gamma).$$

Furthermore, let $R$ be a commutative ring and $\Lambda$ a Frobenius $R$-algebra which is finitely generated free $R$-module:

$$\Lambda = u_{1}R \oplus \cdots \oplus u_{n}R = Ru_{1} \oplus \cdots \oplus Ru_{n};$$

$$yu_{i} = \sum_{j=1}^{n} u_{j}\alpha_{ji}(y), \ v_{j}y = \sum_{i=1}^{n} \alpha_{ji}(y)v_{i} \ (y \in \Lambda, \alpha_{ji}(y) \in R),$$

$$\phi_{\Lambda} : \Lambda \sim \to \text{Hom}_{R}(\Lambda_{\Lambda}, R), \ \phi_{\Lambda}(u_{i})(v_{j}) = \delta_{ij}.$$ We set

$$\mu_{\Lambda} = \phi_{\Lambda}(1), \ N_{\Lambda}(y) = \sum_{i=1}^{n} u_{i}yv_{i},$$

$$y^{\tau'} = \sum_{i=1}^{n} \mu_{\Lambda}(u_{i}y)v_{i} \ (\text{Nakayama automorphism of } \Lambda/R).$$

Then $\Gamma$ is a Frobenius $R$-algebra of rank $mn$ with $R$-bases $(a_{i}u_{j}), (v_{j}b_{i}) (1 \leq i \leq m, 1 \leq j \leq n)$:

$$xa_{i}u_{j} = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{k}u_{l}\beta_{ij}(\alpha_{kl}(x)),$$

$$v_{l}b_{k}x = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij}(\alpha_{ki}(x))v_{j}b_{i} \ (x \in \Gamma),$$

$$\phi_{\Gamma} : \Gamma \sim \to \text{Hom}_{R}(\Gamma_{\Gamma}, R), \ \phi_{\Gamma}(a_{i}u_{j})(v_{l}b_{k}) = \delta_{(i,j),(k,l)}.$$ If we set $\mu_{\Gamma} = \phi_{\Gamma}(1)$, then

$$x = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{\Gamma}(xa_{i}u_{j})v_{j}b_{i} = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{k}u_{l}\mu_{\Gamma}(v_{l}b_{k}x) \ (x \in \Gamma).$$

We set

$$N_{\Gamma}(x) := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i}u_{j}xv_{j}b_{i},$$

$$x^{\tau} := \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_{\Gamma}(a_{i}u_{j}x)v_{j}b_{i} \ (x \in \Gamma).$$

Then we have

$$N_{\Gamma/\Lambda} \circ N_{\Lambda} = N_{\Gamma}, \ \mu_{\Lambda} \circ \mu_{\Gamma/\Lambda} = \mu_{\Gamma}, \ \tau_{|_{\Lambda}} = \tau'.$$
3 Restriction and corestriction maps

We set

\[(X_{\Gamma})_{p} = \Gamma \otimes_{R} \cdots \otimes_{R} \Gamma \quad (p + 2 \text{ times tensor products of } \Gamma),\]
\[(X_{\Lambda})_{p} = \Lambda \otimes_{R} \cdots \otimes_{R} \Lambda \quad (p + 2 \text{ times tensor products of } \Lambda),\]

and we define

\[d_{p} : (X_{\Gamma})_{p} \longrightarrow (X_{\Gamma})_{p-1},\]
\[d_{p}(x_{0} \otimes x_{1} \otimes \cdots \otimes x_{p} \otimes x_{p+1}) = x_{0}x_{1} \otimes \cdots \otimes x_{p} \otimes x_{p+1}\]
\[+ \sum_{i=1}^{p-1} (-1)^{i}x_{0} \otimes \cdots \otimes x_{i}x_{i+1} \otimes \cdots \otimes x_{p} + (-1)^{p}x_{0} \otimes \cdots \otimes x_{p-1} \otimes x_{p}x_{p+1}.\]

Then we have the following commutative diagram:

\[\cdots \longrightarrow \text{Hom}_{\Gamma}((X_{\Gamma})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Gamma}} M \xrightarrow{d_{1} \otimes_{\Gamma}} (X_{\Gamma})_{1}^{\tau} \otimes_{\Gamma} M \longrightarrow \cdots\]
\[\text{res}^{0} \downarrow \quad \text{res}^{1} \downarrow \quad \text{cor}_{1} \downarrow \quad \text{res}_{1}\]
\[\cdots \longrightarrow \text{Hom}_{\Lambda}((X_{\Lambda})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Lambda}} M \xrightarrow{d_{1} \otimes_{\Lambda}} (X_{\Lambda})_{1}^{\tau'} \otimes_{\Lambda} M \longrightarrow \cdots.\]

Here, \((X_{\Gamma})_{\tau}^{r}\) is defined by \(w(x \otimes y^{opp}) = y^{r-1}wx\) for \(w \in (X_{\Gamma})_{p}, x \otimes y^{opp} \in \Gamma^{c}\), and

\[\text{res}^{0} : M \rightarrow M, x \mapsto x, \quad \text{res}_{0} : M \rightarrow M, x \mapsto \sum_{i=1}^{m} b_{i}x_{i}^{\tau},\]
\[\text{res}_{q} : (X_{\Gamma})_{q}^{\tau} \otimes_{\Gamma^{c}} M \rightarrow (X_{\Lambda})_{q}^{\tau'} \otimes_{\Lambda^{c}} M, 1 \otimes y_{1} \otimes \cdots \otimes y_{q} \otimes 1 \otimes_{\Gamma^{c}} x \mapsto \sum_{i_{1}, \ldots, i_{q+1}=1}^{m} 1 \otimes_{\mu_{\Gamma/\Lambda}(b_{i_{1}}y_{1}a_{i_{2}})} \otimes \cdots \otimes_{\mu_{\Gamma/\Lambda}(b_{i_{q}}y_{q}a_{i_{q+1}})} 1 \otimes_{\Lambda^{c}} b_{i_{q+1}}xa_{1}^{\tau},\]
\[\text{res}^{p} (p \geq 1) \text{ is defined to be a natural homomorphism induced by } (X_{\Lambda})_{p} \rightarrow (X_{\Gamma})_{p}.\]

Then we have

\[\text{Res}^{r} : H^{r}(\Gamma, \Gamma; M) \rightarrow H^{r}(\Lambda, M) \quad (r \in \mathbb{Z}).\]

Here, \(H^{r}(\Gamma, -) \) and \(H^{r}(\Lambda, -) \) denotes the complete Hochschild cohomology of \(\Gamma\) and \(\Lambda\), respectively, and these are obtained by the horizontal sequences.

On the other hand, we have the following commutative diagram:

\[\cdots \longrightarrow \text{Hom}_{\Gamma}((X_{\Gamma})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Gamma}} M \xrightarrow{d_{1} \otimes_{\Gamma}} (X_{\Gamma})_{1}^{\tau} \otimes_{\Gamma} M \longrightarrow \cdots\]
\[\text{cor}^{1} \uparrow \quad \text{cor}^{0} \uparrow \quad \text{cor}_{0} \uparrow \quad \text{cor}_{1}\]
\[\cdots \longrightarrow \text{Hom}_{\Lambda}((X_{\Lambda})_{1}, M) \xrightarrow{d_{1}} M \xrightarrow{N_{\Lambda}} M \xrightarrow{d_{1} \otimes_{\Lambda}} (X_{\Lambda})_{1}^{\tau'} \otimes_{\Lambda} M \longrightarrow \cdots.\]

Here,
cor_0: M \rightarrow M, x \mapsto x, \quad cor^0: M \rightarrow M, x \mapsto N_{\Gamma/\Lambda}(x),
cor^p: \text{Hom}_{\Lambda^e}((X_{\Lambda})_p, M) \rightarrow \text{Hom}_{\Gamma^e}((X_{\Gamma})_p, M),
cor^p(g)(y_0 \otimes y_1 \otimes \cdots \otimes y_p \otimes y_{p+1})
= \sum_{i_1, \ldots, i_{p+1}=1}^{m} y_0a_{i_1}g(1 \otimes \mu_{\Gamma/\Lambda}(b_{i_1}y_1a_{i_2}) \otimes \cdots \otimes \mu_{\Gamma/\Lambda}(b_{i_p}y_pa_{i_{p+1}}) \otimes 1)b_{i_{p+1}}y_{p+1},

and cor_q (q \geq 1) is defined to be a natural homomorphism induced by \((X_{\Lambda})_q \rightarrow (X_{\Gamma})_q\). Then we have

\text{Cor}^r : H^r(\Lambda, \Gamma M_T) \rightarrow H^r(\Gamma, M) \quad (r \in \mathbb{Z}).

\textbf{Proposition} We have following fundamental properties for Res and Cor.

(1) Given \(f : \Gamma^* M \rightarrow \Gamma^* N\), we have

\[ f^* \text{Res}^r = \text{Res}^r f^* : H^r(\Gamma, M) \rightarrow H^r(\Lambda, N), \]
\[ f^* \text{Cor}^r = \text{Cor}^r f^* : H^r(\Lambda, M) \rightarrow H^r(\Gamma, N). \]

(2) Given a short exact sequence \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) of \(\Gamma^e\)-modules, we have

\[ \partial \text{Res}^r = \text{Res}^{r+1} \partial : H^r(\Gamma, N) \rightarrow H^{r+1}(\Lambda, L), \]
\[ \partial \text{Cor}^r = \text{Cor}^{r+1} \partial : H^r(\Lambda, N) \rightarrow H^{r+1}(\Gamma, L). \]

(3) Given a \(\Gamma^e\)-module \(M\), we have

\[ \text{Cor}^r \text{Res}^r(w) = N_{\Gamma/\Lambda}(1)w \quad (w \in H^r(\Gamma, M)). \]

Since Res preserves the cup product, it follows that we can define a ring homomorphism
\[ HH^*(\Gamma) \rightarrow HH^*(\Lambda, \Gamma). \]
Moreover, using the embedding of \(\Lambda\)-bimodules \(\Lambda \rightarrow \Gamma\), we can define

\[ HH^*(\Lambda) \rightarrow HH^*(\Lambda, \Gamma) \xrightarrow{\text{Cor}} HH^*(\Gamma). \]

In particular, we have \(Z\Lambda/N_{\Lambda}(\Lambda) \rightarrow Z\Gamma/N_{\Gamma}(\Gamma) : \overline{z} \mapsto \overline{N_{\Gamma/\Lambda}(z)}\) in the zero dimension. Note that the zero dimensional complete Hochschild cohomology is different from the ordinary one (cf. [Br]).

On the other hand, using the \(\Lambda\)-bimodule homomorphism \(\mu_{\Gamma/\Lambda} : \Gamma \rightarrow \Lambda\), we have

\[ HH^*(\Gamma) \xrightarrow{\text{Res}} HH^*(\Lambda, \Gamma) \xrightarrow{\mu_{\Gamma/\Lambda}} HH^*(\Lambda). \]

In particular, we have \(Z\Gamma/N_{\Gamma}(\Gamma) \rightarrow Z\Lambda/N_{\Lambda}(\Lambda) : \overline{z} \mapsto \overline{\mu_{\Gamma/\Lambda}(z)}\) in the zero dimension.

We have some explicit calculations of Res and Cor for twisted group algebras and crossed products (see [S1] and [S2] for twisted group algebras, and [S4] for crossed products).
References


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