The Chow rings of the algebraic groups $E_6, E_7,$ and $E_8$

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1 Introduction

Let $G$ be a simply connected, simple algebraic group over the complex numbers $\mathbb{C}$, $B$ a Borel subgroup and $H$ a maximal torus contained in $B$. Denote by $\hat{H}$ the character group of $H$. By taking the first Chern class of the homogeneous line bundle $L_{\chi}$ over the flag variety $G/B$ associated to each character $\chi$, we define the characteristic homomorphism for $G$,

$$c_G : S(\hat{H}) \rightarrow A(G/B),$$

where $S(\hat{H})$ is the symmetric algebra of $\hat{H}$ and $A(G/B) = \oplus_{i \geq 0} A^i(G/B)$ is the Chow ring of the algebraic variety $G/B$.

According to Grothendieck's remark ([6], p.21, REMARQUES 2°), the Chow ring $A(G)$ of $G$ is obtained as the quotient of $A(G/B)$ by the ideal

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generated by the image of $\hat{H}$ under $c_G$. Following this remark, $A(G)$ for $G = \text{SO}(n), \text{Spin}(n), G_2,$ and $F_4$ were computed by R. Marlin [8]. So the remaining simply connected simple groups are $E_6, E_7,$ and $E_8$.

**Problem 1.1** *Determine the Chow rings of $E_6, E_7,$ and $E_8.*

### 2 Computations of $A(G/B)$

In order to determine the Chow ring $A(G)$ of $G$ following Grothendieck's remark, we have to compute the Chow ring $A(G/B)$ of the corresponding flag variety $G/B$. As for the Chow rings of flag varieties, the following fact is known.

**Fact 2.1** *The Chow ring $A(G/B)$ is isomorphic to the integral cohomology ring $H^*(G/B; \mathbb{Z})$ via the cycle map.*

In what follows, we consider the integral cohomology ring $H^*(G/B; \mathbb{Z})$. As is well known, there are two different ways of describing the cohomology of $G/B$. Namely, the *Borel presentation* and the *Schubert presentation*, which we now recall.

**Borel presentation**

Let $K$ be a maximal compact subgroup of $G$ and $T = K \cap H$ a maximal torus of $K$. Then we have the diffeomorphism $G/B \cong K/T$ by the Iwasawa decomposition of $G$. According to Borel, there exists a fibration

$$K/T \xrightarrow{\iota} BT \xrightarrow{\rho} BK,$$

where $BT$ (resp. $BK$) denotes the classifying space of $T$ (resp. $K$). The induced homomorphism in cohomology,

$$c = \iota^*: H^*(BT; \mathbb{Z}) \longrightarrow H^*(K/T; \mathbb{Z})$$

(2)

is called *Borel's characteristic homomorphism* and can be identified with the characteristic homomorphism (1). The Weyl group $W$ of $K$ acts naturally on $T$, hence on $H^2(BT; \mathbb{Z})$. We extend this action of $W$ to the whole $H^*(BT; \mathbb{Z})$ and also to $H^*(BT; F) = H^*(BT; \mathbb{Z}) \otimes F$, where $F$ is any field. Then one of Borel's results can be stated as follows.
Theorem 2.2 Let $\mathbb{F}$ be a field of characteristic zero. Then Borel's characteristic homomorphism induces an isomorphism,

$$\bar{c}: H^*(BT; \mathbb{F})/(H^+(BT; \mathbb{F})^W) \to H^*(K/T; \mathbb{F}),$$

where $(H^+(BT; \mathbb{F})^W)$ is the ideal of $H^*(BT; \mathbb{F})$ generated by the $W$-invariants of positive degrees.

In particular, one can reduce the computation of the rational cohomology ring $H^*(K/T; \mathbb{Q})$ to that of the ring of invariants $H^*(BT; \mathbb{Q})^W$. In order to determine the integral cohomology ring $H^*(K/T; \mathbb{Z})$, we need further considerations. General description of $H^*(K/T; \mathbb{Z})$ by a minimal system of generators and relations was given by H. Toda [12]. Up to now, the following results have been available.

- $H^*(SU(n+1)/T; \mathbb{Z})$ ... Borel (1953),
- $H^*(SO(2n+1)/T; \mathbb{Z})$ ... Toda-Watanabe (1974),
- $H^*(Sp(n)/T; \mathbb{Z})$ ... Borel (1953),
- $H^*(SO(2n)/T; \mathbb{Z})$ ... Toda-Watanabe (1974),
- $H^*(G_2/T; \mathbb{Z})$ ... Bott-Samelson (1955),
- $H^*(F_4/T; \mathbb{Z})$ ... Toda-Watanabe (1974),
- $H^*(E_6/T; \mathbb{Z})$ ... Toda-Watanabe (1974),
- $H^*(E_7/T; \mathbb{Z})$ ... Nakagawa (2001),
- $H^*(E_8/T; \mathbb{Z})$ ... Nakagawa (2007).

Remark 2.3 In the Borel presentation, the ring structure of $H^*(K/T; \mathbb{Z})$ is relatively easy to obtain. However, the ring generators have little "geometric meaning" in this presentation.

Schubert presentation

As is well known, $G$ has the Bruhat decomposition,

$$G = \coprod_{w \in W} BwB,$$

where $w$ denotes any representative of $w \in W$. It induces a cell decomposition,

$$G/B = \coprod_{w \in W} BwB/B,$$
where \( X^*_w = BwB/B \cong \mathbb{C}^{l(w)} \) is called the Schubert cell. Here \( l(w) \) is the length of the element \( w \in W \). The Schubert variety \( X_w \) is defined to be the closure of \( X^*_w \). Denote by \( [X_w] \in H_{2l(w)}(G/B; \mathbb{Z}) \) the image of the fundamental class \( [X_w] \in H_{2l(w)}(X_w; \mathbb{Z}) \) under the induced homomorphism by the inclusion \( X_w \hookrightarrow G/B \). We define a cohomology class \( Z_w \in H^{2l(w)}(G/B; \mathbb{Z}) \) as the Poincaré dual of \( [X_{w_0w}] \), where \( w_0 \) is the longest element of \( W \). We call \( Z_w \) the Schubert class. Then we have

**Fact 2.4** The Schubert classes \( \{Z_w\}_{w \in W} \) form an additive basis for \( H^*(G/B; \mathbb{Z}) \). We refer to \( \{Z_w\}_{w \in W} \) as the Schubert basis.

**Remark 2.5** In the Schubert presentation, the Schubert classes correspond to the geometric objects – the Schubert varieties. However, the multiplicative structure among them is highly complicated.

Now we consider the following problem.

**Problem 2.6** Establish a connection between the Borel presentation and the Schubert presentation.

Our main tool is the divided difference operators introduced independently by Bernstein-Gelfand-Gelfand [1] and Demazure [5].

**Divided difference operators**

First we need some notation.

\( \Delta \): the root system of \( K \) with respect to \( T \);

\( \Delta^+ \): a set of positive roots;

\( \Pi \): the system of simple roots;

\( s_\alpha \): the reflection corresponding to the simple root \( \alpha \in \Pi \).

**Definition 2.7** (i) For each \( \alpha \in \Delta \), the operator

\[
\Delta_\alpha : H^*(BT; \mathbb{Z}) \longrightarrow H^*(BT; \mathbb{Z})
\]

is defined as

\[
\Delta_\alpha(u) = \frac{u - s_\alpha(u)}{u} \quad \text{for} \quad u \in H^*(BT; \mathbb{Z}).
\]
(ii) For \( w \in W \), the operator \( \Delta_w \) is defined as
\[
\Delta_w = \Delta_{\alpha_1} \circ \Delta_{\alpha_2} \cdots \circ \Delta_{\alpha_k},
\]
where \( w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k} \) (\( \alpha_i \in \Pi \)) is any reduced decomposition of \( w \).

One can show that the definition is well defined, i.e., independent of the choice of a reduced decomposition of \( w \). Then Borel's characteristic homomorphism (2) can be described by the divided difference operators.

**Theorem 2.8 (Bernstein-Gelfand-Gelfand [1], Demazure [5])** For a homogeneous polynomial \( f \in H^{2k}(BT; \mathbb{Z}) \), we have
\[
c(f) = \sum_{w \in W, l(w) = k} \Delta_w(f) Z_w. \tag{3}
\]

In particular, for \( \alpha \in \Pi \), we have
\[
c(\omega_\alpha) = Z_{s_\alpha},
\]
where \( \omega_\alpha \) denotes the fundamental weight corresponding to the simple root \( \alpha \in \Pi \).

### 3 \( H^*(E_l/T; \mathbb{Z}) \) \( (l = 6, 7, 8) \)

Let \( E_l \) \( (l = 6, 7, 8) \) be the simply connected simple complex algebraic group of exceptional type, \( E_l \) its maximal compact subgroup and \( T \) a maximal torus of \( E_l \). According to [4], we take the simple roots \( \{\alpha_i\}_{1 \leq i \leq l} \) and denote by \( \{\omega_i\}_{1 \leq i \leq l} \) the corresponding fundamental weights. Let \( s_i \) \( (1 \leq i \leq l) \) denote the reflection corresponding to the simple root \( \alpha_i \) \( (1 \leq i \leq l) \). Then the Weyl group \( W(E_l) \) of \( E_l \) is generated by \( s_i \) \( (1 \leq i \leq l) \). As usual, we regard roots and weights as elements of \( H^2(BT; \mathbb{Z}) \). Following the notation in [11], [9], and [10], we put
\[
\begin{align*}
t_l &= \omega_l, \\
t_i &= s_{i+1}(t_{i+1}) \quad (2 \leq i \leq l - 1), \\
t_1 &= s_1(t_2), \\
t &= \omega_2, \\
c_i &= \sigma_i(t_1, \ldots, t_i) \quad (1 \leq i \leq l),
\end{align*}
\tag{4}
\]
where $\sigma_i(t_1, \ldots, t_l)$ denotes the $i$-th elementary symmetric function in the variables $t_1, \ldots, t_l$. Then we have

$$H^*(BT; \mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_l] = \mathbb{Z}[t_1, t_2, \ldots, t_l, t]/(c_1 - 3t).$$

Since we consider the simply connected form of the groups, Borel's characteristic homomorphism restricted in degree 2 is an isomorphism:

$$c = i^*: H^2(BT; \mathbb{Z}) \longrightarrow H^2(E_l/T; \mathbb{Z}).$$

Under this isomorphism, we denote the images of $t_i$ ($1 \leq i \leq l$) and $t$ by the same symbols. Thus $H^2(E_l/T; \mathbb{Z})$ is a free $\mathbb{Z}$-module generated by $t_i$ ($1 \leq i \leq l$) and $t$ with a relation $c_1 = 3t$.

Then the integral cohomology ring of $E_6/T$ is given as follows.

**Theorem 3.1 ([11], Theorem B)** The integral cohomology ring of $E_6/T$ is

$$H^*(E_6/T; \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_6, t, \gamma_3, \gamma_4]/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12}),$$

where

$$\rho_1 = c_1 - 3t, \quad \rho_2 = c_2 - 4t^2, \quad \rho_3 = c_3 - 2\gamma_3, \quad \rho_4 = c_4 + 2t^4 - 3\gamma_4, \quad \rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3, \quad \rho_6 = \gamma_3^2 + 2c_6 - 3t^2\gamma_4 + t^6, \quad \rho_8 = 3\gamma_4^2 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8, \quad \rho_9 = 2c_6\gamma_3 - 3t^3c_6, \quad \rho_{12} = 3c_6^2 - 2\gamma_4^3 + 6t\gamma_3\gamma_4^2 + 3t^2c_6\gamma_4 + 5t^3c_6\gamma_3 - 15t^4\gamma_4^2 - 10t^6c_6 + 19t^8\gamma_4 - 6t^9\gamma_3 - 2t^{12}.$$

Similar presentations of $H^*(E_l/T; \mathbb{Z})$ ($l = 7, 8$) are also obtained in [9] and [10].

Now we consider the following problem.
Problem 3.2 Find the relations between the ring generators \( \{t_1, \ldots, t_l, t, \gamma_3, \gamma_4, \ldots\} \) in the Borel presentation and the Schubert basis \( \{Z_w\}_{w \in W(E_l)} \) \( (l = 6, 7, 8) \).

We will show how to do this in the case of \( E_6 \). Since \( c(\omega_i) = Z_i \) by Theorem 2.8, it follows immediately from (4) that

\[
\begin{align*}
t_1 &= -Z_1 + Z_2, \\
t_2 &= Z_1 + Z_2 - Z_3, \\
t_3 &= Z_2 + Z_3 - Z_4, \\
t_4 &= Z_4 - Z_5, \\
t_5 &= Z_5 - Z_6, \\
t_6 &= Z_6, \\
t &= Z_2.
\end{align*}
\]

For \( i = 3, 4 \), we can put

\[
\gamma_i = \sum_{l(w) = i} a_w Z_w
\]

for some integers \( a_w \). We will determine the coefficients \( a_w \). By Theorem 3.1, we have

\[
\begin{align*}
2\gamma_3 &= c_3, \\
3\gamma_4 &= c_4 + 2t^4.
\end{align*}
\]

Therefore \( 2\gamma_3 \) and \( 3\gamma_4 \) are contained in the image of \( c \). Define the polynomials of \( H^*(BT; \mathbb{Z}) \) by

\[
\begin{align*}
\delta_3 &= c_3, \\
\delta_4 &= c_4 + 2t^4,
\end{align*}
\]

so that \( c(\delta_3) = c_3 (= 2\gamma_3) \), \( c(\delta_4) = c_4 + 2t^4 (= 3\gamma_4) \) in \( H^*(E_6/T; \mathbb{Z}) \). We apply the divided difference operators to the polynomials \( \delta_3 \) and \( \delta_4 \).

Thus we obtain
\[ c_3 = 2Z_{342} + 4Z_{542} \]
\[ = 2(Z_{342} + 2Z_{542}), \]
\[ c_4 + 2t^4 = 3Z_{1342} + 6Z_{3542} + 6Z_{6542} \]
\[ = 3(Z_{1342} + 2Z_{3542} + 2Z_{6542}). \]

By (6) and (8), we can express \( \gamma_i \) \((i = 3, 4)\) in terms of Schubert classes. Since \( H^*(E_6/T; \mathbb{Z}) \) is torsion free, we obtain
\[ \gamma_3 = Z_{342} + 2Z_{542}, \]
\[ \gamma_4 = Z_{1342} + 2Z_{3542} + 2Z_{6542}. \]

Moreover, we obtain
\[ Z_{342} = -\gamma_3 + 2t^3, \]
\[ Z_{542} = \gamma_3 - t^3, \]
\[ Z_{1342} = \gamma_4 - 2t\gamma_3 + 2t^4, \]
\[ Z_{3542} = -\gamma_4 + t\gamma_3, \]
\[ Z_{6542} = \gamma_4 - t^4. \]

### 4 Computations of \( A(G) \)

In this section, we determine the Chow rings of the exceptional groups \( E_6, E_7, \) and \( E_8 \). Since we have the following commutative diagram,

\[
\begin{array}{ccc}
S(\hat{H}) & \xrightarrow{\alpha} & A(G/B) \\
\cong \downarrow & & \cong \downarrow \\
H^*(BT; \mathbb{Z}) & \xrightarrow{c} & H^*(G/B; \mathbb{Z}),
\end{array}
\]

we have
\[ A(G) = A(G/B)/(c_G(\hat{H})) \]
\[ = H^*(G/B;\mathbb{Z})/(c(H^2(BT;\mathbb{Z})) \]
\[ = H^*(G/B;\mathbb{Z})/(H^2(G/B;\mathbb{Z})) \]
\[ = H^*(K/T;\mathbb{Z})/(H^2(K/T;\mathbb{Z})). \]

Therefore we have only to compute the quotient ring of \( H^*(K/T;\mathbb{Z}) \) by the ideal generated by \( H^2(K/T;\mathbb{Z}) \). We will show how to do this for the case of \( E_6 \). By Theorem 3.1 and (9), we compute

\[ H^*(E_6/T;\mathbb{Z})/(H^2(E_6/T;\mathbb{Z})) = H^*(E_6/T;\mathbb{Z})/(t_1, \ldots, t_6, t) \]
\[ = \mathbb{Z}[\gamma_3, \gamma_4]/(2\gamma_3, 3\gamma_4, \gamma_3^2, \gamma_4^3) \]
\[ = \mathbb{Z}[Z_{542}, Z_{6542}]/(2Z_{542}, 3Z_{6542}, Z_{542}^2, Z_{6542}^3). \]

In this way, we can compute the Chow rings of \( E_l \) \((l = 6, 7, 8)\). Let \( T_G : A(G/B) \rightarrow A(G) \) denote the natural projection and \( w_0 \) the longest element of the Weyl group \( W(E_l) \) \((l = 6, 7, 8)\). Then we have the following main result.

**Theorem 4.1** (i) The Chow ring of \( E_6 \) is
\[ A(E_6) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3), \]
where \( X_3 = T_{E_6}(X_{w_0\infty\infty\infty\infty\infty}) \) and \( X_4 = T_{E_6}(X_{w_0\infty\infty\infty\infty\infty}). \)

(ii) The Chow ring of \( E_7 \) is
\[ A(E_7) = \mathbb{Z}[X_3, X_4, X_5, X_9] \]
\[ / (2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2), \]
where \( X_3 = T_{E_7}(X_{w_0\infty\infty\infty\infty\infty\infty\infty}) \), \( X_4 = T_{E_7}(X_{w_0\infty\infty\infty\infty\infty\infty\infty}) \), \( X_5 = T_{E_7}(X_{w_0\infty\infty\infty\infty\infty\infty\infty\infty}) \), \( X_9 = T_{E_7}(X_{w_0\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty}). \)

(iii) The Chow ring of \( E_8 \) is
\[ A(E_8) = \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \]
\[ / (2X_3, 3X_4, 2X_5, 5X_6, 2X_9, X_6^2 - 3X_{10}, X_3^3, \]
\[ 2X_{15}, X_9^2, 3X_{10}^2, X_8^2, X_{15}^2 + X_{10}^3 + 2X_6^5), \]
where \( X_i = T_{E_8}(\gamma_i) \) \((i = 3, 4, 5, 6, 9, 10, 15)\).
Remark 4.2 (i) The result of $E_8$ is not satisfactory. We determined merely the ring structure of $A(E_8)$. At present, we are not able to express the ring generators of $H^*(E_8/T; \mathbb{Z})$ in terms of Schubert classes. (ii) For details on the computations for $E_6$ and $E_7$, see [7].

參考文献


