On a probability distribution of a binomial type generated by a mean

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1. Means and paths. In this note, we use operator means, in particular, the Kubo-Ando mean [6] plays a central role: A binary operation $m$ on positive operators on a Hilbert space is called the Kubo-Ando (operator) mean if $m$ satisfies the following axioms:

monotonicity: $A \leq C, B \leq D \Rightarrow A_m B \leq C_m D$.

semicontinuity: $A_n \downarrow A, B_n \downarrow B \Rightarrow A_m B_n \downarrow A_m B$.

transformer inequality: $T^*(A_m B)T \leq T^* AT m T^* BT$.

normalization: $A_m A = A$.

By semicontinuity, we may assume positive operators are invertible. The representing function $f_m(x) = 1_m x$ for a Kubo-Ando mean $m$ is operator monotone (concave) on $(0, \infty)$ and $m$ is represented by

$$A_m B = A^{\frac{1}{2}} f_m(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

A path $A_m t B$ means parametrized operator means which is usually differentiable for $t$ with $A_m 0 B = A$ and $A_m 1 B = 0$. A path is called symmetric if

$$A_m t B = B_m 1-t A$$

holds for all $t \in [0,1]$. Typical example is (quasi-arithmetic) power means for $r \in [-1,1]$:

$$A \#_{r,t} B = A^{\frac{1}{2}} \left((1-t)I + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r\right)^{\frac{1}{2}} A^{\frac{1}{2}},$$

which include important means:

arithmetic mean: $A \nabla_t B = A \#_{1,t} B = (1-t) A + t B$

gometric mean: $A \#_{0,t} B \equiv \lim_{\epsilon \to 0} A \#_{\epsilon,t} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$

harmonic mean: $A \! t B = A \#_{-1,t} B = ((1-t) A^{-1} + t B^{-1})^{-1}$. 
Moreover the above paths are **interpolational** in the sense that

\[(A \#_{r,q} B) \#_{r,t} (A \#_{r,q} B) = A \#_{r,(1-t)p+tq} B\]

for all \(p, q, t \in [0, 1]\).

2. **Thompson metric.** Let \(\mathcal{A}^+\) be the positive invertible elements in a unital C*-algebra \(\mathcal{A}\), which is discussed as differentiable manifold by Corach-Porta-Recht [3, ?]. Corach himself reformulated it in [4]. They showed the above manifold \(\mathcal{A}^+\) is the Finsler space with a Finsler metric

\[L(X; A) = \|X\|_A = \|A^{-1/2}XA^{-1/2}\| :\]

Then the geodesic is the shortest path with respect to this metric: The length \(\ell(\gamma)\) of path \(\gamma(t)\) is defined by

\[\ell(\gamma) \equiv \int_0^1 L(\gamma'(t); \gamma(t)) dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\| dt.\]

If \(\gamma(t)\) is a path from \(A\) to \(B\), then

\[d(A, B) \equiv \inf_{\gamma} \ell(\gamma) = \ell(A \#_t B) = \|\log(A^{-1/2}BA^{-1/2})\| = \log \left( \max\{\|A^{-1/2}BA^{-1/2}\|, \|B^{-1/2}AB^{-1/2}\|\} \right) = \log \left( \max\{r(A^{-1}B), r(B^{-1}A)\} \right).\]

Also the homogeneity of \(\mathcal{A}^+\) implies

\[d(A, B) = d(X^*AX, X^*BX) = d(I, A^{-1/2}BA^{-1/2})\]

for invertible \(X\). The metric \(d\) makes \(\mathcal{A}^+\) a complete metric space and it is called the **Thompson (part) one** [12, 10].

3. **Lawson-Lim's operator mean.** Recently, Lawson-Lim [8, 9, 7] defines multivariable operator means parametrized by \(t \in [0, 1]\) which is an extension of Ando-Li-Mathius' geometric operator mean [1]: For a symmetric path \(\mathbf{m}_t\) in Kubo-Ando means, it is defined inductively:

\[(n = 2): \quad \mathbf{m}[2,t](A_1, A_2) = A_1 \mathbf{m}_t A_2\]

\[(n + 1): \quad \mathbf{m}[n + 1,t](A_1, \cdots, A_{n+1}) = \lim_{r \to \infty} A_m(r)_{k} \text{iff the limit exists}\]

where

\[\begin{cases} A_m(r)_{k} = \mathbf{m}[n,t]((A_m(r-1))_{j \neq k}) \\ (A_m(1)_{k} = A_k). \end{cases}\]
Then they showed that $\#[n,t](A_1, \cdots, A_n)$ always exists making use of the Thompson metric and that it coincides with Ando-Li-Mathias’ one for $t = 1/2$. In [5], we pointed out that the arithmetic mean plays an essential part. In fact, it is expressed by the weight $\{t[n]_k\}$:

$$\nabla[n,t](A_1, \cdots, A_n) = \sum_{k=1}^{n} t[n]_k A_k.$$

Also the harmonic mean is

$$![n,t](A_1, \cdots, A_n) = \left( \sum_{k=1}^{n} t[n]_k A_k^{-1} \right)^{-1}.$$

If $A_k$ are commuting, then the geometric mean is

$$\#[n,t](A_1, \cdots, A_n) = \prod_{k=1}^{n} A_k^{t[n]_k}.$$

Moreover we extend the convexity

$$d(A_1 \# B_1, A_2 \# B_2) \leq d(A_1, B_1) \nabla_t d(A_2, B_2)$$

of the Thompson metric:

$$d(\#[n,t](A_1, \cdots, A_n), \#[n,t](B_1, \cdots, B_n)) \leq \nabla[n,t](d(A_1, B_1), \cdots, d(A_n, B_n)) = \sum_{k=1}^{n} t[n]_k d(A_k, B_k),$$

which shows the existence of the Lawson-Lim geometric mean. Then we obtain the formulae for $t[n]_k$ in [5]:

**Lemma.**

$$t[n]_n = \frac{t}{1 + (n-2)t},$$

$$t[n]_1 = \frac{1-t}{1 + (n-2)(1-t)} = \frac{1-t}{(n-1)-(n-2)t}.$$

**Theorem.**

(i) $$t[n]_{n-m} = \frac{m(m+1)+2m(n-2m-2)t + (n^2 - (4m+1)n + 4m(m+1)) t^2}{(n-1)(m + (n-2m)t)(m + 1 + (n-2(m+1))t)},$$

(ii) $$\sum_{j>n-m-1} t[n]_j = t[n]_n + \cdots + t[n]_{n-m} = \frac{(m+1)(m + (n-2m-1)t)}{(n-1)(m + 1 + (n-2m-2)t)}.$$
Here we give another short proof of the above to show the probability distribution distribution function

$$F_n(k) = \sum_{j<k+1} t[n]_j = 1 - \frac{(n-k)(n-k-1+(2k-n+1)t)}{(n-1)(n-k+(2k-n)t)}.$$ 

**Proof.** Suppose the formula for $F_N(k)$ is valid for all $k$. Putting $v = F_N(k-1)$ and $w = F_N(k)$, we have

$$a_{n+1} = va_n + (1-v)b_n \quad \text{and} \quad b_{n+1} = wa_n + (1-w)b_n.$$ 

Thereby

$$a_{n+1} - b_{n+1} = (v-w)a_n + (w-v)b_n = (v-w)(a_n - b_n) = \cdots = (v-w)^n,$$

and hence $b_n = a_n - (v-w)^n$. Then we have $a_{n+1} - a_n = -(1-v)(v-w)^n$ and

$$a_{n+1} = a_1 - (1-v)\sum_{k=0}^{n-1}(v-w)^k \rightarrow 1 - \frac{1-v}{1-v+w},$$

which coincides with $F_{N+1}(k)$. Therefore, the formulae $F_n(k)$ are valid by induction. Thus (ii) in Theorem is obtained by $1 - F_n(k)$ and (i) by $t[n]_k = F_n(k) - F_n(k-1)$. \(\square\)

Now we give the table for the density function $t[n]_k$:

<table>
<thead>
<tr>
<th></th>
<th>$1-t$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{1-t}{2-t}$</td>
<td>$\frac{1-t+t^2}{(2-t)(1+t)}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1-t}{3-2t}$</td>
<td>$\frac{3-4t+4t^2}{3(3-2t)}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1-t}{4-3t}$</td>
<td>$\frac{6-9t+4t^2}{2(4-3t)(3-t)}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1-t}{5-4t}$</td>
<td>$\frac{10-16t+7t^2}{5(5-4t)(2-t)}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1-t}{6-5t}$</td>
<td>$\frac{15-25t+11t^2}{3(5-3t)(6-5t)}$</td>
</tr>
</tbody>
</table>

The table for $t[n]_k$
Appendix: binomial mean $m[n]_t$ for $m_t$. From the viewpoint of probability distribution, a simple one-parameter extension of symmetric path can be defined inductively:

\[
m[2]_t(A_1, A_2) = A_1 m_t A_2 \\
m[3]_t(A_1, A_2, A_3) = (m[2]_t(A_1, A_2)) m_t (m[2]_t(A_2, A_3)) \\
m[n + 1]_t(A_1, \cdots, A_{n+1}) = (m[n]_t(A_1, \cdots, A_n)) m_t (m[n]_t(A_2, \cdots, A_{n+1})).
\]

This path is symmetric in the sense of

\[
m[n]_t(A_1, \cdots, A_n) = m[n]_{1-t}(A_n, \cdots, A_1)
\]

The binomial arithmetic mean is

\[
\nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} \binom{n}{k-1} (1-t)^{n-k} t^{k-1} A_k,
\]

and the barycenter is the usual arithmetic mean:

\[
\int_{0}^{1} \nabla[n]_t(A_1, \cdots, A_n) = \sum_{k=1}^{n} \binom{n}{k-1} B(n-k+1, k) A_k = \frac{1}{n} \sum_{k=1}^{n} A_k
\]

where $B(p, q)$ is the beta function. As in [11], a multivariable extension of logarithmic mean

\[
L[2](a, b) = \frac{b - a}{\log b - \log a}
\]

is a fascinating one. Considering

\[
L[2](A, B) = \int_{0}^{1} A \#_t B dt
\]

holds in Kubo-Ando means, we might define

\[
L[n](A_1, \cdots, A_n) = \int_{0}^{1} \#[n]_t(A_1, \cdots, A_n) dt.
\]
参考文献


