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# $\alpha$-Conservative Approximation for Probabilistically Constrained Convex Programs *

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**Abstract**

In this paper, we address an approximate solution of a probabilistically constrained convex program (PCCP), where a convex objective function is minimized over solutions satisfying, with a given probability, convex constraints that are parameterized by random variables. In order to approach to a solution, we set forth a conservative approximation problem by introducing a parameter $\alpha$ which indicates an approximate accuracy, and formulate it as a D.C. optimization problem.

As an example of the PCCP, the Value-at-Risk (VaR) minimization is considered under the assumption that the support of the probability of the associated random loss is given by a finitely large number of scenarios. It is advantageous in solving the D.C. optimization that the numbers of variables and constraints are independent of the number of scenarios, and a simplicial branch-and-bound algorithm is posed to find a solution of the D.C. optimization. Numerical experiments demonstrate the following: (i) by adjusting a parameter $\alpha$, the proposed problem can achieve a smaller VaR than the other convex approximation approaches; (ii) when the number of scenarios is large, a typical 0-1 mixed integer formulation for the VaR minimization cannot be solved in a reasonable time and the improvement of the incumbent values is slow, whereas the proposed method can achieve a good solution.

**KeyWords:** chance constraint, D.C. optimization, branch-and-bound, value-at-risk minimization, probabilistically constrained program.

## 1 Introduction

In this paper, we consider an approach to a solution of the probabilistically constrained convex program (PCCP), in which a convex objective function is minimized over constraints including a probabilistic constraint which imposes that the solution would satisfy a designated portion of given convex constraints. Since Charnes, Cooper and Symonds [5] introduced a model involving probabilistic constraints, enormous number of such models have been studied (e.g., [9, 13]), and most of them are in the form of the PCCP.

*This article is the digest version of [7], and refer to [7] for detailed explanation.

Many methods have been proposed to solve general PCCP problems, and they can be roughly classified into three types: (a) nonlinear programming methods (see [6] for references), (b) scenario approximation based on Monte Carlo sampling techniques (e.g., [3, 4]), and (c) conservative approximation (e.g., [1, 2, 10, 11]). The type (c) approaches build an alternative tractable optimization problem whose feasible set is contained in that of the PCCP. In particular, Nemirovski and Shapiro [11] consider a convex conservative approach to the general PCCP. However, such a conservative approach faces a criticism that the solution is excessively conservative. For example, in the numerical illustration of [11], although a solution is allowed to take up to 5% of the associated probability, the obtained solution achieves only less than 1%, which indicates that the solution was too conservative to be a good approximation for the global optimality of the original problem.

In this research, motivated by [11], we consider a conservative approximation approach to the PCCP, and apply it to the minimization of the Value-at-Risk (VaR) of a financial portfolio by employing deterministic global optimization algorithms. By introducing a parameter which indicates the conservativeness (or, equivalently, approximation accuracy), the resulting problem has a nonconvex feasible region represented by the difference of two convex sets, or an inequality constraint whose left-hand side is given by the difference of two convex functions. These formulations are known as the D.C. formulation, and several global or local solution algorithms have been developed (see Tuy [15], for example). Many of D.C. algorithms can achieve a globally optimal solution in practical time only when the number of variables associated with the nonconvexity is relatively small ([8]). A nice point of the proposed D.C. formulation is that the degree of the nonconvexity is almost independent of the number of scenarios, which contrasts with the fact that the typical MIP formulation requires 0-1 variables with the number of scenarios. A branch-and-bound algorithm is posed to solve the nonconvex program, and some comparative computational results will be given, presenting the performance and characteristics of the proposed algorithm.

The rest of the paper is organized as follows. In Section 2, the convex conservative approximation of [11] is briefly explained, and a new conservative approximation is introduced. Section 3 explains the VaR minimization and presents the formulation of the proposed conservative approximation. Section 4 is devoted to a branch-and-bound algorithm for solving the new approximation problem of the VaR minimization. Also, a remark will be provided on the application of an outer approximation algorithm. In Section 5, computational experiments are presented, showing the comparative superiority of the proposed approach.

## 2 Probabilistically Constrained Convex Program and $\alpha$-Conservative Approximation

In this section, we first formulate the probabilistically constrained convex program and next introduce its $\alpha$-conservative approximation problem.

## 2.1 $\alpha$-conservative approximation problem of PCCP

A probabilistically constrained convex program (PCCP) is formulated as the following optimization problem:

$$\text{(PCCP)} \left| \begin{array}{ll} \underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in X \\ & \text{Prob}\Big\{g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) > 0\Big\} \leq 1-\beta, \end{array} \right. \tag{1}$$

where

- $\boldsymbol{x}$ : decision variable, $\boldsymbol{x} \in \mathbb{R}^n$
- $X$ : closed convex set which represents a feasible set of $\boldsymbol{x}$, $X \subseteq \mathbb{R}^n$
- $f$ : objective function which is assumed to be convex in $\boldsymbol{x}$, $f : \mathbb{R}^n \to \mathbb{R}$
- $\tilde{\boldsymbol{\xi}}$ : $d$ dimensional real random vector (tilde ( $\tilde{}$ ) denotes random variables)
- $\Xi$ : support of random variable $\tilde{\boldsymbol{\xi}}$, $\Xi \subseteq \mathbb{R}^d$
- Prob : probability measure, $\text{Prob}\{F\}$ denotes probability of an event $F$
- $g$ : function which is convex in $\boldsymbol{x}$ for any fixed $\tilde{\boldsymbol{\xi}} \in \Xi$, $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$
- $\beta$ : user-defined parameter for representing a confidence level, $\beta \in (0,1)$

and the constraint

$$\text{VP}(\boldsymbol{x}) := \text{Prob}\Big\{g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) > 0\Big\} \leq 1-\beta \tag{2}$$

is referred to as the *probabilistic constraint* or the *chance constraint*, and the left-hand side of the constraint is called the *violation probability*. Intuitively, this constraint forces $g(\boldsymbol{x},\tilde{\boldsymbol{\xi}})$ to be non-positive with probability $\beta$. The function $g$ is here assumed to be scalar-valued without loss of generality. Indeed, if the probabilistic constraint is represented as $\text{Prob}\{g_i(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \leq 0,\ \forall i = 1,\dots,\ell\} \geq \beta$, and the functions $g_i$ are convex in $\boldsymbol{x}$ for any fixed $\tilde{\boldsymbol{\xi}}$, then this can be converted into a constraint (2) by putting $g(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) := \max\{g_i(\boldsymbol{x},\tilde{\boldsymbol{\xi}}) \mid i = 1,\dots,\ell\}$.

Though functions $f$ and $g$ are convex in $\boldsymbol{x}$ (for any fixed $\tilde{\boldsymbol{\xi}}$), this problem has nonconvex feasible region in general, and consequently, is intractable as mentioned in Introduction. In particular, it may have multiple local minima when the support of the associated probabilities is given by a finite set of scenarios. In order to tame such a difficulty arising from the nonconvexity, Nemirovski and Shapiro [11] introduce a convex conservative approximation, presenting a convex optimization problem which provides a feasible solution of the original problem (1). Although their approach enjoys the convex structure, the distance to the original problem (1) is not clear. In this paper, we extend the conservative approach by relinquishing to keep convexity, and next explain the approach to PCCP (1).

In [11], an conservative constraint of the form

$$\inf\left\{t\,\mathbb{E}\left[\psi\left(\frac{1}{t}g(\boldsymbol{x},\tilde{\boldsymbol{\xi}})\right)\right]-t(1-\beta)\,\middle|\,t>0\right\}\leq 0 \tag{3}$$

is adopted in place of the probabilistic constraint (2) of Problem (1), and this is shown to be a convex conservative constraint in $\boldsymbol{x}$. Though the left-hand side of the constraint (3) is represented via an optimization over $t > 0$, the resulting conservative approximation problem can be solved via an one-level (nonlinear) convex optimization due to the convexity:

$$\left|\ \underset{(\boldsymbol{x},t)\in\mathbb{R}^n\times\mathbb{R}}{\text{minimize}}\ f(\boldsymbol{x}) \quad \text{subject to}\ \ \boldsymbol{x}\in X,\ t\,\mathbb{E}\left[\psi\left(\frac{1}{t}g(\boldsymbol{x},\tilde{\boldsymbol{\xi}})\right)\right]-t(1-\beta)\leq 0,\ t\geq 0. \right. \tag{4}$$

A criticism of this approach focuses on the fact that the obtained solution can be too conservative in terms of violation probability $\mathrm{VP}(\cdot)$, that is, it can provide a solution with violation probability much smaller than $1-\beta$. In order to overcome this drawback, they propose a strategy of iteratively solving this problem by replacing $\beta$ with a smaller value $\beta^-$ in (4) until the violation probability will become as close as $1-\beta$. Although this strategy may succeed in finding a feasible solution to the original problem (1) with higher violation probability, there is a possibility that the obtained objective value has much larger than the optimal value of the original problem (1).

In contrast to their strategy, we below introduce a new approximation approach to Problem (1). For a parameter $\alpha > 0$, let us define $\Psi_\alpha : \mathbb{R} \to \mathbb{R}$ by

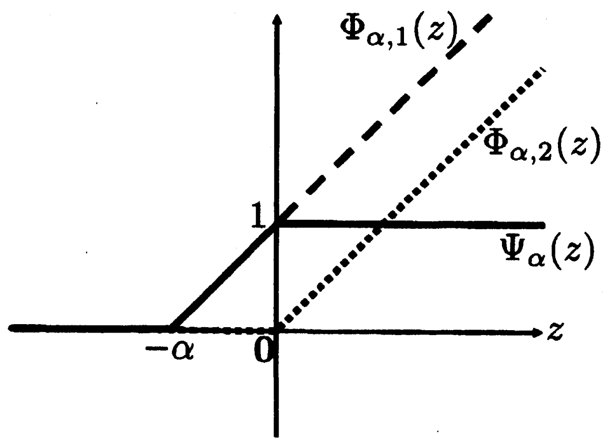


Figure 1: Graphs of $\Psi_\alpha$, $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$.

$$\Psi_\alpha(z) := \Phi_{\alpha,1}(z) - \Phi_{\alpha,2}(z),$$

where

$$\Phi_{\alpha,1}(z) := \max\left\{0,\ 1+\frac{1}{\alpha}z\right\},\quad \Phi_{\alpha,2}(z) := \max\left\{0,\ \frac{1}{\alpha}z\right\}. \tag{5}$$

For a scalar valued random variable $\tilde{Z}$, one then has

$$\mathbb{E}\left[\Psi_\alpha(\tilde{Z})\right] \geq \mathbb{E}\left[\mathbf{1}_{[0,+\infty)}(\tilde{Z})\right] = \mathrm{Prob}\{\tilde{Z} \geq 0\} \geq \mathrm{Prob}\{\tilde{Z} > 0\},$$

where $\mathbb{E}[\cdot]$ is the mathematical expectation operator, and $\mathbf{1}_A : \mathbb{R} \to \{0,1\}$ is the indicator function of a set $A$, i.e.,

$$\mathbf{1}_A(z) := \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{if } z \notin A. \end{cases}$$

From this relation, by taking $\tilde{Z} = g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}})$, it is clear that

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \mathbb{E}\left[\Psi_\alpha(g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))\right] \leq 1 - \beta \right\} \subseteq \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \mathrm{VP}(\boldsymbol{x}) \leq 1 - \beta \right\}.$$

Consequently, we obtain an another conservative approximation problem:

$$(\mathrm{CAP}(\alpha)) \quad \left| \begin{array}{ll} \underset{\boldsymbol{x} \in \mathbb{R}^n}{\text{minimize}} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in X \\ & \mathbb{E}\left[\Psi_\alpha(g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))\right] = \mathbb{E}\left[\Phi_{\alpha,1}(g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))\right] - \mathbb{E}\left[\Phi_{\alpha,2}(g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))\right] \leq 1 - \beta. \end{array} \right. \tag{6}$$

We refer to this problem as *$\alpha$-conservative approximation* problem of (1), and the new constraint as *$\alpha$-conservative approximation constraint* of (2). It should be noted that both $\mathbb{E}\left[\Phi_{\alpha,1}(g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))\right]$ and $\mathbb{E}\left[\Phi_{\alpha,2}(g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))\right]$ are convex in $\boldsymbol{x}$ since both $\Phi_{\alpha,1}$ and $\Phi_{\alpha,2}$ are nondecreasing convex functions, and accordingly, $\mathbb{E}\left[\Psi_\alpha(g(\boldsymbol{x}, \tilde{\boldsymbol{\xi}}))\right]$ is a D.C. function and Problem (6) is a D.C. optimization problem, for which several global optimization algorithms have been developed (e.g. Tuy [15]).

## 3 Portfolio Selection via Value-at-Risk Minimization

In this section, we formulate the minimization of the Value-at-Risk (VaR) of a financial asset portfolio as an example of the PCCP.

The VaR minimization of a financial asset portfolio is to determine the amount of investment (or investment ratio) to $N$ kinds of financial assets so that it achieves the minimum $\beta$-VaR, which is defined as the $\beta$-quantile of the loss distribution of the portfolio. Formally, it is formulated as the following optimization problem:

$$\left| \begin{array}{ll} \underset{(\boldsymbol{x},m) \in \mathbb{R}^N \times \mathbb{R}}{\text{minimize}} & m \\ \text{subject to} & \boldsymbol{x} \in X \\ & \mathrm{Prob}\left\{\boldsymbol{x}^{\mathsf{T}} \tilde{\boldsymbol{y}} - m > 0\right\} \leq 1 - \beta, \end{array} \right. \tag{7}$$

where

$x$ : investment ratio to $N$ kinds of financial assets (decision variable), $x \in \mathbb{R}^N$

$m$ : VaR (decision variable), $m \in \mathbb{R}$

$X$ : set of feasible portfolio $x$, $X \subseteq \mathbb{R}^N$

$\tilde{y}$ : $N$ dimensional random vector representing the loss associated with the financial assets

$\beta$ : confidence level, $\beta \in (0,1)$.

The random loss $\tilde{y}$ is sometimes defined as "$(-1)\times$(rate of return)," and besides, the probabilistic constraint in Problem (7) imposes that the probability of the portfolio loss being greater than $m$ is no more than $1-\beta$.

In the rest part of the paper, we assume that the support of the random loss $\tilde{y}$ is given by a finite set of scenarios $\{y^s\}_{s\in S}$, and let

**Assumption 1** $p_s := \text{Prob}\{\tilde{y} = y^s\}$, *where* $\sum_{s\in S} p_s = 1$ *and* $p_s > 0$ *for all* $s \in S$, *and* $|S| < \infty$.★

It is worth noting that this assumption is practical especially when the scenarios are generated from a (non-normal) distribution. Furthermore, we assume the following:

**Assumption 2** *The feasible region $X$ of $x$ is a polytope.* ★

This assumption seems reasonable since the constraints $\mathbf{1}^\top x = 1$ and $x \geq \mathbf{0}$ are included in many practical situations, and many other constraints are representable by linear inequalities.

The most typical way to an exact solution of Problem (7) is to equivalently formulate it as a 0-1 mixed integer program:

$$\left|\begin{array}{ll} \underset{(x,m,u)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^{|S|}}{\text{minimize}} & m \\ \text{subject to} & x \in X \\ & \sum_{s\in S} p_s u_s \leq 1-\beta \\ & x^\top y^s - m \leq \bar{M} u_s, \ u_s \in \{0,1\}, \ \ \forall s \in S, \end{array}\right. \tag{8}$$

where $\bar{M}$ is a sufficiently large number satisfying $\bar{M} > \max\{x^\top y^s \mid x \in X, \ s \in S\} - \min\{x^\top y^s \mid x \in X, \ s \in S\}$. It should be noted that the number of 0-1 variables $u_s$ is equal to that of scenarios, i.e., $|S|$.

In the following sections, we consider the $\alpha$-conservative approximation of Problem (7) under Assumptions 1 and 2.

$$\left|\begin{array}{ll} \underset{(x,m)\in\mathbb{R}^N\times\mathbb{R}}{\text{minimize}} & m \\ \text{subject to} & x \in X \\ & \sum_{s\in S} p_s \Phi_{\alpha,1}(x^\top y^s - m) - \sum_{s\in S} p_s \Phi_{\alpha,2}(x^\top y^s - m) \leq 1-\beta. \end{array}\right. \tag{9}$$

# 4 Global Optimization Algorithms

In this section, a simplicial branch-and-bound algorithm is presented for computing a globally optimal solution of Problem (9). Also, a remark on application of an outer approximation algorithm will be provided.

## 4.1 Simplicial Branch-and-Bound Algorithms

By denoting

$$h^D(\boldsymbol{x}, m) := \sum_{s\in S} p_s \Phi_{\alpha,1}(\boldsymbol{x}^\top \boldsymbol{y}^s - m), \quad h^C(\boldsymbol{x}, m) := \sum_{s\in S} p_s \Phi_{\alpha,2}(\boldsymbol{x}^\top \boldsymbol{y}^s - m),$$

Problem (9) can be rewritten as

$$\left|\ \underset{(\boldsymbol{x},m)\in\mathbb{R}^N\times\mathbb{R}}{\text{minimize}}\ m \quad \text{subject to}\ \ \boldsymbol{x}\in X,\ h^D(\boldsymbol{x},m) - h^C(\boldsymbol{x},m) \le 1-\beta. \right. \tag{10}$$

Let $M \subset \mathbb{R}^{N+1}$ be a simplex, and let $\{\boldsymbol{v}^{M,1}, \boldsymbol{v}^{M,2}, ..., \boldsymbol{v}^{M,N+2}\}$ be a set of vertices of $M$. For $M$, we consider

$$(\mathrm{RSP}(M)) \left|\ \begin{array}{ll} \underset{\boldsymbol{\lambda}\in\mathbb{R}^{N+2}}{\text{minimize}} & \displaystyle\sum_{i=1}^{N+2} \lambda_i v_{N+1}^{M,i} \\ \text{subject to} & \displaystyle\sum_{i=1}^{N+2} \lambda_i \boldsymbol{v}^{M,i} \in X\times[m_L,\, m_U],\ \boldsymbol{\lambda}\ge \boldsymbol{0},\ \boldsymbol{1}^\top\boldsymbol{\lambda} = 1 \\ & h^D\Big(\displaystyle\sum_{i=1}^{N+2} \lambda_i \boldsymbol{v}^{M,i}\Big) - \displaystyle\sum_{i=1}^{N+2} \lambda_i h^C(\boldsymbol{v}^{M,i}) \le 1-\beta, \end{array}\right. \tag{11}$$

where $m_L$ and $m_U$ are, respectively, lower and upper bounds on the optimal objective value of Problem (10).

It is easy to see that RSP($M$) is a relaxed subproblem of Problem (10) over a simplex $M$, providing a lower bound on the objective value of Problem (10) over $M$. Technically, $m_L$ can be computed via an algorithm of Pang and Leyffer [12], for example, and $m$ of any feasible solution $(\boldsymbol{x}, m)$ can be employed as $m_U$, whereas, in the experiments reported in Section 5, we used sufficiently small and large numbers as $m_L$ and $m_U$, respectively.

The initial simplex $M_0$ is set up so that $M_0 \supseteq X \times [m_L,\, m_U]$ and an optimal solution of Problem (10) is contained in $M_0$. For such $M_0$, we solve RSP($M_0$), obtaining a lower bound on the optimal value of Problem (10). It should be noted that one can easily find a feasible solution of Problem (10) if any $\boldsymbol{x} \in X$ is available because, for any $\boldsymbol{x} \in X$, sufficiently large $m$ satisfies the D.C. inequality. In addition, due to the monotonicity of the left-hand side of the D.C. inequality with respect to $m$, we can find $m$ satisfying the inequality at equality and such an $m$ can be employed as the incumbent value (i.e., the best known upper bound on the optimal value). We then split the simplex $M_0$ into two simplices $M_1$ and $M_2$ by dividing at the middle point of the longest edge, and compute lower bounds over $M_1$ and $M_2$ by solving RSP($M_1$) and RSP($M_2$), respectively. If one of the two subproblems finds a feasible solution of Problem (10) with objective value smaller than the incumbent, the incumbent is updated with the better solution. If the lower bound on each simplex is no less than the incumbent value, the corresponding simplex is discarded because such a simplex is guaranteed to have no better solution.

In the following step of the algorithm, as long as any simplex remains to be considered, we choose a simplex $M$ with the lowest lower bound and bisect $M$, i.e., split $M$ at the middle of the longest edge, generating two simplices in place of $M$ *(branching procedure)*. For the two simplices, say, $M'$ and $M''$, the lower bounds are computed by solving RSP($M'$) and RSP($M''$). If one of them attains a better feasible solution of Problem (10), the incumbent solution is updated by the solution. Let $\gamma$ be the incumbent objective value, i.e., the best objective value obtained so far. If the lower bound on a simplex $M$ is no less than $\gamma$, we discard it from further consideration *(bounding procedure)*. If there is no simplex to be considered, the algorithm terminates and the global optimality is guaranteed.

## 4.2 On the Computation of the Relaxed Problem

In the above branch-and-bound scheme, each relaxed problem RSP($M$) on a simplex $M$ is a convex program with a single nonlinear constraint $h^D(\sum_{i=1}^{N+2} \lambda_i \boldsymbol{v}^{M,i}) - \sum_{i=1}^{N+2} \lambda_i h^C(\boldsymbol{v}^{M,i}) \leq 1-\beta$ where $h^D(\sum_{i=1}^{N+2} \lambda_i \boldsymbol{v}^{M,i})$ is a convex and piecewise linear function in $\boldsymbol{\lambda}$.

Accordingly, we employ LP based subroutines for computing the lower bound. The first strategy "Linear Relaxation" uses a part of linear functions which coincides with $h^D$ at extreme points and the center of each simplex. This strategy provides a relaxed solution of the relaxed problem RSP($M$) while the size of the resulting LP is still independent of the number of scenarios.

Another strategy "Kelly's Method" is a straightforward application of the well-known Kelly's cutting plane idea. This strategy can compute the relaxed problem RSP($M$) in an exact manner, and accordingly, it may deal with a number of constraints in the order of scenarios. However, this strategy is expected to work efficiently because it brings in needed constraints effectively, and the efficient dual simplex algorithm can be adopted when a linear constraint is added at each iteration.

**Remark 1** *(On the Application of the Outer Approximation Method)* The second approach to solve the D.C. problem (9) is an outer approximation algorithm. By introducing a new variable

$\pi$, Problem (9) can be rewritten as follows:

$$\left|\begin{array}{ll} \underset{(\boldsymbol{x},m,\pi)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}}{\text{minimize}} & m \\ \text{subject to} & \boldsymbol{x}\in X \\ & \sum_{s\in S} p_s\Phi_{\alpha,1}(\boldsymbol{x}^\top\boldsymbol{y}^s - m) - \pi \le 1-\beta \\ & \sum_{s\in S} p_s\Phi_{\alpha,2}(\boldsymbol{x}^\top\boldsymbol{y}^s - m) - \pi \ge 0. \end{array}\right. \tag{12}$$

By introducing two sets in $\mathbb{R}^{N+2}$ defined by

$$D := \left\{(\boldsymbol{x},m,\pi)\,\middle|\,\boldsymbol{x}\in X,\ g^D(\boldsymbol{x},m,\pi)\le 0\right\},\quad C := \left\{(\boldsymbol{x},m,\pi)\,\middle|\,g^C(\boldsymbol{x},m,\pi)\le 0\right\},$$

where $g^D(\boldsymbol{x},m,\pi) := h^D(\boldsymbol{x},m) - \pi - (1-\beta)$ and $g^C(\boldsymbol{x},m,\pi) := h^C(\boldsymbol{x},m) - \pi$, Problem (12) can be considered as the following D.C. program:

$$\left|\ \underset{(\boldsymbol{x},m,\pi)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}}{\text{minimize}}\quad m \quad \text{subject to}\ \ (\boldsymbol{x},m,\pi)\in D\setminus \text{int}C,\right. \tag{13}$$

where $\text{int}\,C$ is the interior of $C$. We apply an outer approximation method described in [15] to the formulation (13). Through some preliminary computational experiment, this method is found to be inferior to the simplicial branch-and-bound method which is combined with several strategies, and therefore, the explanation and experimental result of this method are omitted in this article. ★

## 5 Computational Experiments

In this section, we report some numerical results of the VaR minimization algorithms. We consider the minimization of the VaR of a portfolio consisting of five assets where the loss $\tilde{y}_i$ of asset $i$ is given as an independent random variable, and it is formulated as the following PCCP:

$$\left|\begin{array}{ll} \underset{(\boldsymbol{x},m)\in\mathbb{R}^N\times\mathbb{R}}{\text{minimize}} & m \\ \text{subject to} & 0\le x_i\le 0.49,\quad i=1,\dots,N \\ & \sum_{i=1}^N x_i = 1,\ \sum_{i=1}^N \mu_i x_i \ge 1.2 \\ & \text{Prob}\left\{\sum_{i=1}^N x_i\tilde{y}_i - m > 0\right\} \le 0.1, \end{array}\right. \tag{14}$$

where $N=5$, and $\mu_i$ is the expected return of asset $i$, and we set $\mu_i = 1.25$ ($i$ is odd) or 1.1 ($i$ is even). The loss scenarios of assets 1 and 2 are generated from a Cauchy distribution where the location of the peak of the density is 0, and the half-width at half-maximum is 2. On the other hand, the loss scenarios of assets 3, 4 and 5 are generated from a uniform distribution on the interval $[-12.5, 12.5]$. We consider three cases 100, 1,000, 10,000 for the scenario size $|S|$, and assume $p_s := \frac{1}{|S|}$ for all $s \in S$.

We implemented five approaches to a solution of Problem (14): (a) the proposed branch-and-bound algorithm with linear relaxation, (b) the proposed branch-and-bound algorithm with Kelly's method, (c) the convex approximation (4) by Nemirovski and Shapiro [11] using $\psi(z) = \max\{0, 1+z\}$, (d) the CVaR minimization, (e) the typical MIP formulation (8) to Problem (14), and we compare these in terms of the resulting $\mathrm{VaR}(\boldsymbol{x}^*)$ and the violation probability $\mathrm{VP}(\boldsymbol{x}^*, m^*) := \mathrm{Prob}\{(\boldsymbol{x}^*)^\top \tilde{\boldsymbol{y}} - m^* > 0\}$ of the obtained solution $(\boldsymbol{x}^*, m^*)$. (a) and (b) are the proposed simplicial branch-and-bound algorithms, and solve the relaxed subproblem by the two relaxation strategies, and we set $\varepsilon = 0.5$ as the tolerance for optimality. Incumbent solutions are updated via the VaR evaluation rule and the subroutine for searching a feasible solution is employed in the proposed algorithms. Detailed explanation of them is omitted in this article for lack of space. (d) is the Conditional Value-at-Risk (CVaR) minimization formulated as the following LP ([14]):

$$\left|\begin{array}{ll} \underset{(\boldsymbol{x},m,\boldsymbol{\tau})\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^{|S|}}{\text{minimize}} & m + \dfrac{1}{1-\beta}\displaystyle\sum_{s\in S} p_s\tau_s \\ \text{subject to} & \boldsymbol{x} \in X \\ & \tau_s \geq 0,\ \tau_s \geq \boldsymbol{x}^\top \boldsymbol{y}^s - m, \quad \forall s \in S. \end{array}\right. \tag{15}$$

According to [14], the $\beta$-CVaR can be approximately regarded as the conditional expectation of the loss exceeding the $\beta$-VaR, and for $\beta$ close to one, the $\beta$-CVaR minimizer is expected to be similar to the $\beta$-VaR minimizer.

All computations are conducted on a personal computer with Pentium4 processor (3.4 GHz) and 2 GB memory. MATLAB R2006b with optimization toolbox is employed for implementing the proposed algorithms and the convex approximation, while the LP (15) for the CVaR minimization and the MIP formulation are solved by using Xpress-MP release 2006B.

Tables 1 (i) to (iii) show the computational results, each table corresponding to one of the three scenario sizes $|S|$=100, 1,000 and 10,000. All the values show the average of five experiments, each using different scenario set, but generated from the identical distribution mentioned above. When $|S|$=100, the MIP formulation quickly achieves the VaR in an exact manner. However, when $|S|$=1,000 and 10,000, the MIP formulation cannot be solved within 10 hours or results in memory shortage. On the other hand, the proposed algorithms which attain better solutions than that of the convex approximation (c) and the CVaR minimization (d). Moreover, if approximation accuracy $\alpha$ is relaxed from 2 to 5, CPU time decreases sharply whereas the difference of the achieved VaRs is small. This tendency motivates us to reduce the computation time by relaxing the approximation accuracy. When $|S|$=10,000, CPU time of the proposed algorithms does not change so much compared with that in the case of $|S| = 1{,}000$.

It may be worth mentioning that when approximation accuracy $\alpha$ is so small or the number of assets is larger, the Kelly's method (b) is expected to be superior to the linear relaxation (a)

Table 1: The VaR, the violation probability, and the computation time ($N = 5$)

The column "VaR" displays the value of $\mathrm{VaR}(\boldsymbol{x}^*)$ for the obtained solution $(\boldsymbol{x}^*, m^*)$ via each approach, while the column "VP" displays the violation probability $\mathrm{VP}(\boldsymbol{x}^*, m^*) := \mathrm{Prob}\{(\boldsymbol{x}^*)^\top \tilde{\boldsymbol{y}} - m^* > 0\}$.

(i) $|S|$=100

| | VaR | VP | CPU time (sec) |
|---|---|---|---|
| (a) BB with linear relaxation, $\alpha = 2$ | 3.53 | 0.076 | 408.1 |
| (a) BB with linear relaxation, $\alpha = 5$ | 3.57 | 0.044 | 57.3 |
| (b) BB with Kelly's method, $\alpha = 2$ | 3.45 | 0.074 | 573.1 |
| (b) BB with Kelly's method, $\alpha = 5$ | 3.58 | 0.040 | 80.8 |
| (c) Convex Approximation | 5.18 | 0.038 | 0.3 |
| (d) CVaR minimization | 4.89 | - | 0.1 |
| (e) MIP formulation | 3.24 | 0.100 | 2.1 |

(ii) $|S|$=1,000

| | VaR | VP | CPU time (sec) |
|---|---|---|---|
| (a) BB with linear relaxation, $\alpha = 2$ | 4.04 | 0.070 | 6816.3 |
| (a) BB with linear relaxation, $\alpha = 5$ | 4.08 | 0.050 | 239.4 |
| (b) BB with Kelly's method, $\alpha = 2$ | 4.03 | 0.071 | 6332.6 |
| (b) BB with Kelly's method, $\alpha = 5$ | 4.10 | 0.047 | 386.9 |
| (c) Convex Approximation | 5.26 | 0.037 | 0.3 |
| (d) CVaR minimization | 5.22 | - | 0.1 |
| (e) MIP formulation | - | - | over 10 hours |

(iii) $|S|$=10,000

| | VaR | VP | CPU time (sec) |
|---|---|---|---|
| (a) BB with linear relaxation, $\alpha = 2$ | 4.38 | 0.073 | 4730.9 |
| (a) BB with linear relaxation, $\alpha = 5$ | 4.40 | 0.048 | 331.7 |
| (b) BB with Kelly's method, $\alpha = 2$ | 4.38 | 0.073 | 6539.1 |
| (b) BB with Kelly's method, $\alpha = 5$ | 4.39 | 0.050 | 618.5 |
| (c) Convex Approximation | 5.45 | 0.039 | 0.5 |
| (d) CVaR minimization | 5.45 | - | 0.9 |
| (e) MIP formulation | - | - | memory shortage |

since the size of branch-and-bound tree becomes larger owing to the excessively relaxed linear relaxation. Besides, it is noted that the resulting VaR of the convex approximation is no less than that of the CVaR minimization in all the results.

## 6 Conclusion

In this paper, we construct the $\alpha$-conservative approximation problem of the probabilistically constrained convex program (PCCP), and show that it can be formulated as a D.C optimization problem. It is advantageous that the number of (sampled) scenarios does not affect the number of variables or constraints while it does not hold in the MIP formulation which requires a number of 0-1 variables, each corresponding to one scenario.

Although solving a genuine D.C. problem in a deterministic manner is known to be very hard, several algorithms are shown to have a potential to solve the problem especially when the number of variables concerned with the nonconvexity is up to ten (see, e.g., [8]).

In this paper, the simplicial branch-and-bound method which is a famous deterministic algorithms for achieving a globally optimal solution is mainly investigated, and is applied to the VaR minimization of a financial portfolio. Through the numerical experiments, we show the comparative superiority of the proposed approach. Although, when the number of assets is hundred or more, the problem clearly becomes (prohibitively) hard and this nonconvex approach may not look appealing, it is worth noting that the number of scenarios is critical for the accuracy of the solution, and the number of scenarios required for sufficient accuracy drastically increases as the number of assets grows.

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## References

[1] A. Ben-Tal and A. Nemirovski, "Robust solutions of Linear Programming problems contaminated with uncertain data," *Mathematical Programming*, **88**, 411–424 (2000).

[2] D. Bertsimas and M. Sim, "Price of Robustness," *Operations Research*, **52**, 35–53 (2004).

[3] G. Calafiore and M. C. Campi, "Decision making in an uncertain environment: the scenario-based optimization approach," In *Multiple Participant Decision Making*, J. Andrysek, M. Karny and J. Kracik (Eds.), Advanced Knowledge International, 99–111 (2004).

[4] G. Calafiore and M. C. Campi, "Uncertain convex programs: randomized solutions and confidence levels," *Mathematical Programming*, **102**, 25–46 (2005).

[5] A. Charnes, W. W. Cooper and G. H. Symonds, "Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil," *Management Science*, **4**, 235–263 (1958).

[6] D. Dentcheva, "Optimization Models with Probabilistic Constraints," In *Probabilistic and Randomized Methods for Design under Uncertainty*, G. Calafiore and F. Dabbene (Eds.), Springer, London, 49–97 (2006).

[7] J. Gotoh and Y. Takano, "$\alpha$-Conservative Approximation for Probabilistically Constrained Convex Programs," Discussion paper, Chuo University (2007).

[8] H. Konno, P. T. Thach and H. Tuy, *Optimization on Low Rank Nonconvex Structures*, Kluwer Academic Publishers (1997).

[9] L. B. Miller and H. Wagner, "Chance constrained programming with Joint constraints," *Operations Research*, **13**, 930–945 (1965).

[10] A. Nemirovski, "On tractable approximations of randomly perturbed convex constraints," *Proceedings of the 42nd IEEE Conference on Decision and Control*, **3**, 2419–2422 (2003).

[11] A. Nemirovski and A. Shapiro, "Convex Approximations of Chance Constrained Programs," *SIAM Journal on Optimization*, **17**, 969–996 (2006).

[12] J. S. Pang and S. Leyffer, "On the Global Minimization of the Value-at-Risk," *Optimization Methods and Software*, **19**, 611–631 (2004).

[13] A. Prékopa, On Probabilistic Constrained Programming, *Proceedings of the Princeton Symposium on Mathematical Programming*, Princeton University Press, Princeton, 113–138 (1970).

[14] R. T. Rockafellar and S. Uryasev, "Conditional value-at-risk for general loss distributions," *Journal of Banking and Finance*, **26**, 1443–1471 (2002).

[15] H. Tuy, *Convex Analysis and Global Optimization*, Kluwer Academic Publishers (1998).