数理解析研究所講究録
第 1585 巻 2008 年 51-64

STRASSEN'S MARGINAL PROBLEMS FOR VECTOR VALUED-MEASURES—A SHORT SURVEY

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1. Introduction

In a celebrated paper, V. Strassen (1965) gave necessary and sufficient conditions for the existence of probability measures with given marginals in the context of Polish spaces. Strassen's theorem and some of its offspring have been extended for real or vector-valued measures in more general settings. In this note, we will do a survey of those results succinctly and give an imperfect but helpful list of papers on Strassen's marginal problems.

2. Notation and preliminaries

Notation 2.1. Let S be a Hausdorff space.

- $\mathcal{B}(S)$: the Borel σ-field of all Borel subsets of S, that is, the σ-field generated by the open subsets of S
- $C_b(S)$: the set of all bounded, continuous, real functions on S
- $L_b(S)$: the set of all bounded, lower semicontinuous, real functions on S
- $\mathcal{P}(S)$: the set of all Borel probability measures on S
- $\mathcal{P}_t(S)$: the set of all $\mu \in \mathcal{P}(S)$ which are Radon, that is, for every $A \in \mathcal{B}(S)$, it holds that $\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ is compact}\}$
- $\mathcal{P}_\tau(S)$: the set of all $\mu \in \mathcal{P}(S)$ which are τ-smooth, that is, for every increasing net $\{G_\alpha\}_{\alpha\in\Gamma}$ of open subsets of S with $G = \bigcup_{\alpha\in\Gamma} G_\alpha$, it holds that $\mu(G) = \sup_{\alpha\in\Gamma} \mu(G_\alpha)$.

2000 *Mathematics Subject Classification.* Primary 28B05; Secondary 28A33, 46A40.
Key words and phrases. Strassen's marginal problem, vector measure, locally convex space, Riesz space, weak topology of measures.
*Research supported by Grant-in-Aid for General Scientific Research No. 18540166, Ministry of Education, Culture, Sports, Science and Technology, Japan.

Definition 2.2. Let S be a Hausdorff space. We endow $\mathcal{P}(S)$ with the *weak topology of measures*, that is, the weakest topology for which all mappings

$$\mu \in \mathcal{P}(S) \mapsto \int_S f d\mu$$

are lower semicontinuous whenever $f \in L_b(S)$.

Fact 2.3. When S is a completely regular Hausdorff space, the weak topology of measures on $\mathcal{P}(S)$ is the weakest topology for which all mappings $\mu \in \mathcal{P}(S) \mapsto \int_S f d\mu$ are continuous whenever $f \in C_b(S)$.

Hahn-Banach Theorem: Let X be a real vector space and L a subspace of X. Let q be a sublinear functional on X, that is, q is a real function on X such that $q(x+y) \leq q(x) + q(y)$ and $q(cx) = cq(x)$ for all $x, y \in X$ and $c \geq 0$. Let φ be a real linear functional on L such that $\varphi(x) \leq q(x)$ for all $x \in L$. Then there is a real linear functional $\bar{\varphi}$ on X extending φ and such that $\bar{\varphi}(x) \leq q(x)$ for all $x \in X$.

Separation Theorem: Let X be a real locally convex space and F a non-empty, closed, convex subset of X. Let $x \notin F$. Then there is a real continuous linear functional φ on X such that $\varphi(x) > \sup\{\varphi(y) : y \in F\}$.

Weak* continuous linear functionals: Let X be a topological vector space. A linear functional Φ on the topological dual X' of X is $\sigma(X', X)$-continuous if and only if it is the evaluation at some point of X, that is, there is a point $x \in X$ such that $\Phi(\varphi) = \varphi(x)$ for all $\varphi \in X'$.

Extension of positive operators (Kantrovič): Let U and V be two Riesz spaces with V Dedekind complete. Let L be a vector subspace of U. Assume that L is majorizing U, that is, for each $u \in U$ there is $v \in L$ such that $u \leq v$. If $T : L \to V$ is a positive linear operator, then T has a positive extension to all of U.

3. Marginal problem: function-type

MP1 (Marginal problem; function-type): *Let S and T be Hausdorff spaces. Let $\mu \in \mathcal{P}(S)$ and $\nu \in \mathcal{P}(T)$. Assume that $\mathcal{Q}$ is a non-empty, closed, convex subset of $\mathcal{P}(S \times T)$. The following conditions are equivalent:*

(i) There is $\lambda \in \mathcal{Q}$ with marginals μ and ν.

(ii) For every $f \in C_b(S)$ and $g \in C_b(T)$, it holds that

$$\int_S f d\mu + \int_T g d\nu \leq \sup\left\{\int_{S\times T} f \oplus g d\lambda : \lambda \in \mathcal{Q}\right\},$$

where $(f \oplus g)(s,t) := f(s) + g(t)$ for all $(s,t) \in S \times T$.

Remark 3.1. The proof of implication (i) $\Rightarrow$ (ii) is easy. Indeed, the inequality in condition (ii) holds for every bounded, Borel functions f and g.

Strassen (1965) [34, Theorem 7]: The assertion (MP1) holds whenever

- S and T are complete separable metric spaces.
- $\mu \in \mathcal{P}(S)$ and $\nu \in \mathcal{P}(T)$.
- $\mathcal{Q}$ is a non-empty, closed, convex subset of $\mathcal{P}(S \times T)$.

Edwards (1979) [7, Theorem 5.2]: The assertion (MP1) holds whenever

- S and T are completely regular Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.
- $\mathcal{Q}$ is a non-empty, closed, convex subset of $\mathcal{P}_t(S \times T)$.

Tahata (1984) [35, Theorem 2.3]: The assertion (MP1) holds whenever

- S and T are Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.
- $\mathcal{Q}$ is a non-empty, closed, convex subset of $\mathcal{P}_t(S \times T)$.
- $f \in L_b(S)$ and $g \in L_b(T)$ instead of $f \in C_b(S)$ and $g \in C_b(T)$.

Skala (1993) [33, Theorem 1]: The assertion (MP1) holds whenever

- S and T are Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.
- $\mathcal{Q}$ is a non-empty, closed, convex subset of $\mathcal{P}_t(S \times T)$.
- $f \in L_b(S)$ and $g \in L_b(T)$ instead of $f \in C_b(S)$ and $g \in C_b(T)$.

Khurana (2005) [18, Theorem 5]: The assertion (MP1) holds whenever

- S and T are completely regular Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_\tau(T)$, and vice versa.
- $\mathcal{Q}$ is a non-empty, closed, convex subset of $\mathcal{P}_\tau(S \times T)$.

The proof of (MP1): compact case.

Denote by π_S and π_T the projections from $S \times T$ onto S and T, respectively. Denote by $\mathcal{M}_t(S)$, $\mathcal{M}_t(T)$, and $\mathcal{M}_t(S \times T)$ the set of all Radon real measures on S, T, and $S \times T$, respectively. Consider the weak topology $\sigma_0 := \sigma(\mathcal{M}_t(S) \times \mathcal{M}_t(T), C_b(S) \oplus C_b(T))$ defined by the natural duality

$$\langle (\varphi, \psi), (f, g) \rangle := \int_S f d\varphi + \int_T g d\psi.$$

Then σ_0 coincides with the product topology $\sigma(\mathcal{M}_t(S), C_b(S))$ and $\sigma(\mathcal{M}_t(T), C_b(T))$, that is,

$$\sigma_0 = \sigma(\mathcal{M}_t(S), C_b(S)) \times \sigma(\mathcal{M}_t(T), C_b(T)).$$

Put

$$M_Q := \{(\varphi, \psi) \in \mathcal{M}_t(S) \times \mathcal{M}_t(T) : \exists\gamma \in Q;\ \pi_S(\gamma) = \varphi, \pi_T(\gamma) = \psi\}.$$

Then we have only to prove that $(\mu, \nu) \in M_Q$.

It is easy to show that M_Q is a non-empty, convex subset of $\mathcal{M}_t(S) \times \mathcal{M}_t(T)$. Further, M_Q is σ_0-closed since Q is $\sigma(\mathcal{M}_t(S\times T), C_b(S\times T))$-compact by the Banach-Alaoglue theorem. Assume to the contrary that $(\mu, \nu) \notin M_Q$. By the separation theorem, there is $\Phi \in (\mathcal{M}_t(S) \times \mathcal{M}_t(T))'$ such that

$$\Phi(\mu, \nu) > \sup\{\Phi(\varphi, \psi) : (\varphi, \psi) \in M_Q\}. \tag{*}$$

Since Φ is σ_0-continuous, there is $(f_0, g_0) \in C_b(S) \oplus C_b(T)$ such that

$$\Phi(\varphi, \psi) = \langle(\varphi, \psi), (f_0, g_0)\rangle = \int_S f_0 d\varphi + \int_T g_0 d\psi$$

for all $(\varphi, \psi) \in \mathcal{M}_t(S) \times \mathcal{M}_t(T)$. Thus, it follows from (*) that

$$\begin{aligned} \int_S f_0 d\mu + \int_T g_0 d\nu &> \sup\left\{\int_S f_0 d\varphi + \int_T g_0 d\psi : (\varphi, \psi) \in M_Q\right\} \\ &\geq \sup\left\{\int_{S\times T} (f_0 \oplus g_0) d\gamma : \gamma \in Q\right\}, \end{aligned}$$

which leads us to a contradiction!

The proof of (MP1): general case.

• *Approach 1*: Use the Stone-Čech compactification! We need the following lemma to pull back arguments in the compact space into the original space: Let S be a completely regular Hausdorff space. Let $\check{S}$ be the Stone-Čech compactification of S and $\kappa : S \to \kappa(S) \subset \check{S}$ the associated homeomorphism. Let $\check{\mu} \in \mathcal{P}_t(\check{S})$. Then, there is $\mu \in \mathcal{P}_t(S)$ such that $\kappa\mu = \check{\mu}$ if and only if for every $\varepsilon > 0$, there is a compact subset K of S such that $\check{\mu}(\check{S} - \kappa(K)) < \varepsilon$.

• *Approach 2*: We shall divide the proof into steps.

(1) Prove $(\mu, \nu) \in \overline{M_Q}^{\sigma_0}$ by the separation theorem.

(2) Then there is a net $\{(\mu_\alpha, \nu_\alpha)\}_{\alpha\in\Gamma} \in M_Q$ with $(\mu_\alpha, \nu_\alpha) \xrightarrow{\sigma_0} (\mu, \nu)$, so that $\mu_\alpha \xrightarrow{w} \mu$ and $\nu_\alpha \xrightarrow{w} \nu$. Since each (μ_α, ν_α) is an element of M_Q, there is $\gamma_\alpha \in Q$ such that $\pi_S(\gamma_\alpha) = \mu_\alpha$ and $\pi_T(\gamma_\alpha) = \nu_\alpha$. Thus, it holds that $\pi_S(\gamma_\alpha) \xrightarrow{w} \mu$ and $\pi_T(\gamma_\alpha) \xrightarrow{w} \nu$.

(3) Prove the **Key Lemma**: Let $\{\gamma_\alpha\}_{\alpha\in\Gamma}$ be a uniformly bounded net in $\mathcal{P}_t(S\times T)$. If $\pi_S(\gamma_\alpha) \xrightarrow{w} \mu \in \mathcal{P}_t(S)$ and $\pi_T(\gamma_\alpha) \xrightarrow{w} \nu \in \mathcal{P}_t(T)$, then every subnet of $\{\gamma_\alpha\}_{\alpha\in\Gamma}$ has a subnet converging weakly to $\gamma \in \mathcal{P}_t(S\times T)$ such that $\pi_S(\gamma) = \mu$ and $\pi_T(\gamma) = \nu$.

(4) By the Key Lemma, there is a subnet $\{\gamma_\beta\}_{\beta\in\Lambda}$ of $\{\gamma_\alpha\}_{\alpha\in\Gamma}$ and $\gamma \in \mathcal{P}_t(S\times T)$ such that $\gamma_\beta \xrightarrow{w} \gamma$. Since $\mathcal{Q}$ is closed for the weak topology of measures, $\gamma \in \mathcal{Q}$. Further, it follows from the continuity of π_S and π_T that $\pi_S(\gamma) = \mu$ and $\pi_T(\gamma) = \nu$, and the proof is complete!

4. Marginal problem: set-type

MP2 (Marginal problem; set-type): *Let S and T be Hausdorff spaces. Let $\mu \in \mathcal{P}(S)$ and $\nu \in \mathcal{P}(T)$. Assume that D is a non-empty, closed subset of $S\times T$. Fix $\varepsilon \geq 0$. The following conditions are equivalent:*

(i) There is $\lambda \in \mathcal{P}(S\times T)$ with marginals μ and ν such that $\lambda(D) \geq 1-\varepsilon$.

(ii) It holds that $\mu(A)+\nu(B) \leq 1+\varepsilon$ whenever $A \in \mathcal{B}(S)$ and $B \in \mathcal{B}(T)$ satisfy $(A\times B)\cap D = \emptyset$.

Remark 4.1. The proof of implication (i) $\Rightarrow$ (ii) is easy.

Strassen (1965) [34, Theorem 11]: The assertion (MP2) holds whenever

- S and T are complete separable metric spaces.
- $\mu \in \mathcal{P}(S)$ and $\nu \in \mathcal{P}(T)$.

Edwards (1979) [7, Proposition 5.4]: The assertion (MP2) for $\varepsilon = 0$ holds whenever

- S and T are completely regular Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.

Kellerer (1984) [17, Proposition 3.8]: The assertion (MP2) for $\varepsilon = 0$ holds whenever

- S and T are Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.

Tahata (1984) [35, Proposition 2.5]: The assertion (MP2) for $\varepsilon = 0$ holds whenever

- S and T are Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.

Hansel-Troallic (1986) [11, Theorem 4.4]: The assertion (MP2) holds whenever

- S and T are Hausdorff spaces.

- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.

Plebanek (1989) [25, Corollary to Theorem 4]: The assertion (MP2) holds whenever

- S and T are completely regular Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}(T)$, and vice versa.

Skala (1993) [33, Corollary 6]: The assertion (MP2) holds whenever

- S and T are Hausdorff spaces.
- $\mu \in \mathcal{P}_t(S)$ and $\nu \in \mathcal{P}_t(T)$.

5. Marginal problem for vector measures: function-type

Definition 5.1. A vector space V is called an *ordered vector space* if the following axioms are satisfied:

(i) $u \leq v$ implies $u + w \leq v + w$ for all $u, v, w \in V$

(ii) $u \leq v$ implies $cu \leq cv$ for all $u, v \in V$ and $c > 0$.

Definition 5.2. Let $(\Omega, \mathcal{A})$ be a measurable space. Let V be an ordered vector space. Let $\mu : \mathcal{A} \to V$ be a set function.

(1) μ is called a *vector measure* if it is finitely additive.

(2) μ is said to be *positive* if $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

(3) Let S be a Hausdorff space. A finitely additive set function $\mu : \mathcal{B}(S) \to V$ is called a *Borel* vector measure on S.

Notation 5.3. Let S be a Hausdorff space. Let V be a locally convex space which is an ordered vector space.

- $\mathcal{M}^+(S; V)$: the set of all positive, Borel vector measures $\mu : \mathcal{B}(S) \to V$ which is countably additive for the locally convex topology on V
- $\mathcal{M}_t^+(S; V)$: the set of all $\mu \in \mathcal{M}^+(S; V)$ which are Radon, that is, for each $\varepsilon > 0$, $A \in \mathcal{B}(S)$, and continuous seminorm q on V, there is a compact subset K of S such that $\|\mu\|_q(A - K) < \varepsilon$, where

$$\|\mu\|_q(A) := \sup\{|v'\mu|(A) : v' \in V', |\langle u, v' \rangle| \leq q(u) \text{ for all } u \in V\}$$

denotes the q-semivariation of μ.

- A subset $\mathcal{V}$ of $\mathcal{M}^+(S; V)$ is said to be *uniformly bounded* if $\sup_{\mu \in \mathcal{V}} \|\mu\|_q(S) < \infty$ for every continuous seminorm q on V.

Definition 5.4. Let S be a completely regular Hausdorff space. We endow $\mathcal{M}^+(S;V)$ with the *weak topology of vector measures*, that is, the weakest topology for which all mappings

$$\mu \in \mathcal{M}^+(S;V) \mapsto \int_S f d\mu$$

are continuous for the locally convex topology on V whenever $f \in C_b(S)$.

VMP1 (Marginal problem for vector measures; function-type I): *Let S and T be Hausdorff spaces. Let V be a locally convex space which is an ordered vector space. Let $\mu \in \mathcal{M}^+(S;V)$ and $\nu \in \mathcal{M}^+(T;V)$. Assume that $\mathcal{Q}$ is a non-empty, uniformly bounded, closed, convex subset of $\mathcal{M}^+(S \times T;V)$. The following conditions are equivalent:*

(i) There is $\lambda \in \mathcal{Q}$ with marginals μ and ν.
(ii) For any $n \in \mathbb{N}$ and any $\{f_i\}_{1\le i\le n} \subset C_b(S)$, $\{g_i\}_{1\le i\le n} \subset C_b(T)$, and $\{u_i'\}_{1\le i\le n} \subset V'$, it holds that

$$\sum_{i=1}^n \left\langle \int_S f_i d\mu + \int_T g_i d\nu, u_i' \right\rangle \le \sup \left\{ \sum_{i=1}^n \left\langle \int_{S\times T} f_i \oplus g_i d\lambda, u_i' \right\rangle : \lambda \in \mathcal{Q} \right\},$$

where $(f \oplus g)(s,t) := f(t) + g(s)$ for all $(s,t) \in S \times T$.

Kawabe (2000) [15, Theorem 1]: The assertion (VMP1) holds whenever

- S and T are completely regular Hausdorff spaces.
- $V = U_\sigma'$; the topological dual of U with the weak topology $\sigma(V,U)$, where U is a barreled locally convex space which is an ordered vector space, each of whose element can be decomposed into the difference of two positive elements.
- $\mu \in \mathcal{M}_t^+(S;V)$ and $\nu \in \mathcal{M}_t^+(T;V)$.

Why do we need finitely many $\{f_i\}$, $\{g_i\}$ and $\{u_i'\}$?

In the proof of [Kawabe (2000)] we consider the duality between $\mathcal{M}_t(S;V) \times \mathcal{M}_t(T;V)$ and $(C(S) \otimes U) \oplus (C(T) \otimes U)$ defined by

$$\langle (\mu,\nu),(f,g) \rangle := \sum_{i=1}^n \int_S f_i d(u_i \mu) + \sum_{j=1}^m \int_T g_j d(v_j \nu),$$

where $f = \sum_{i=1}^n f_i \otimes u_i \in C(S) \otimes U$ and $g = \sum_{j=1}^m g_j \otimes v_j \in C(T) \otimes U$. That is the reason!

Khurana (2006) [19, Theorem 2]: The assertion (VMP1) holds whenever

- S and T are completely regular Hausdorff spaces.

- V is a semi-reflexive, ordered locally convex space whose positive cone is normal.
- $\mu \in \mathcal{M}_t^+(S;V)$ and $\nu \in \mathcal{M}_t^+(T;V)$.

Remark 5.5. The ordered locally convex space $V = U'_\sigma$ in [Kawabe (2000)] is semi-reflexive, since U is assumed to be barreled.

Definition 5.6. (1) A topological vector space V is said to be *quasi-complete* if every bounded, closed subset of V is complete.

(2) Let V be a locally convex space which is an ordered vector space. V is said to be an *ordered locally convex space* whose positive cone is *normal* if the positive cone $C := \{u \in V : u \geq 0\}$ is closed in V and there is a generating family Q of semi-norms on V such that $q(u) \leq q(u+v)$ whenever $u \geq 0$, $v \geq 0$ and $q \in Q$.

(3) A Riesz space V is called a *locally convex Riesz space* if it is a locally convex space that possesses a 0-neighborhood base of solid sets.

Fact 5.7. Every locally convex Riesz space is an ordered topological vector space whose positive cone is normal [27, p.235].

Khurana (2006) [19, Theorem 4]: The assertion (VMP1) holds whenever

- S and T are completely regular Hausdorff spaces.
- V is a Dedekind complete and quasi-complete locally convex Riesz space such that if an order bounded net $\{u_\alpha\}_{\alpha\in\Gamma}$ of elements of V order converges to $u \in V$, then $u_\alpha \to u$ for the locally convex topology on V.
- $\mu \in \mathcal{M}_t^+(S;V)$ and $\nu \in \mathcal{M}_t^+(T;V)$.

Notation 5.8. Let S be a Hausdorff space. Let V be a Riesz space.

- $\mathcal{M}_o^+(S;V)$: the set of all positive, Borel vector measures $\mu : \mathcal{B}(S) \to V$ which are countably additive for the order convergence on V
- $\mathcal{M}_{o,t}^+(S;V)$: the set of all $\mu \in \mathcal{M}_o^+(S;V)$ which are quasi-Radon, that is, for every open subset G of S, it holds that $\mu(G) = \sup\{\mu(K) : K \subset G, K \text{ is compact}\}$.
- $\mathcal{M}_{o,\tau}^+(S;V)$: the set of all $\mu \in \mathcal{M}_o^+(S;V)$ which are τ-smooth, that is, for every increasing net $\{G_\alpha\}_{\alpha\in\Gamma}$ of open subsets of S with $G = \bigcup_{\alpha\in\Gamma} G_\alpha$, it holds that $\mu(G) = \sup_{\alpha\in\Gamma} \mu(G_\alpha)$.

VMP2 (Marginal problem for vector measures; function-type II): *Let S and T be Hausdorff spaces. Let V be a Riesz space. Let $\mu \in \mathcal{M}_o^+(S;V)$ and $\nu \in \mathcal{M}_o^+(T;V)$. Assume that $\mu(S) = \nu(T) = e$. Let D be a non-empty, closed subset of $S \times T$. Fix $u \in V^+$ with $u \leq e$. The following conditions are equivalent:*

(i) There is $\lambda \in \mathcal{M}_o^+(S \times T; V)$ with marginals μ and ν such that $\lambda(D) \geq u$.

(ii) For any $f \in C_b(S)$ and $g \in C_b(T)$, it holds that $\int_S f d\mu + \int_T g d\nu \geq u$ whenever $f(s) + g(t) \geq 1$ for all $(s,t) \in D$.

Khurana (2007) [20, Theorem 3]: The assertion (VMP2) holds whenever

- S and T are completely regular Hausdorff spaces.
- V is a Dedekind complete Riesz space.
- $\mu \in \mathcal{M}_{o,t}^+(S; V)$ and $\nu \in \mathcal{M}_{o,t}^+(T; V)$.

Definition 5.9. Let V be a Dedekind σ-complete Riesz space.

(1) V is said to be *weakly σ-distributive* if whenever $\{v_{i,j}\}_{(i,j)\in\mathbb{N}^2}$ is an order bounded subset of V with $v_{i,j+1} \leq v_{i,j}$ for each $(i,j) \in \mathbb{N}^2$ then it holds that $\sup_{i\in\mathbb{N}} \inf_{j\in\mathbb{N}} v_{i,j} = \inf_{\theta\in\mathbb{N}^\mathbb{N}} \sup_{i\in\mathbb{N}} v_{i,\theta(i)}$.

(2) Assume that V is Dedekind complete. V is said to be *weakly (σ,∞)-distributive* if whenever $\mathcal{L}$ is an infinite set with $\operatorname{card}\mathcal{L} \leq \aleph$ and $\{v_{n,\lambda}\}_{(n,\lambda)\in\mathbb{N}\times\mathcal{L}}$ is an order bounded family of elements of V, then it holds that

$$\sup_{n\in\mathbb{N}} \inf_{\lambda\in\mathcal{L}} v_{n,\lambda} = \inf_{\zeta\in L^\mathbb{N}} \sup_{n\in\mathbb{N}} v_{n,\zeta(n)},$$

where L is the set of all non-empty finite subsets of $\mathcal{L}$ and, for each $n \in \mathbb{N}$ and $\zeta \in L^\mathbb{N}$, $v_{n,\zeta(n)}$ is defined to be $\inf_{\lambda\in\zeta(n)} v_{n,\lambda}$.

Khurana (2007) [20, Theorem 3]: The assertion (VMP2) holds whenever

- S and T are completely regular Hausdorff spaces.
- V is a Dedekind complete and (σ,∞)-distributive Riesz space.
- $\mu \in \mathcal{M}_{o,t}^+(S; V)$ and $\nu \in \mathcal{M}_{o,\tau}^+(T; V)$, and vice versa.

6. Marginal problem for vector measures: set-type

VMP3 (Marginal problem for vector measures; set-type I): *Let $(\Omega, \mathcal{A})$ and $(\Lambda, \mathcal{B})$ be measurable spaces. Let V be a Riesz space or a Riesz space with locally convex topology. Let $\mu : \mathcal{A} \to V^+$ and $\nu : \mathcal{B} \to V^+$ are countably additive vector measures such that $\mu(\Omega) = \nu(\Lambda) = e$. Let $D \in \mathcal{A} \otimes \mathcal{B}$ be a countable intersection of sets in $\mathcal{A} \times \mathcal{B}$. Fix $u \in V^+$ with $u \leq e$. The following conditions are equivalent:*

(i) There is a countably additive vector measure $\lambda : \mathcal{A}\otimes\mathcal{B} \to V^+$ with marginals μ and ν such that $\lambda(D) \geq u$.

(ii) It holds that $\mu(A) + \nu(B) \leq 2e - u$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$ satisfy $(A \times B) \cap D = \emptyset$.

Remark 6.1. In (VMP3), the countable additivity of the involved vector measures means the countable additivity for the order convergence or the locally convex topology on V in context.

Strassen's theorem for finitely additive vector measures (Hirshberg-Shortt (1997) [12, Theorem 2]; D'Aniello-Wright (2000) [3, Lemma 3.6]): Let $(\Omega, \mathcal{A})$ and $(\Lambda, \mathcal{B})$ be measurable spaces. Let V be a Dedekind σ-complete Riesz space. Let $\mu : \mathcal{A} \to V^+$ and $\nu : \mathcal{B} \to V^+$ be vector measures such that $\mu(\Omega) = \nu(\Lambda) = e$. Let $D \in \mathcal{A} \otimes \mathcal{B}$ be a countable intersection of sets in $\mathcal{A} \times \mathcal{B}$. Let $\mathcal{C}$ be the field generated by $\mathcal{A} \times \mathcal{B}$ and D. Fix $u \in V^+$ with $u \leq e$. The following conditions are equivalent:

(i) There is a vector measure $\lambda : \mathcal{C} \to V^+$ with marginals μ and ν such that $\lambda(D) \geq u$.

(ii) It holds that $\mu(A) + \nu(B) \leq 2e - u$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$ satisfy $(A \times B) \cap D = \emptyset$.

Countable additivity of indirect product measures (D'Aniello-Wright [3, Theorem 3.3]: Let $(\Omega, \mathcal{A})$ and $(\Lambda, \mathcal{B})$ be measurable spaces. Let V be a Dedekind σ-complete and weakly σ-distributive Riesz space. Let $\mu : \mathcal{A} \to V^+$ be a countably additive vector measure for the order convergence on V and $\nu : \mathcal{B} \to V^+$ a σ-compact (see Definition 6.5) vector measure such that $\mu(\Omega) = \nu(\Lambda)$. Let $\lambda_0 : \mathcal{A} \times \mathcal{B} \to V^+$ be a vector measure with marginals μ and ν. Then λ_0 is countably additive and extends to a countably additive vector measure $\lambda : \mathcal{A} \otimes \mathcal{B} \to V^+$ for the order convergence on V.

Definition 6.2. Let V be a Banach lattice.

(1) V is called a *KB-space* if each norm bounded increasing sequence of elements of V is norm convergent.

(2) V is said to have *order continuous norm* if every order convergent net of elements of V norm converges.

Fact 6.3. (1) Every KB-space has order continuous norm.

(2) Every Banach lattice having order continuous norm is Dedekind complete.

Definition 6.4. Let $(\Omega, \mathcal{A})$ be a measurable space and V a Banach lattice. Let $\mu : \mathcal{A} \to V^+$ be a vector measure.

(1) A class $\mathcal{K}$ of subsets of Ω is said to be *compact* if whenever $\{K_n\}_{n \in \mathbb{N}}$ is a sequence of elements of $\mathcal{K}$ such that $K_1 \cap K_2 \cap \ldots K_n \neq \emptyset$ for each $n \in \mathbb{N}$, then it holds that $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

(2) μ is said to be *compact* if there is a compact class $\mathcal{K}$ of subsets of Ω such that for any $A \in \mathcal{A}$ and $\varepsilon > 0$, there are sets $B \in \mathcal{A}$ and $K \in \mathcal{K}$ with $B \subset K \subset A$ and $\|\mu(A - B)\| < \varepsilon$.

(3) μ is said to be *perfect* if the restriction of μ to every countably generated sub σ-field of $\mathcal{A}$ is compact.

Hirshberg-Shortt (1998) [13, Theorem 2]: The assertion (VMP3) holds whenever

- V is a KB-space.
- $\mu : \mathcal{A} \to V^+$ and $\nu : \mathcal{B} \to V^+$ are countably additive vector measures for the norm topology on V, one of which is perfect.

D'Aniello (1999/2000) [2, Theorem 3.10]: The assertion (VMP3) holds whenever

- V is a Banach lattice with order continuous norm.
- $\mu : \mathcal{A} \to V^+$ and $\nu : \mathcal{B} \to V^+$ are countably additive vector measures for the norm topology on V, one of which is perfect or compact.

Definition 6.5. Let $(\Omega, \mathcal{A})$ be a measurable space and V a Dedekind complete Riesz space. Let $\mu : \mathcal{A} \to V^+$ be a vector measure.

(1) μ is said to be *σ-compact* if there is a compact class $\mathcal{K}$ of subsets of Ω such that, for each $A \in \mathcal{A}$, there is a monotone increasing sequence $\{B_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{A}$ with the following properties:

(i) for each $n \in \mathbb{N}$, there is $K_n \in \mathcal{K}_s$ such that $B_n \subset K_n \subset A$, where $\mathcal{K}_s$ is the class of all finite unions of sets in $\mathcal{K}$,

(ii) $\mu(A) = \sup_{n\in\mathbb{N}} \mu(B_n)$.

(2) μ is said to be *completely compact* if there is a compact class $\mathcal{K}$ of subsets of Ω such that for each $A \in \mathcal{A}$, it holds that $\mu(A) = \sup\{\mu(B) : B \in \mathcal{A}$ is such that there is $K \in \mathcal{K}_s$ with $B \subset K \subset A\}$.

D'Aniello-Wright (2000) [3, Theorem 3.7]: The assertion (VMP3) holds whenever

- V is a Dedekind σ-complete and weakly σ-distributive Riesz space.
- $\mu : \mathcal{A} \to V^+$ and $\nu : \mathcal{B} \to V^+$ are countably additive vector measures for the order convergence on V, one of which is σ-compact.

D'Aniello-Wright (2000) [3, Theorem 3.13]: The assertion (VMP3) holds whenever

- V is a Dedekind complete and weakly (σ, ∞)-distributive Riesz space.

- $\mu : \mathcal{A} \to V^+$ and $\nu : \mathcal{B} \to V^+$ are countably additive vector measures for the order convergence on V, one of which is completely compact.

Definition 6.6. Let V be a Riesz space. A locally convex topology on V is said to be *sequentially Lebesgue* if every monotone decreasing sequence with infimum 0 converges to 0 for the locally convex topology on V.

Definition 6.7. Let $(\Omega, \mathcal{A})$ be a measurable space. Let V be a locally convex space and a Riesz space. Assume that $V' \subset V^\sim$, where $V^\sim$ is the order dual of V, that is, the set of all linear functionals on V which are bounded on order bounded sets. A vector measure $\mu : \mathcal{A} \to V^+$ is said to be *weakly perfect* if for every $u' \in V'$, the real measure $|u'|\mu$ is perfect.

Guerra and Muñoz-Bouzo (2002) [9, Theorem 1]: The assertion (VMP3) holds whenever

- V is a Dedekind complete Riesz space with a sequentially Lebesgue locally convex topology.
- $\mu : \mathcal{A} \to V^+$ and $\nu : \mathcal{B} \to V^+$ are countably additive vector measures for the locally convex topology on V, one of which is weakly perfect.

VMP4 (Marginal problem for vector measures; set-type II): *Let S and T be Hausdorff spaces. Let V be a locally convex space which is an ordered vector space. Let $\mu \in \mathcal{M}^+(S;V)$ and $\nu \in \mathcal{M}^+(T;V)$. Assume that $\mu(S) = \nu(T) = e$. Let D be a non-empty, closed subset of $S \times T$. Fix $u \in V^+$ with $u \leq e$. The following conditions are equivalent:*

(i) There is $\lambda \in \mathcal{M}^+(S \times T; V)$ with marginals μ and ν such that $\lambda(D) \geq u$.
(ii) It holds that $\mu(A) + \nu(B) \leq 2e - u$ whenever $A \in \mathcal{B}(S)$ and $B \in \mathcal{B}(T)$ satisfy $(A \times B) \cap D = \emptyset$.

Khurana (2006) [19, Theorem 5]: The assertion (VMP4) holds whenever

- S and T are completely regular Hausdorff spaces.
- V is a Dedekind complete locally convex Riesz space such that if an order bounded net $\{u_\alpha\}_{\alpha \in \Gamma}$ of elements of V order converges to $u \in V$, then $u_\alpha \to u$ for the locally convex topology on V.
- $\mu \in \mathcal{M}_t^+(S;V)$ and $\nu \in \mathcal{M}_t^+(T;V)$.

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