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Nonprincipal Block of $SL(2, q)$

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Abstract

We shall claim that Broué's abelian defect group conjecture holds for the nonprincipal p-block of $SL(2, p^n)$.

1 Introduction

Let G be a finite group and P a p-subgroup of G. The next theorem is one of the most important theorems on the block theory of finite groups:

Brauer's First Main Theorem. *There is one to one correspondence between the blocks of kG with defect group P and the blocks of $kN_G(P)$ with defect group P.*

The correspondence is called *Brauer correspondence.* The following conjecture is our main problem:

Broué's Abelian Defect Group Conjecture. *Suppose that A is a block of kG with an abelian defect group P and that B is the Brauer correspondent of A (in $N_G(P)$). Then is A derived equivalent to B?*

If $G = SL(2, q)$ where $q = p^n$, it has been proved that the conjecture is true for the principal block by T.Okuyama (see [6]). Even in the nonprincipal case, the conjecture was proved to be true for $n = 2$ by M.Holloway (see [4]), but it has not been known if the conjecture is true for $n \geq 3$ yet. However, it has turned out that it can be proved to be true even for $n \geq 3$ by imitating Okuyama's proof [6].

The Main Result. *If $G = SL(2, q)$ where $q = p^n$, Broué's abelian defect group conjecture is true for the nonprincipal block of kG.*

We shall explain about derived equivalences. Let k be an algebraically closed field of characteristic $p > 0$, let A and B be finite dimensional k-algebras, mod-A the category consisting of all finite dimensional right A-modules, proj-A the full subcategory of mod-A consisting of all finite dimensional right projective A-modules, $K^b(\text{mod-}A)$ the homotopy category consisting of all bounded complexes of finite dimensional right A-modules, and $K^b(\text{proj-}A)$ the homotopy category consisting of all bounded complexes of finite dimensional right projective A-modules. We say that A is *derived equivalent* to B if $K^b(\text{proj-}A)$ is equivalent to $K^b(\text{proj-}B)$ as triangulated categories. The next theorem is a criterion for derived equivalence:

Theorem(Rickard [7]). *The following are equivalent.*

(a) A is derived equivalent to B.

(b) There is a complex $T^\bullet \in K^b(\text{proj-}A)$ with $B \cong \operatorname{End}_{K^b(\text{proj-}A)}(T^\bullet)$ such that

(i) $\operatorname{Hom}_{K^b(\text{proj-}A)}(T^\bullet, T^\bullet[i]) = 0$ for any $i \neq 0$.

(ii) If $\operatorname{add}(T^\bullet)$ is the full subcategory of $K^b(\text{proj-}A)$ consisting of all direct summands of all direct sums of $T^\bullet$, then it generates the triangulated category $K^b(\text{proj-}A)$.

We call $T^\bullet$ a *tilting complex* for A.

2 $SL(2, q)$

Set $G = SL(2, q)$ where $q = p^n$. In this section, we shall state some facts of representations of kG. Set

$$P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{F}_q \right\},$$

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{F}_q^\times \right\},$$

and

$$H = N_G(P) = \left\{ \left(\begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right) \middle| a \in \mathbb{F}_q^{\times},\ b \in \mathbb{F}_q \right\},$$

where P is a Sylow p-subgroup of G and hence is isomorphic to the elementary abelian group $C_p \times \cdots \times C_p$ (n times), D is isomorphic to C_{q-1}, and H is the semidirect product $P \rtimes D$.

Considering a nonprincipal block, we assume $p \neq 2$ in the rest of the article (if $p = 2$, kG has no nonprincipal blocks with full defect). Now we have the block decompositions $kG = A_0 \oplus A_1 \oplus A_2$, where A_0 is the principal block, A_1 is a nonprincipal block with full defect, and A_2 has defect zero, and $kN_G(P) = B_0 \oplus B_1$, where B_0 and B_1 are the Brauer correspondents of A_0 and A_1 respectively. It is well known that all nonisomorphic simple kG-modules are indexed by $\{0, 1, 2, \cdots, q-1\}$, where $\{0, 2, \cdots, q-3\}$, $\{1, 3, \cdots, q-2\}$ and $\{q-1\}$ correspond to A_0, A_1 and A_2 respectively, and all nonisomorphic simple $kN_G(P)$-modules are indexed by $\{0, 1, 2, \cdots, q-2\}$, where $\{0, 2, \cdots, q-3\}$ and $\{1, 3, \cdots, q-2\}$ correspond to B_0 and B_1 respectively (see [3] or [6]).

3 Outline of Proof

Set $\Lambda = \{0, 1, 2, \cdots, q-1\}$, $I = I_{odd} = \{1, 3, 5, \cdots, q-2\}$. For $\lambda \in \Lambda - \{q-1\}$, set

$$\widetilde{\lambda} = \begin{cases} 0 & (\text{if } \lambda = 0) \\ q-1-\lambda & (\text{if } \lambda \neq 0), \end{cases}$$

and for a subset $\Omega \subseteq \Lambda - \{q-1\}$, set $\widetilde{\Omega} = \{\widetilde{\lambda} | \lambda \in \Omega\}$. Then for any simple $kN_G(P)$-module, $T_{\widetilde{\lambda}}$ is isomorphic to the dual module T_λ^* of T_λ, and note that " $\widetilde{\cdot}$ " is a permutation on $\Lambda - \{q-1\}$ of order 2. Moreover, we define an equivalence relation " $\sim$ " on $\Lambda - \{q-1\}$ by

$$\lambda \sim \mu \overset{def}{\Leftrightarrow} \text{There exists some } j \in \{0, 1, \cdots, n-1\} \text{ such that } \lambda \equiv p^j \mu \pmod{q-1}.$$

Note that I is closed under the equivalence relation.

We define equivalence classes (with respect to " $\sim$ ") $J_{-1}, J_0, J_1, \cdots, J_s$ as follows (cf. Okuyama [6, §2]):
Let $J_{-1}, \widetilde{J_{-1}}$ be empty sets (by convention), J_0 the class containing 1, and J_i the class containing the smallest $\lambda_i \notin \bigcup_{u=-1}^{i-1}(J_u \cup \widetilde{J_u})$ for $i \geq 1$. We repeat this procedure until s satisfies $I = \bigcup_{u=-1}^{s}(J_u \cup \widetilde{J_u})$.

Now we can construct derived equivalent k-algebras $A^0, A^1, \cdots, A^s, A^{s+1}$ as follows (cf. Okuyama [6, §3]):
First, set $A^0 = A$. Then for $1 \leq t \leq s+1$, we define A^t as an endomorphism algebra of a tilting complex for A^{t-1} determined by J_{t-1} which is seen in [6, §1].

Then, we can show that A^{s+1} is isomorphic to B as k-algebras like Okuyama [6, §3], so we obtain the main result.

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