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Graded centers and p-blocks of finite groups

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1 The center of a category

The center of a category is a notion which goes back to work of P. Gabriel [3]. Given a commutative ring k and a k-linear category $\mathcal{C}$, the *center of* $\mathcal{C}$ is the k-algebra $Z(\mathcal{C})$ consisting of all natural transformations $\varphi : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$. Explicitly, an element $\varphi \in Z(\mathcal{C})$ is a family of morphisms

$$\{\varphi(X) : X \to X\}_{X \in \mathrm{Ob}(\mathcal{C})}$$

such that for any morphism $\psi : X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi(X)"] \arrow[d, "\psi"'] & X \arrow[d, "\psi"] \\ Y \arrow[r, "\varphi(Y)"'] & Y \end{tikzcd}$$

is commutative. It is easy to see that $Z(\mathcal{C})$ is a commutative k-algebra with unit element $\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$ (the identity transormation on the identity functor on $\mathcal{C}$). Note that we ignore set theoretic issues (we implicitly assume that $\mathcal{C}$ is equivalent to a small category; this is sufficient for the applications below).

Examples 1.1. Let k be a commutative ring and A a k-algebra.

(a) Denote by $\mathrm{mod}(A)$ the category of finitely generated left A-modules. We have an isomorphism

$$Z(A) \cong Z(\mathrm{mod}(A))$$

sending $z \in Z(A)$ to the natural transformation φ_z given by left multiplication with z; that is, $\varphi_z(M)(m) = zm$ for any finitely generated A-module M and $m \in M$. The inverse of this map sends $\varphi \in Z(\mathrm{mod}(A))$ to $\varphi(A)(1_A)$, where here A is viewed as left A-module.

(b) Denote by $D^b(A)$ the bounded derived category of finitely generated left A-modules. The map sending $z \in Z(A)$ to left multiplication by z on the components of a complex of A-modules induces an injective k-algebra homomorphism

$$Z(A) \longrightarrow Z(D^b(A))$$

but this map need not be surjective (there are examples due to Rickard and Künzer).

(c) Suppose that k is a field and that A is a finite-dimensional k-algebra. The *stable category* $\overline{\text{mod}}(A)$ has the same objects as $\text{mod}(A)$, and morphisms in $\overline{\text{mod}}(A)$ are quotients $\overline{\text{Hom}}_A(U,V) = \text{Hom}_A(U,V)/\text{Hom}_A^{pr}(U,V)$, where U, V are finitely generated A-modules and where $\text{Hom}_A^{pr}(U,V)$ is the space of all A-homomorphisms from U to V which factor through a projective A-module. Again, the map sending $z \in Z(A)$ to left multiplication by z on each A-module induces a k-algebra homomorphism $Z(A) \to Z(\overline{\text{mod}}(A))$. This map is not injective, and it is not known whether it is surjective. Neither the kernel nor the image of this map are understood in general.

2 The graded center of a graded category

The graded center of a graded category is being considered by a growing number of authors including Buchweitz, Flenner, Benson, Iyengar, Krause; see for instance [2], [1], [7].

Definition 2.1. Let k be a commutative ring, let $\mathcal{C}$ be a k-linear category, and suppose that $\mathcal{C}$ is *graded*; that is, $\mathcal{C}$ is endowed with a k-linear equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$. The *graded center of* $(\mathcal{C}, \Sigma)$ is the graded k-module $Z^*(\mathcal{C}) = Z^*(\mathcal{C}, \Sigma)$ whose degree n component $Z^n(\mathcal{C})$ consists of all natural transformations

$$\varphi : \text{Id} \longrightarrow \Sigma^n$$

with the property that $\Sigma\varphi = (-1)^n \varphi\Sigma$, for any integer n. Explicitly, an element $\varphi \in Z^n(\mathcal{C})$ is a familiy of morphisms

$$\{\varphi(X) : X \to \Sigma^n(X)\}_{X \in \text{Ob}(\mathcal{C})}$$

such that for any morphism $\psi : X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi(X)} & \Sigma^n(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\Sigma^n(\psi)} \\ Y & \xrightarrow[\varphi(Y)]{} & \Sigma^n(Y) \end{array}$$

is commutative and such that for any object X in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} \Sigma(X) & \xrightarrow{\Sigma(\varphi(X))} & \Sigma^{n+1}(X) \\ \| & & \downarrow{\scriptstyle(-1)^n \text{Id}} \\ \Sigma(X) & \xrightarrow[\varphi(\Sigma(X))]{} & \Sigma^{n+1}(X) \end{array}$$

is commutative.

Remarks 2.2. Let k be a commutative ring and let $(\mathcal{C}, \Sigma)$ be a graded k-linear category.
(a) The graded k-module $Z^*(\mathcal{C})$ is in fact a graded commutative k-algebra, with product defined as follows: for m, n integers and $\varphi \in Z^m(\mathcal{C})$, $\psi \in Z^n(\mathcal{C})$ we define $\varphi\psi \in Z^{m+n}(\mathcal{C})$ as the familiy of compositions of maps

$$\varphi\psi(X) = (X \xrightarrow{\psi(X)} \Sigma^n(X) \xrightarrow{\Sigma^m(\varphi(X))} \Sigma^{m+n}(X))$$

One easily checks that then $\varphi\psi = (-1)^{mn}\psi\varphi$.
(b) Note that $Z^0(\mathcal{C}) \subseteq Z(\mathcal{C})$ is an inclusion of commutative k-algebras. This inclusion need not be an equality in general because the elements in $\varphi \in Z^0(\mathcal{C})$ satisfy the additional condition $\Sigma\varphi = \varphi\Sigma$.
(c) The relevant examples of graded categories in the context of block theory are actually *triangulated categories*, a concept introduced by Verdier and Puppe. The graded category $(\mathcal{C}, \Sigma)$ is triangulated if for any morphism $f : X \to Y$ in $\mathcal{C}$ there is a *distinguished* or *exact triangle*; that is, a sequence of morphisms of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X)$$

satisfying a list of properties, one of which is that then the "shifted" triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma(X) \xrightarrow{-\Sigma(f)} \Sigma(Y)$$

is also exact, implying in particular that

$$\Sigma^n(X) \xrightarrow{(-1)^n\Sigma^n(f)} \Sigma^n(Y) \xrightarrow{(-1)^n\Sigma^n(g)} \Sigma^n(Z) \xrightarrow{(-1)^n\Sigma^n(h)} \Sigma^{n+1}(X)$$

is exact. The point of the additional commutation $\Sigma\varphi = (-1)^n\varphi\Sigma$ in the definition of an element $\varphi \in Z^n(\mathcal{C})$ is that then φ induces morphisms of exact triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi(X)"'] & Y \arrow[r, "g"] \arrow[d, "(-1)^n\varphi(Y)"'] & Z \arrow[r, "h"] \arrow[d, "\varphi(Z)"] & \Sigma(X) \arrow[d, "(-1)^n\varphi(\Sigma(X))=\Sigma(\varphi(X))"] \\
\Sigma^n(X) \arrow[r, "(-1)^n\Sigma^n(f)"'] & \Sigma^n(Y) \arrow[r, "(-1)^n\Sigma^n(g)"'] & \Sigma^n(Z) \arrow[r, "(-1)^n\Sigma^n(h)"'] & \Sigma^{n+1}(X)
\end{tikzcd}$$

3 Examples

3.1 Derived categories and Hochschild cohomology

Let A be an algebra over a commutative ring k such that A is finitely generated projective as k-module. The derived bounded category $D^b(A)$ of finitely generated A-modules is triangulated, with the shift functor [1] as self-equivalence and mapping cone sequences as exact triangles. Since

A is finitely generated projective as k-module, the Hochschild cohomology $HH^*(A)$ of A can be identified with the Ext-algebra of A as $A \otimes_k A^{op}$-module. There is a canonical graded k-algebra homomorphism

$$HH^*(A) \longrightarrow Z^*(D^b(A))$$

sending an element in $HH^n(A)$ represented by a morphism $\zeta : A \to A[n]$ in $D^b(A)$ to the family of chain maps $\zeta \otimes \mathrm{Id}_X : X \longrightarrow X[n]$, where X is a bounded complex of left A-modules and where we identify $A \otimes_A X \cong X$ and $A[n] \otimes_A X \cong X[n]$, for any integer n (note though that $HH^n(A) = \{0\}$ for n negative). This graded algebra homomorphism is neither injective nor surjective, in general.

3.2 Finite p-group algebras

Let p be a prime, P a finite p-group and suppose that k is a field of characteristic p. Evaluation at the trivial kP-module k induces a graded algebra homomorphism

$$Z(D^b(kP)) \longrightarrow H^*(P; k)$$

which is surjective and whose kernel $\mathcal{N}$ is a nilpotent ideal (cf. [7, 1.3]). For the surjectivity one observes that this map has a section sending $\zeta \in H^n(P, k)$ to the family $\zeta \otimes \mathrm{Id}_M$, with M runnig over the finitely generated left kP-modules (so we make use of the Hopf algebra structure of kP). The nilpotency of $\mathcal{N}$ follows from the fact that $D^b(kP)$ is a triangulated category of finite dimension, in the sense of Rouquier.

3.3 Stable categories of symmertic algebras

Let k be a field and let A be a finite-dimensional symmetric k-algebra; that is, the k-dual $A^* = \mathrm{Hom}_k(A, k)$ of A is isomorphic to A as A-A-bimodule. Examples of symmetric algebras include group algebras of finite groups and Iwahori-Hecke algebras. Then the stable category $\overline{\mathrm{mod}}(A)$ is triangulated, with a shift functor Σ which sends an A-module U to the cokernel $\mathrm{coker}(U \to I)$ of an injective envelope $U \to I$ of U. By a theorem of Rickard, there is a canonical functor of triangulated categories

$$D^b(A) \longrightarrow \overline{\mathrm{mod}}(A)$$

The Tate analogue $\hat{HH}^*(A)$ of Hochschild cohomology has the properties that $\hat{HH}^n(A) = HH^n(A)$ for n positive, $\hat{HH}^0(A)$ is the quotient of $HH^0(A)$ by the ideal generated by the projective ideal $Z^{pr}(A)$ of $Z(A)$ consisting of all $z \in Z(A)$ such that left (or right) multiplication by z on A induces an $A \otimes_k A^{op}$-endomorphism of A belonging to $\mathrm{End}^{pr}_{A \otimes_k A^{op}}(A)$, and for n negative we have Tate duality $\hat{HH}^n(A) \cong \hat{HH}^{-n-1}(A)^*$ while $HH^n(A) = \{0\}$. As in the case of Hochschild cohomology, there is a canonical graded k-algebra homomorphism

$$\hat{HH}^*(A) \longrightarrow Z^*(\overline{\mathrm{mod}}(A))$$

about which very little is known.

3.4 Brauer tree algebras

Let A be a Brauer tree algebra over a field k. The canonical map

$$\hat{HH}^*(A) \longrightarrow Z^*(\overline{\mathrm{mod}}(A))$$

is surjective in even degrees and zero in odd degrees. In particular, this map induces an isomorphism modulo nilpotent ideals, the degree zero component $Z(A) \to Z^0(\overline{\mathrm{mod}}(A))$ is surjective and $Z^0(\overline{\mathrm{mod}}(A))$ is a uniserial algebra. This is proved by explicit calculations, using first that A can be replaced by a serial algebra and then the fact that there are only finitely many isomorphism classes of indecomposable modules. See [4] for details.

3.5 Degree -1 and almost split sequences

Given a symmetric algebra A over a field k, it is not known whether $Z^0(\overline{\mathrm{mod}}(A))$ is even finite-dimensional. It turns out that almost split sequences determine elements in $Z^{-1}(\overline{\mathrm{mod}}(A))$: any almost split sequence is of the form

$$0 \longrightarrow \Sigma^{-2}(U) \longrightarrow E \longrightarrow U \longrightarrow 0$$

hence determines an element ζ_U in $\mathrm{Ext}^1_A(U, \Sigma^{-2}(U)) \cong \overline{\mathrm{Hom}}_A(U, \Sigma^{-1}(U))$, and then the family $(\zeta_U)_{U \in \overline{\mathrm{mod}}(A)}$ is an element in $Z^{-1}(\overline{\mathrm{mod}}(A))$. As a consequence we get that $Z^{-1}(\overline{\mathrm{mod}}(A))$ is infinite dimensional whenever $\overline{\mathrm{mod}}(A)$ has infinitely many non periodic Σ-orbits; see [7, §2]. If Tate duality could be extended to $Z^*(\overline{\mathrm{mod}}(A))$ in some sense this would imply that $Z^0(\overline{\mathrm{mod}}(A))$ would also be infinite dimensional in that case.

4 Transfer

The group theoretic notions of tranfer between the comology rings of a finite group G and a subgroup H of G over some commutative ring k is based on the fact that restriction and induction are both left and right adjoint functors between $\mathrm{mod}(kG)$ and $\mathrm{mod}(kH)$. Similarly, any pair of biadjoint functors between module categories $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$ of symmetric k-algebras A, B yields transfer maps between their Hochschild cohomology rings $HH^*(A)$ and $HH^*(B)$; cf. [5]. The same principle extends to centers of graded categories (cf. [7]):

Definition 4.1. Let $(\mathcal{C}, \Sigma)$, $(\mathcal{D}, \Delta)$ be graded k-linear categories, where k is a commutative ring, and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be two biadjoint functors commuting with Σ and Δ. We define the transfer map

$$\mathrm{tr}_{\mathcal{F}} : Z^*(\mathcal{C}) \longrightarrow Z^*(\mathcal{D})$$

by sending an element $\varphi \in Z^n(\mathcal{C})$ to the composition of natural transformations

$$\mathrm{Id}_{\mathcal{D}} \longrightarrow \mathcal{F}\mathcal{G} = \mathcal{F}\mathrm{Id}_{\mathcal{C}}\mathcal{G} \xrightarrow{\mathcal{F}\varphi\mathcal{G}} \mathcal{F}\Sigma^n\mathcal{G} = \mathcal{F}\mathcal{G}\Delta^n \longrightarrow \Delta^n$$

where the first and last arrows are induced by adjunction units and counits, respectively. Analogously we define

$$\mathrm{tr}_{\mathcal{G}} : Z^*(\mathcal{D}) \longrightarrow Z^*(\mathcal{C})$$

An element $\varphi \in Z^n(\mathcal{C})$ is called *$\mathcal{F}$-stable* if there is $\psi \in Z^n(\mathcal{D})$ such that $\mathcal{F}\varphi = \psi\mathcal{F}$ as natural transformations from $\mathcal{F}$ to $\mathcal{F}\Sigma^n = \Delta^n\mathcal{F}$. An element in $Z^*(\mathcal{C})$ is $\mathcal{F}$-stable if all its components are $\mathcal{F}$-stable. We denote by $Z^*_{\mathcal{F}}(\mathcal{C})$ the set of $\mathcal{F}$-stable elements in $\mathbb{Z}^*(\mathcal{C})$; this is a graded subalgebra of $Z^*(\mathcal{C})$.

The transfer maps defined above depend on a choice of adjunction isomorphisms. These maps are graded k-linear, but not multiplicative in general. One can use them under certain circumstances to get isomorphisms between subalgebras of stable elements:

Theorem 4.2. *With the notation of 4.1, if* $\mathrm{tr}_{\mathcal{F}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}) \in H^0(\mathcal{D})$ *and* $\mathrm{tr}_{\mathcal{G}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}) \in H^0(\mathcal{C})$ *are invertible then there is a canonical isomorphism of graded algebras*

$$Z^*_{\mathcal{F}}(\mathcal{C}) \cong Z^*_{\mathcal{G}}(\mathcal{D})$$

The word canonical in the above theorem refers to the fact that the isomorphism does no longer depend on the choice of adjunctions so long as the elements $\mathrm{tr}_{\mathcal{F}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}) \in H^0(\mathcal{D})$ and $\mathrm{tr}_{\mathcal{G}}(\mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}) \in H^0(\mathcal{C})$ are invertible.

5 Applications to block theory

Let p be a prime number, k and algebraically closed field of characteristic p and and let G be a finite group. A *block of kG* is an indecomposable direct factor B of kG as k-algebra, or, which amounts to the same, an indecomposable direct summand of kG as kG-kG-bimodule. A block B of kG gives rise to two types of invariants, associated with either

- the module category $\mathrm{mod}(B)$, or
- the *fusion system* $\mathcal{F}$ of B on a defect group P of B.

The relationship between the two types of invariants is one of the mysteries which drives block theory. For instance, it is not known whether two blocks B, B' (of possibly different finite groups) with equivalent module categories will have isomorphic defect groups and fusion systems. Conversely, some of the deepest conjectures in block theory such as Alperin's weight conjecture predict that the number of isomorphism classes of simple B-modules can be expressed in terms of the fusion system together with a certain cohomological invariant of $\mathcal{F}$. One of the invariants of the fusion system $\mathcal{F}$ of the block B is the *block cohomology* $H^*(B)$ defined as inverse limit over $\mathcal{F}$ of the contravariant functor sending a subgroup Q of the defect group P to its cohomology ring $H^*(Q;k)$. This is a finitely generated graded commutative k-algebra, hence defines a variety $V(B)$, called *block variety* (cf. [6]). The next observation, which relates block cohomology $H^*(B)$ and the derived category of B is again based on the fact that bounded derived categories of finite dimensional algebras are finite dimensional.

Proposition 5.1. *There is a canonical graded algebra homomorphism*

$$H^*(B) \longrightarrow Z^*(D^b(B))$$

and a nilpotent ideal $\mathcal{N}$ in $Z^*(D^b(B))$ such that $Z^*(D^b(B))/\mathcal{N}$ becomes noetherian as $H^*(B)$-module; in particular, $Z^*(D^b(B))/\mathcal{N}$ is finitely generated as k-algebra.

One would very much like a more precise result: is it true that actually $H^*(B) \cong Z^*(D^b(B))/\mathcal{N}$ for some nilpotent ideal $\mathcal{N}$? If true, it would have the consequence that any two derived equivalent blockalgebras B, B' would automatically have homeomorphic block varieties. The relevance of this type of statement, if true, lies precisely in the fact that $D^b(B)$ is an invariant of the module category of H while the block variety $V(B)$ is an invariant of the fusion system $\mathcal{F}$ of B. Using the transfer technology from the previous section one can show the following weaker result:

Theorem 5.2. *Denote by $\mathcal{G} : D^b(B) \to D^b(kP)$ the functor induced by restriction. The canonical map $H^*(B) \to Z^*(D^b(B))$ sends $H^*(B)$ to $Z^*_{\mathcal{G}}(D^b(B))$, and there is a nilpotent ideal $\mathcal{N}$ in $Z^*_{\mathcal{G}}(D^b(B))$ such that*

$$H^*(B) \cong Z^*_{\mathcal{G}}(D^b(B))/\mathcal{N}$$

See [7] for proofs. While certainly a step in the right direction, the above result is not satisfactory as yet because we do not know "how far" the subalgebra $Z^*_{\mathcal{G}}(D^b(B))$ is from $Z^*(D^b(B))$.

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