Nonprincipal Block of $SL(2, q)$

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Abstract

We shall claim that Broué's abelian defect group conjecture holds for the nonprincipal $p$-block of $SL(2, p^n)$.

1 Introduction

Let $G$ be a finite group and $P$ a $p$-subgroup of $G$. The next theorem is one of the most important theorems on the block theory of finite groups:

Brauer's First Main Theorem. There is one to one correspondence between the blocks of $kG$ with defect group $P$ and the blocks of $kN_G(P)$ with defect group $P$.

The correspondence is called Brauer correspondence. The following conjecture is our main problem:

Broué's Abelian Defect Group Conjecture. Suppose that $A$ is a block of $kG$ with an abelian defect group $P$ and that $B$ is the Brauer correspondent of $A$ (in $N_G(P)$). Then is $A$ derived equivalent to $B$?

If $G = SL(2, q)$ where $q = p^n$, it has been proved that the conjecture is true for the principal block by T.Okuyama (see [6]). Even in the nonprincipal case, the conjecture was proved to be true for $n = 2$ by M.Holloway (see [4]), but it has not been known if the conjecture is true for $n \geq 3$ yet. However, it has turned out that it can be proved to be true even for $n \geq 3$ by imitating Okuyama's proof [6].

The Main Result. If $G = SL(2, q)$ where $q = p^n$, Broué's abelian defect group conjecture is true for the nonprincipal block of $kG$. 
We shall explain about derived equivalences. Let $k$ be an algebraically closed field of characteristic $p > 0$, let $A$ and $B$ be finite dimensional $k$-algebras, mod-$A$ the category consisting of all finite dimensional right $A$-modules, proj-$A$ the full subcategory of mod-$A$ consisting of all finite dimensional right projective $A$-modules, $\mathcal{K}^b$ (mod-$A$) the homotopy category consisting of all bounded complexes of finite dimensional right $A$-modules, and $\mathcal{K}^b$ (proj-$A$) the homotopy category consisting of all bounded complexes of finite dimensional right projective $A$-modules. We say that $A$ is derived equivalent to $B$ if $\mathcal{K}^b$ (proj-$A$) is equivalent to $\mathcal{K}^b$ (proj-$B$) as triangulated categories. The next theorem is a criterion for derived equivalence:

**Theorem (Rickard [7]).** The following are equivalent.

(a) $A$ is derived equivalent to $B$.

(b) There is a complex $T^\bullet \in \mathcal{K}^b$ (proj-$A$) with $B \cong \text{End}_{\mathcal{K}^b$(proj-$A$)}(T^\bullet)$ such that

(i) $\text{Hom}_{\mathcal{K}^b$(proj-$A$)}(T^\bullet, T^\bullet[i]) = 0$ for any $i \neq 0$.

(ii) If $\text{add}(T^\bullet)$ is the full subcategory of $\mathcal{K}^b$ (proj-$A$) consisting of all direct summands of all direct sums of $T^\bullet$, then it generates the triangulated category $\mathcal{K}^b$(proj-$A$).

We call $T^\bullet$ a tilting complex for $A$.

2 $SL(2, q)$

Set $G = SL(2, q)$ where $q = p^n$. In this section, we shall state some facts of representations of $kG$. Set

$$P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F}_q \right\},$$

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}_q^\times \right\},$$
and 

\[ H = N_G(P) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_{q}^\times, \; b \in \mathbb{F}_{q} \right\}, \]

where \( P \) is a Sylow \( p \)-subgroup of \( G \) and hence is isomorphic to the elementary abelian group \( C_p \times \cdots \times C_p \) (\( n \) times), \( D \) is isomorphic to \( C_{q-1} \), and \( H \) is the semidirect product \( P \rtimes D \).

Considering a nonprincipal block, we assume \( p \neq 2 \) in the rest of the article (if \( p = 2 \), \( kG \) has no nonprincipal blocks with full defect). Now we have the block decompositions \( kG = A_0 \oplus A_1 \oplus A_2 \), where \( A_0 \) is the principal block, \( A_1 \) is a nonprincipal block with full defect, and \( A_2 \) has defect zero, and \( kN_G(P) = B_0 \oplus B_1 \), where \( B_0 \) and \( B_1 \) are the Brauer correspondents of \( A_0 \) and \( A_1 \) respectively. It is well known that all nonisomorphic simple \( kG \)-modules are indexed by \( \{0, 1, 2, \cdots , q-1\} \), where \( \{0, 2, \cdots , q-3\} \) and \( \{1, 3, \cdots , q-2\} \) correspond to \( A_0 \), \( A_1 \) and \( A_2 \) respectively; and all nonisomorphic simple \( kN_G(P) \)-modules are indexed by \( \{0, 1, 2, \cdots , q-2\} \), where \( \{0, 2, \cdots , q-3\} \) and \( \{1, 3, \cdots , q-2\} \) correspond to \( B_0 \) and \( B_1 \) respectively (see [3] or [6]).

### 3 Outline of Proof

Set \( \Lambda = \{0, 1, 2, \cdots , q-1\} \), \( I = I_{odd} = \{1, 3, 5, \cdots , q-2\} \). For \( \lambda \in \Lambda - \{q-1\} \), set

\[ \sim \lambda = \begin{cases} 0 & \text{(if } \lambda = 0) \\ q - 1 - \lambda & \text{(if } \lambda \neq 0) \end{cases} \]

and for a subset \( \Omega \subseteq \Lambda - \{q-1\} \), set \( \tilde{\Omega} = \{ \sim \lambda \mid \lambda \in \Omega \} \). Then for any simple \( kN_G(P) \)-module, \( T_\lambda \) is isomorphic to the dual module \( T_\lambda^* \) of \( T_\lambda \), and note that "\( \sim \)" is a permutation on \( \Lambda - \{q-1\} \) of order 2. Moreover, we define an equivalence relation "\( \sim \)" on \( \Lambda - \{q-1\} \) by

\[ \lambda \sim \mu \overset{\text{def}}{=} \text{There exists some } j \in \{0, 1, \cdots , n-1\} \text{ such that } \lambda \equiv p^j \mu \pmod{q-1}. \]

Note that \( I \) is closed under the equivalence relation.

We define equivalence classes (with respect to "\( \sim \)") \( J_{-1}, J_0, J_1, \cdots , J_s \) as follows (cf. Okuyama [6, §2]):

Let \( J_{-1}, \tilde{J}_{-1} \) be empty sets (by convention), \( J_0 \) the class containing 1, and \( J_i \) the class containing the smallest \( \lambda_i \notin \bigcup_{u=-1}^{i-1}(J_u \cup \tilde{J}_u) \) for \( i \geq 1 \). We repeat this procedure until \( s \) satisfies \( I = \bigcup_{u=-1}^{s}(J_u \cup \tilde{J}_u) \).
Now we can construct derived equivalent $k$-algebras $A^0, A^1, \ldots, A^s, A^{s+1}$ as follows (cf. Okuyama [6, §3]):

First, set $A^0 = A$. Then for $1 \leq t \leq s + 1$, we define $A^t$ as an endomorphism algebra of a tilting complex for $A^{t-1}$ determined by $J_{t-1}$ which is seen in [6, §1].

Then, we can show that $A^{s+1}$ is isomorphic to $B$ as $k$-algebras like Okuyama [6, §3], so we obtain the main result.

References


[10] R.Rouquier, From stable equivalences to Rickard equivalences for blocks with cyclic defect, Groups '93 Galway/St Andrews II (C.M.Campbell
