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Analytic Solutions of a nonlinear two variables Difference System

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Abstract

For nonlinear difference equations, it is difficult to have analytic solutions of it. Especially, when all the absolute values of the equation are equal to 1, it is quite difficult to have an analytic solution of it.

We consider a second order nonlinear difference equation which can be transformed into the following simultaneous system of nonlinear difference equations,

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases}$$

where $X(x, y) = x + y + \sum_{i+j\geqq 2} c_{ij}x^i y^j$, $Y(x, y) = y + \sum_{i+j\geqq 2} d_{ij}x^i y^j$ and we assume some conditions. For these equations, we will obtain analytic solutions.

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1 Introduction

At first we consider the following second order nonlinear difference equation,

$$\begin{cases} u(t+1) = U(u(t), v(t)), \\ v(t+1) = V(u(t), v(t)), \end{cases} \tag{1.1}$$

where $U(u, v)$ and $V(u, v)$ are entire functions for u and v. We suppose that the equation (1.1) admits an equilibrium point (u^*, v^*) : $\begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} U(u^*, v^*) \\ V(u^*, v^*) \end{pmatrix}$. We can assume, without losing generality, that $(u^*, v^*) = (0, 0)$. Furthermore we suppose that U and V are written in the following form

$$\begin{pmatrix} u(t+1) \\ v(t+1) \end{pmatrix} = M \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} U_1(u(t), v(t)) \\ V_1(u(t), v(t)) \end{pmatrix},$$

where $U_1(u,v)$ and $V_1(u,v)$ are higher order terms of u and v. Let λ_1, λ_2 be characteristic values of matrix M. For some regular matrix P which decided by M, put $\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$, then we can transform the system (1.1) into the following simultaneous system of first order difference equations (1.2):

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \tag{1.2}$$

where $X(x,y)$ and $Y(x,y)$ are supposed to be holomorphic and expanded in a neighborhood of $(0,0)$ in the following form,

$$\begin{cases} X(x,y) = \lambda_1 x + \sum_{i+j\geqq 2} c_{ij}x^i y^j = \lambda_1 x + X_1(x,y), \\ Y(x,y) = \lambda_2 y + \sum_{i+j\geqq 2} d_{ij}x^i y^j = \lambda_2 y + Y_1(x,y), \end{cases} \tag{1.3}$$

or

$$\begin{cases} X(x,y) = \lambda x + y + \sum_{i+j\geqq 2} c'_{ij}x^i y^j = \lambda x + X'_1(x,y), \\ Y(x,y) = \lambda y + \sum_{i+j\geqq 2} d'_{ij}x^i y^j = \lambda y + Y'_1(x,y), \quad (\lambda = \lambda_1 = \lambda_2.) \end{cases} \tag{1.4}$$

In this paper we consider analytic solutions of difference system (1.2) in which X, Y are defined by (1.4). In [7] and [8], we have obtained general analytic solutions of (1.2) in the case $|\lambda_1| \neq 1$ or $|\lambda_2| \neq 1$. But in the case $|\lambda_1| = |\lambda_2| = 1$, it is difficult to prove an existence of analytic solution or seek an analytic solution of the equation. For a long time we have not be able to derive a solution of the equation (1.2) under the condition.

For analytic solutions of a nonlinear first order difference equations, Kimura [2] has studied the cases in which eigenvalue equal to 1, furthermore Yanagihara [10] has studied the cases in which the absolute value of the eigenvalue equal to 1. Then we will study for analytic solutions of nonlinear second order difference equation in which the absolute value of the eigenvalues of the matrix M equal to 1.

In this present paper, making use of theorems in [2], [5], and [9] we will seek an analytic solution of (1.2), in which X, Y are defined by (1.4) and $\lambda = 1$ such that $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$, i.e., we suppose that

$$\begin{cases} X(x,y) = x + y + \sum_{i+j\geqq 2} c_{ij}x^i y^j = x + X_1(x,y), \\ Y(x,y) = y + \sum_{i+j\geqq 2} d_{ij}x^i y^j = y + Y_1(x,y). \end{cases} \tag{1.5}$$

Further we assume $d_{20} = 0$.

Next we consider a functional equation

$$\Psi(X(x,\Psi(x))) = Y(x,\Psi(x)), \tag{1.6}$$

where $X(x,y)$ and $Y(x,y)$ are holomorphic functions in $|x| < \delta_1$, $|y| < \delta_1$. We assume that $X(x,y)$ and $Y(x,y)$ are expanded there as in (1.5).

Consider the simultaneous system of difference equations (1.2). Suppose (1.2) admits a solution $(x(t), y(t))$. If $\frac{dx}{dt} \neq 0$, then we can write $t = \psi(x)$ with a function ψ in a neighborhood of $x_0 = x(t_0)$, and we can write

$$y = y(t) = y(\psi(x)) = \Psi(x), \tag{1.7}$$

as far as $\frac{dx}{dt} \neq 0$. Then the function Ψ satisfies the equation (1.6).

Conversely we assume that a function Ψ is a solution of the functional equation (1.6). If the first order difference equation

$$x(t+1) = X(x(t), \Psi(x(t))), \tag{1.8}$$

has a solution $x(t)$, we put $y(t) = \Psi(x(t))$. Then the $(x(t), y(t))$ is a solution of (1.2). Hence if there is a solution Ψ of (1.6), then we can reduce the system (1.2) to a single equation (1.8).

We have proved the existence of solutions Ψ of (1.6) in [3] ([4]), [5] and [8], and we have proved the existence of solutions in the case which X and Y are defined by (1.5) in [7] and [8]. in other conditions. Hereafter we consider t to be a complex variable, and concentrate on the difference system (1.2). We define domain $D_1(\kappa_0, R_0)$ by

$$D_1(\kappa_0, R_0) = \{t \,:\, |t| > R_0, |\arg[t]| < \kappa_0\}, \tag{1.9}$$

where κ_0 is any constant such that $0 < \kappa_0 \leqq \frac{\pi}{4}$ and R_0 is sufficiently large number which may depend on X and Y. Further we define domain $D^*(\kappa, \delta)$ by

$$D^*(\kappa, \delta) = \{x \,;\, |\arg[x]| < \kappa,\ 0 < |x| < \delta\}, \tag{1.10}$$

where δ is a small constant and κ is a constant such that $\kappa = 2\kappa_0$, i.e., $0 < \kappa \leqq \frac{\pi}{2}$.

Further we defined $g_0^{\pm}$ as following for the cofficients of $X(x,y$ and $Y(x,y)$

$$g_0^+(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4}, \tag{1.11}$$

$$g_0^-(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) - \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4}, \tag{1.12}$$

respectively.

Our aim in this paper is to show the following Theorem 1.

Theorem 1 *Suppose $X(x,y)$ and $Y(x,y)$ are expanded in the forms (1.5). We defined $A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20}$, $A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20}$. We suppose*

$$d_{20} = 0,\ A_2 < 0, \tag{1.13}$$

and we assume the following conditions,

$$(g_0^+(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}) \tag{1.14}$$

$$(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) \tag{1.15}$$

for all $n \in \mathbb{N}$, $(n \geqq 4)$. *Then we have formal solutions* $x(t)$ *of (1.2) the following form*

$$-\frac{1}{A_2 t}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j}\left(\frac{\log t}{t}\right)^k\right)^{-1}, \quad -\frac{1}{A_1 t}\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j}\left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{1.16}$$

where $\hat{q}_{jk}$ *are constants defined by* X *and* Y.

Further suppose $R_1 = \max(R_0, 2/(|A_2|\delta)))$, *then there are two solutions* $x_1(t)$ *and* $x_2(t)$ *of (1.2) such that*

(i) $x_s(t)$ *are holomorphic and* $x_s(t) \in D^*(\kappa, \delta)$ *for* $t \in D_1(\kappa_0, R_1)$, $s = 1, 2$,

(ii) $x_s(t)$ *are expressible in the following form*

$$x_1(t) = -\frac{1}{A_1 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}, \quad x_2(t) = -\frac{1}{A_2 t}\left(1 + b_2\left(t, \frac{\log t}{t}\right)\right)^{-1}, \tag{1.17}$$

where $b_1(t, \log t/t)$, $b_1(t, \log t/t)$ *are asymptotically expanded in* $D_1(\kappa_0, R_1)$ *as*

$$b_1\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geqq 1} \hat{q}_{jk(1)} t^{-j}\left(\frac{\log t}{t}\right)^k, \quad b_2\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geqq 1} \hat{q}_{jk(2)} t^{-j}\left(\frac{\log t}{t}\right)^k,$$

as $t \to \infty$ *through* $D_1(\kappa_0, R_1)$.

2 Proof of Theorem 1

In [2], Kimura considered the following first order difference equation

$$w(t + \lambda) = F(w(t)), \tag{D1}$$

where F is represented in a neighborhood of ∞ by a Laurent series

$$F(z) = z\left(1 + \sum_{j=m}^{\infty} b_j z^{-j}\right), \quad b_m = \lambda \neq 0. \tag{2.1}$$

He defined the following domains

$$\begin{aligned} D(\epsilon, R) = \{t \,:\, |t| > R, |\arg[t] - \theta| < \frac{\pi}{2} - \epsilon, \text{ or } \mathrm{Im}(e^{i(\theta-\epsilon)}t) > R, \\ \text{or } \mathrm{Im}(e^{i(\theta+\epsilon)}t) < -R\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \hat{D}(\epsilon, R) = \{t \,:\, |t| > R, |\arg[t] - \theta - \pi| < \frac{\pi}{2} - \epsilon \text{ or } \mathrm{Im}(e^{-i(\theta+\pi-\epsilon)}t) > R \\ \text{or } \mathrm{Im}(e^{-i(\theta+\pi+\epsilon)}t) < -R\}, \end{aligned} \tag{2.3}$$

where ϵ is an arbitrarily small positive number and R is a sufficiently large number which may depend on ϵ and F, $\theta = \arg\lambda$, (in this present paper, we consider the case $\lambda = 1$ in (D1)). He proved the following Theorem A and B.

Theorem A. *Equation (D1) admits a formal solution of the form*

$$t\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right) \tag{2.4}$$

containing an arbitrary constant, where $\hat{q}_{jk}$ are constants defined by F.

Theorem B. *Given a formal solution of the form (2.4) of (D1), there exists a unique solution $w(t)$ satisfying the following conditions:*
(i) $w(t)$ is holomorphic in $D(\epsilon, R)$,
(ii) $w(t)$ is expressible in the form

$$w(t) = t\left(1 + b\left(t, \frac{\log t}{t}\right)\right), \tag{2.5}$$

where the domain $D(\epsilon, R)$ is defined by (2.2) and $b(t, \eta)$ is holomorphic for $t \in D(\epsilon, R)$, $|\eta| < 1/R$, and in the expansion

$$b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t)\eta^k,$$

$b_k(t)$ is asymptotically develop-able into

$$b_k(t) \sim \sum_{j+k\geqq 1}^{\infty} \hat{q}_{jk} t^{-j},$$

as $t \to \infty$ through $D(\epsilon, R)$, where $\hat{q}_{jk}$ are constants which are defined by F.
Also there exists a unique solution $\hat{w}$ which is holomorphic in $\hat{D}(\epsilon, R)$ and satisfies a condition analogous to (ii), where the domain $\hat{D}(\epsilon, R)$ is defined by (2.3).

In Theorem A and B, he defined the function F as in (2.1). In our method, we can not have a Laurent series of the function F. Hence we derive following Propositions.
In the following, A_2 and A_1 denote the constants $A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20} < 0$, $A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20} < 0$, in Theorem 1, where c_{20} is the coefficient in (1.5), and $g_0^{\pm}(c_{20}, d_{11}, d_{30})$ are defined by the coefficients in (1.5) as in (1.11) and (1.12).

Proposition 2. *Suppose $\tilde{F}(t)$ is formally expanded such that*

$$\tilde{F}(t) = t\left(1 + \sum_{j=1}^{\infty} b_j t^{-j}\right), \qquad b_1 = \lambda \neq 0. \tag{2.6}$$

Then the equation

$$\psi(\tilde{F}(t)) = \psi(t) + \lambda \tag{2.7}$$

has a formal solution

$$\psi(t) = t\left(1 + \sum_{j=1}^{\infty} q_j t^{-j} + q\frac{\log t}{t}\right), \tag{2.8}$$

where q_1 can be arbitrarily prescribed while other coefficients q_j $(j \geqq 2)$ and q are uniquely determined by b_j, $(j = 1, 2, \cdots)$, independently of q_1.

Proposition 3. *Suppose $\tilde{F}(t)$ is holomorphic and expanded asymptotically in $\{t;\ -1/(A_2 t) \in D^*(\kappa, \delta),\ A_2 < 0\}$ as*

$$\tilde{F}(t) \sim t\left(1 + \sum_{j=1}^{\infty} b_j t^{-j}\right), \qquad b_1 = \lambda \neq 0,$$

where $D^*(\kappa, \delta)$ is defined in (1.10). Then the equation (2.7) has a solution $w = \psi(t)$, which is holomorphic in $\{t;\ -1/(A_2 t) \in D^*(\kappa/2, \delta/2),\ A_2 < 0\}$ and has an asymptotic expansion

$$\psi(t) \sim t\left(1 + \sum_{j=1}^{\infty} q_j t^{-j} + q\frac{\log t}{t}\right),$$

there.

These Propositions are proved as in [2] pp.212–222. Since $A_1 \leqq A_2 < 0$ and $\kappa_0 = \kappa/2$, we see that $t \in D_1(\kappa_0, 2/(|A_2|\delta))$ equivalent to $x \in D^*(\kappa_0, \delta/2)$.

We define a function ϕ to be the inverse of ψ such that $w = \psi^{-1}(t) = \phi(t)$. Then we have $\phi \circ \psi(w) = w, \psi \circ \phi(t) = t$, furthermore ϕ is a solution of the following difference equation

$$w(t + \lambda) = \tilde{F}(w(t)), \tag{D}$$

where $\tilde{F}$ is defined as in Propositions 2 and 3 (see pp.236 in [2]). Hereafter, we put $\lambda = 1$, since $\theta = 0$, then we have the following Propositions 4 and 5 similarly to Theorem A and B.

Proposition 4. *Suppose $\tilde{F}(t)$ is formally expanded as in (2.6). Then the equation (D) has a formal solution*

$$w = \phi(t) = t\left(1 + \sum_{j+k \geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right). \tag{2.9}$$

where $\hat{q}_{jk}$ are constants which are defined by $\tilde{F}$ as in Theorem A.

Proposition 5. *Suppose a function ϕ is the inverse of ψ such that $w = \psi^{-1}(t) = \phi(t)$. Given a formal solution of the form (2.9) of (D) where $\tilde{F}(t)$ is defined as in Propositions 3, there exists a unique solution $w(t) = \phi(t)$ which is holomorphic and asymptotically expanded in $\{t;\ t \in D_1(\kappa_0, 2/(|A_2|\delta))\}$ as*

$$w = \phi(t) = t\left(1 + b\left(t, \frac{\log t}{t}\right)\right), \tag{2.10}$$

where

$$b\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k \geqq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k.$$

This function $\phi(t)$ is a solution of difference equation of (D).

In [9], we have proved the following Theorem C.

Theorem C. *Suppose $X(x,y)$ and $Y(x,y)$ are defined in (1.5). Suppose $d_{20} = 0$,*

$$\frac{2c_{20} + d_{11} \pm \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4} \in \mathbb{R}, \quad \frac{2c_{20} + d_{11} + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4} < 0, \tag{2.11}$$

and we assume the following conditions,

$$(g_0^+(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}) \tag{2.12}$$
$$(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) \tag{2.13}$$

for all $n \in \mathbb{N}$, $(n \geqq 4)$, where

$$g_0^+(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4},$$
$$g_0^-(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) - \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4},$$

respectively, then we have a formal solution $\Psi(x) = \sum_{n\geqq 2}^{\infty} a_n x^n$ of (1.6). Further, for any κ, $0 < \kappa \leqq \frac{\pi}{2}$, there are a $\delta > 0$ and a solution $\Psi(x)$ of (1.6), which is holomorphic and can be expanded asymptotically as

$$\Psi(x) \sim \sum_{n=2}^{\infty} a_n x^n, \tag{2.14}$$

in the domain $D^*(\kappa, \delta)$ which is defined in (1.10).

Proof of Theorem 1. At first we will have formal solutions. From Theorem C, we have a formal solution $\Psi(x)$ of (1.6) which can be formally expanded such that

$$\Psi(x) = \sum_{n=2}^{\infty} a_n x^j. \tag{2.15}$$

where $a_2 = g_0^{\pm}(c_{20}, d_{11}, d_{30})$. Hence we suppose the formal solution $\Psi_s(x)$ of (1.6) such that

$$\Psi_s(x) = \sum_{n=2}^{\infty} a_{n(s)} x^n, \quad (s = 1, 2) \tag{2.16}$$

where $a_{2(1)} = g_0^{+}(c_{20}, d_{11}, d_{30})$, $a_{2(2)} = g_0^{-}(c_{20}, d_{11}, d_{30})$.

On the other hand putting $w_1(t) = -\frac{1}{A_1 x(t)}$, $w_2(t) = -\frac{1}{A_2 x(t)}$, in (1.8), we have

$$w_s(t+1) = -\frac{1}{A_s X\Big(x(t), \Psi_s(x(t))\Big)}, \quad (s = 1, 2), \tag{2.17}$$

and

$$-\frac{1}{A_s X\Big(x, \Psi_s(x)\Big)} = w_s\Big[1 + \frac{a_{2(s)} + c_{20}}{A_s} w_i^{-1} + \sum_{k \geqq 2} \tilde{c}_{k(s)} (w_s)^{-k}\Big], \tag{2.18}$$

where $\tilde{c}_{k(s)}$ are defined by c_{ij} and $a_k(s)$ $(i + j \geqq 2,\ i \geqq 1,\ k \geqq 2,\ s = 1, 2)$. From (2.18) and definition of A_s, we have $a_{2(s)} + c_{20} = A_s$. Therefore we can write (2.17) into the following form (2.19),

$$w_s(t+1) = \tilde{F}_s(w_s(t)) = w_s(t)\Big\{1 + w_s(t)^{-1} + \sum_{k \geqq 2} \tilde{c}_{k(s)} (w_s(t))^{-k}\Big\}, \quad (s = 1, 2). \tag{2.19}$$

On the other hand, putting $\lambda = 1$ and $m = 1$ in (2.1), i.e. $\theta = 0$, then making use of the Proposition 4, we have the following formal solutions (2.20) of (2.19),

$$w_s(t) = t\left(1 + \sum_{j+k \geqq 1} \hat{q}_{jk(s)} t^{-j} \Big(\frac{\log t}{t}\Big)^k\right), \quad (s = 1, 2), \tag{2.20}$$

where $\hat{q}_{jk(s)}$ are defined by $\tilde{F}_s$ in (2.19). From (2.18), (2.19) and (1.6), $\tilde{F}_s$ is defined by X and Y. Hence $\hat{q}_{jk(s)}$ are defined by X and Y.

Since $x(t) = -\frac{1}{A_s w_s(t)}$, From the relation of (1.2) and (1.8) with (1.6) in page 3, we have formal solutions $x(t)$ of (1.2) such that

$$x(t) = -\frac{1}{A_s t}\left(1 + \sum_{j+k \geqq 1} \hat{q}_{jk(s)} t^{-j} \Big(\frac{\log t}{t}\Big)^k\right)^{-1}, \quad (s = 1, 2). \tag{2.21}$$

Next we prove the existence of solutions $x^+(t)$ and $x^-(t)$ of (1.2). We suppose that $R_0 > R$ and $\kappa_0 < \frac{\pi}{4} - \epsilon$. Since $\theta = \arg[\lambda] = \arg[1] = 0$, we have

$$D_1(\kappa_0, R_0) \subset D(\epsilon, R). \tag{2.22}$$

For a $x \in D^*(\kappa, \delta)$, making use of Theorem C, we have a solution $\Psi(x)$ of (1.6) which is holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that as in (2.14).

From the assumption $R_1 = \max(R_0, 2/(|A_2|\delta))$ in Theorem 1, making use of Proposition 5, we have a solution $w_s(t)$ $(s = 1, 2)$ of (2.19) which has an asymptotic expansion

$$w_s(t) = t\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right).$$

in $t \in D_1(\kappa_0, R_1)$, where $b_s\left(t, \frac{\log t}{t}\right) \sim t\left(1 + \sum_{j+k\geqq 1} \hat{q}_{jk(s)} t^{-j} \left(\frac{\log t}{t}\right)^k\right)$, $(s = 1, 2)$, respectively. Thus we have solutions $x(t)$ of (1.2) which has the following asymptotic expansions

$$x(t) = -\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1}, \ (s = 1, 2),$$

there. At first we take a small $\delta > 0$. For sufficiently large R, since $R_1 \geqq R_0 > R$, we can have

$$\left|\frac{1}{A_1 t}\right|\left|1 + b_1\left(t, \frac{\log t}{t}\right)\right|^{-1}, \ \left|\frac{1}{A_2 t}\right|\left|1 + b_2\left(t, \frac{\log t}{t}\right)\right|^{-1} < \delta. \tag{2.23}$$

for $t \in D_1(\kappa_0, R_1)$. Since $A_1 \leqq A_2 < 0$ and $\kappa = 2\kappa_0$, for sufficiently large R_1, we have

$$\left|\arg\left[-\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1}\right]\right| < \kappa \leqq \frac{\pi}{2} \text{ for } t \in D_1(\kappa_0, R_1), \ (s = 1, 2). \tag{2.24}$$

From (2.23) and (2.24), we have that

$$x_1(t) = -\frac{1}{A_1 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}, \ x_2(t) = -\frac{1}{A_2 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}$$

such that $x_s(t) \in D^*(\kappa, \delta)$ for a some κ, $(0 < \kappa \leqq \frac{\pi}{2})$. Hence we have $\Psi_s(x(t))$ $(s = 1, 2)$ which satisfies the equation (1.6).

Therefore from existence of a solution Ψ of (1.6), and making use of Proposition 5, we have a holomorphic solution $w(t)$ of first order difference equation (2.19) for $t \in D_1(\kappa_0, R_1)$, i.e., we have a solution $x(t)$ of (1.2) for t at there, in which satisfying following conditions:
(i) $x_s(t)$ are holomorphic in $D_1(\kappa_0, R_1)$ and $x_s(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$, $(s = 1, 2)$,
(ii) $x_s(t)$ are expressible in the form

$$x_s(t) = -\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1}, \tag{2.25}$$

where $b_s(t, \log t/t)$ is asymptotically expanded in $D_1(\kappa_0, R_1)$ as

$$b_s\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geqq 1} \hat{q}_{jk(s)} t^{-j} \left(\frac{\log t}{t}\right)^k,$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$, $s = 1, 2$. $\square$

Finally, we have a solution $(u(t),\ v(t))$ of (1.1) by the transformation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = P \begin{pmatrix} x_1(t) \\ \Psi(x_1(t)) \end{pmatrix},\ P \begin{pmatrix} x_2(t) \\ \Psi(x_2(t)) \end{pmatrix}.$$

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