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Strong and Weak Convergence Theorems for Equilibrium Problems and Nonlinear Mappings in Banach Spaces

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Abstract. In this article, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Next, we prove two strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method and a new hybrid method called the shrinking projection method. Further, we obtain a necessary and sufficient condition for the existence of solutions of the euquilibrium problem by using the metric resolvents. Finally, we prove a strong convergence theorem for finding a solution of an equilibrium problem in a Banach space by using the shrinking projection method.

1 Introduction

Let E be a real Banach space and let E^* be a dual space of E. Let C be a closed convex subset of E and let f be a bifunction from $C \times C$ to R, where R is the set of real numbers. The equilibrium problem is formulated as follows: Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \quad \text{for all } y \in C.$$

In this case, such a point $\hat{x} \in C$ is called a solution of the problem. The set of such solutions $\hat{x}$ is denoted by $EP(f)$. Many problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Blum and Oettli [5] widely discussed the existence of solutions of such an equilibrium problem. Combettes-Hirstoaga [9], Tada and Takahashi [46], and Takahashi and Takahashi [48] proposed some methods for approximation of solutions of the equilibrium problem in a Hilbert space. However, the problem of approximating solutions of the equilibrium problem in a Banach space is difficult. We also know the problem of finding a point $u \in E$ satisfying

$$0 \in Au,$$

where A is a maximal monotone operator from E to E^*. Such a problem contains numerous problems in physics, optimization and economics. A well-known method to solve this problem is called the proximal point algorithm: $x_1 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \; n = 1, 2, \ldots,$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} are the resolvents of A. Many researchers have studied this algorithm in a Hilbert space, see, for instance, [11, 18, 27, 42, 45] and in a Banach space, see, for instance, [17, 19, 20, 33]. A mapping S of C into E is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. There are some methods for approximation of fixed points of a nonexpansive mapping; see, for instance, [12, 26, 36, 43, 64]. In particular, in 2003 Nakajo–Takahashi [32] proved the following strong convergence theorem by using the hybrid method:

Theorem 1.1 (Nakajo and Takahashi [32]). *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ u_{n+1} = P_{C_n \cap Q_n} x, \quad n \in N, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$ and $\{\alpha_n\}$ is chosen so that $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.

Let us call the hybrid method in Theorem 1.1 the normal hybrid method. Very recently, Takahashi, Takeuchi and Kubota [61] proved the following theorem by using another hybrid method called the shrinking projection method.

Theorem 1.2 (Takahashi, Takeuchi and Kubota [61]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n) T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \quad n \in N, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in N$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.

In this article, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Next, using the normal hybrid method and a new hybrid method called the shrinking projection method, we study two strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space. Further, we obtain a necessary and sufficient condition for the existence of solutions of the euquilibrium problem by using the metric resolvents. Finally, we prove a strong convergence theorem for finding a solution of an equilibrium problem in a Banach space by using the shrinking projection method.

2 Preliminaries

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the topological dual of E. For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^* \rangle$. Then, the duality mapping J on E is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty; see [51] for more details. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \overset{*}{\rightharpoonup} x^*$. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. The space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$. We know that if E is smooth, strictly convex and reflexive, then the duality mapping J is single-valued, one-to-one and onto; see [51, 52] for more details.

Let E be a smooth Banach space and define the real valued function ϕ by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $y, x \in E$. Then, we have that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) - 2\langle x - z, Jz - Jy \rangle$$

for all $x, y, z \in E$. Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E. Following Alber [1], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x)$$

for all $x \in E$. If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C is the metric projection of H onto C. We know the following lemmas for generalized projections.

Lemma 2.1 (Alber [1], Kamimura and Takahashi [20]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \text{for all } x \in C \text{ and } y \in E.$$

Lemma 2.2 (Alber [1], Kamimura and Takahashi [20]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0 \quad \text{for all } y \in C.$$

Let E be a smooth, strictly convex and reflexive Banach space, and let A be a set-valued mapping from E to E^* with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$. We denote a set-valued operator A from E to E^* by $A \subset E \times E^*$. A is said to be monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0$$

for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be maximal monotone if its graph is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is closed and convex; see [51, 52] for more details. The following theorem is well-known.

Theorem 2.3 (Rockafellar [41]). *Let E be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right).$$

Then we can define the resolvent $J_r : C \to D(A)$ of A by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\}$$

for all $x \in C$. We know that $J_r x$ consists of one point. For all $r > 0$, the Yosida approximation $A_r : C \to E^*$ is defined by $A_r x = \frac{Jx - JJ_r x}{r}$ for all $x \in C$. We also know the following lemma; see, for instance, [24].

Lemma 2.4. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $A \subset E \times E^*$ be a monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right).$$

Let $r > 0$ and let J_r and A_r be the resolvent and the Yosida approximation of A, respectively. Then, the following hold:

(1) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$ for all $x \in C$ and $u \in A^{-1}0$;
(2) $(J_r x, A_r x) \in A$ for all $x \in C$;
(3) $F(J_r) = A^{-1}0$.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let T be a mapping from C into itself. We denoted by $F(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. Following Matsushita and Takahashi [29], a mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;
(3) $\hat{F}(T) = F(T)$.

The following lemma is due to Matsushita and Takahashi [28].

Lemma 2.5 (Matsushita and Takahashi [28]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

We also know the following lemma.

Lemma 2.6 (Kamimura and Takahashi [20]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_n \phi(x_n, y_n) = 0$, then $\lim_n \|x_n - y_n\| = 0$.*

3 Equilibrium Problems and Relatively Nonexpansive Mappings

In this section, we prove strong and weak convergence theorems for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Let E be a Banach space and let C be a nonempty closed convex subset of E. A function $f : C \times C \to R$ is said to be maximal monotone with respect to C if, for every $x \in C$ and $x^* \in E^*$,

$$f(x,y) + \langle y - x, x^* \rangle \geq 0$$

for all $y \in C$, whenever $\langle z - x, x^* \rangle \geq f(z,x)$ for all $z \in C$.

In this article, we assume that a bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) for all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous;
(A4) $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$ for all $x, y, z \in C$.

Assume that f satisfies (A1)–(A4). Then, f is maximal monotone. In fact, for every $x \in C$ and $x^* \in E^*$, suppose that

$$\langle z - x, x^* \rangle \geq f(z,x)$$

for all $z \in C$. Putting $z_t = (1-t)x + ty$ with $y \in C$ and $t \in (0,1)$, we have

$$\begin{aligned} 0 =& f(z_t, z_t) \\ \leq& (1-t) f(z_t, x) + t f(z_t, y) \\ \leq& (1-t)\langle z_t - x, x^* \rangle + t f(z_t, y) \\ \leq& t(1-t)\langle y - x, x^* \rangle + t f(z_t, y). \end{aligned}$$

Hence, we have $0 \leq (1-t)\langle y - x, x^* \rangle + f(z_t, y)$. Since f is upper hemicontinuous, we have

$$0 \leq \langle y - x, x^* \rangle + f(x,y).$$

Hence, f is maximal monotone. The following result is in Blum and Oettlli [5]. See [2] for the proof.

Lemma 3.1 (Blum and Oettli [5]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0 \quad \text{for all } y \in C.$$

Motivated by Combettes and Hirstoaga [9] in a Hilbert space, we obtain the following lemma.

Lemma 3.2. *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C \right\}$$

for all $x \in E$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is a firmly nonexpansive-type mapping [24], i.e., for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

We claim that T_r is single-valued. Indeed, for $x \in C$ and $r > 0$, let $z_1, z_2 \in T_r x$. Then,

$$f(z_1, z_2) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx\rangle \geq 0$$

and

$$f(z_2, z_1) + \frac{1}{r}\langle z_1 - z_2, Jz_2 - Jx\rangle \geq 0.$$

Adding two inequalities, we have

$$f(z_1, z_2) + f(z_2, z_1) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq 0.$$

From (A2) and $r > 0$, we have

$$\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq 0.$$

Since E is strictly convex, we have $z_1 = z_2$.

Next, we claim that T_r is a firmly nonexpansive-type mapping. Indeed, for $x, y \in C$, we have

$$f(T_r x, T_r y) + \frac{1}{r}\langle T_r y - T_r x, JT_r x - Jx\rangle \geq 0,$$

and

$$f(T_r y, T_r x) + \frac{1}{r}\langle T_r x - T_r y, JT_r y - Jy\rangle \geq 0.$$

Adding two inequalities, we have

$$f(T_r x, T_r y) + f(T_r y, T_r x) + \frac{1}{r}\langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy\rangle \geq 0.$$

From (A2) and $r > 0$, we have

$$\langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy\rangle \geq 0.$$

Therefore, we have

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle.$$

We call such T_r the relative resolvent of f for $r > 0$. Using Lemma 3.2, we have the following result.

Lemma 3.3. *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Proof. From Lemma 3.2 (2), we have, for all $x, y \in E$,

$$\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \leq \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y).$$

Letting $y = q \in F(T_r)$, we have

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

This completes the proof. □

Now, we prove a strong convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space.

Theorem 3.4 (Takahashi and Zembayashi [63]). *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \ \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in N \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap EP(f)} x$, where $\Pi_{F(S)\cap EP(f)}$ is the generalized projection of E onto $F(S) \cap EP(f)$.

Further, we prove a weak convergence theorem for finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping in a Banach space. Before proving the theorem, we need the following proposition.

Proposition 3.5. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u_1 \in E$,*

$$\begin{cases} x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, J x_n - J u_n \rangle \geq 0, \ \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n) \end{cases}$$

for every $n \in N$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [0, \infty)$. Then, $\{\Pi_{F(S)\cap EP(f)} x_n\}$ converges strongly to $z \in F(S) \cap EP(f)$, where $\Pi_{F(S)\cap EP(f)}$ is the generalized projection of E onto $F(S) \cap EP(f)$.

Using Proposition 3.5, we can prove the following theorem.

Theorem 3.6 (Takahashi and Zembayashi [63]). *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequences generated by $u_1 \in E$,*

$$\begin{cases} x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \ \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n) \end{cases}$$

for every $n \in N$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in F(S) \cap EP(f)$, where $z = \lim_{n\to\infty} \Pi_{F(S)\cap EP(f)} x_n$.

4 Maximal Monotone Operators and Relatively Nonexpansive Mappings

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the normal hybrid method.

Theorem 4.1 (Inoue, Takahashi and Zembayashi [25]). *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right)$$

and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSJ_{r_n}x_n), \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in N \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1)$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap A^{-1}0} x$, where $\Pi_{F(S)\cap A^{-1}0}$ is the generalized projection of E onto $F(S) \cap A^{-1}0$.

As direct consequences of Theorem 4.1, we can obtain the following corollaries.

Corollary 4.2. *Let E be a uniformly smooth and uniformly convex Banach space, let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all*

$r > 0$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C$ *and*

$$\begin{cases} u_n = J_{r_n} x_n, \\ H_n = \{z \in E : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in N \cup \{0\}$, *where* J *is the duality mapping on* E *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{A^{-1}0} x$, *where* $\Pi_{A^{-1}0}$ *is the generalized projection of* E *onto* $A^{-1}0$.

Proof. Putting $S = I$, $C = E$ and $\alpha_n = 0$ in Theorem 4.1, we obtain Corollary 4.2. □

Corollary 4.3 (Matsushita and Takahashi [29]). *Let* E *be a uniformly smooth and uniformly convex Banach space, let* C *be a nonempty closed convex subset of* E, *and let* S *be a relatively nonexpansive mapping from* C *into itself such that* $F(S) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C$ *and*

$$\begin{cases} u_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases}$$

for every $n \in N \cup \{0\}$, *where* J *is the duality mapping on* E, $\{\alpha_n\} \subset [0, 1)$ *satisfies* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)} x$, *where* $\Pi_{F(S)}$ *is the generalized projection of* E *onto* $F(S)$.

Proof. Set $A = \partial i_C$ in Theorem 4.1, where i_C is the indicator function of C and ∂i_C is the subdifferential of i_C. Then, we have that A is a maximal monotone operator and $J_r = \Pi_C$, where J_r is the resolvent of $A = \partial i_C$ for $r > 0$. So, from Theorem 4.1, we obtain Corollary 4.3. □

Using an idea of [61], we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping in a Banach space by using the shrinking projection method.

Theorem 4.4 (Inoue, Takahashi and Zembayashi [25]). *Let* E *be a uniformly smooth and uniformly convex Banach space, and let* C *be a nonempty closed convex subset of* E. *Let* $A \subset E \times E^*$ *be a maximal monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\cap_{r>0} R(J + rA)\right)$$

and let $J_r = (J + rA)^{-1} J$ *for all* $r > 0$. *Let* S *be a relatively nonexpansive mapping from* C *into itself such that* $F(S) \cap A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 = x \in C$, $H_0 = C$ *and*

$$\begin{cases} u_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S J_{r_n} x_n), \\ H_{n+1} = \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{H_{n+1}} x \end{cases}$$

for every $n \in N \cup \{0\}$, *where* J *is the duality mapping on* E, $\{\alpha_n\} \subset [0, 1)$ *satisfies* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *Then,* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)\cap A^{-1}0} x$, *where* $\Pi_{F(S)\cap A^{-1}0}$ *is the generalized projection of* E *onto* $F(S) \cap A^{-1}0$.

As direct consequences of Theorem 4.4, we can obtain the following corollaries.

Corollary 4.5. *Let E be a uniformly smooth and uniformly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$, $H_0 = E$ and*

$$\begin{cases} u_n = J_{r_n} x_n, \\ H_{n+1} = \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{H_{n+1}} x \end{cases}$$

for every $n \in N \cup \{0\}$, where J is the duality mapping on E and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x$.

Proof. Putting $S = I$, $C = H_0 = E$ and $\alpha_n = 0$ in Theorem 4.4, we obtain Corollary 4.5. □

Corollary 4.6. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ H_{n+1} = \{z \in H_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{H_{n+1}} x \end{cases}$$

for every $n \in N \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1)$ satisfies $\liminf_{n\to\infty}(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S)}x$, where $\Pi_{F(S)}$ is the generalized projection of E onto $F(S)$.

Proof. Putting $A = \partial i_C$ in Theorem 4.4, we obtain Corollary 4.6. □

5 Equilibrium Problems and Metric Resolvents

In this section, we prove a strong convergence theorem for finding a solution of the equilibrium problem by using the metric resolvents. Using Lemma 3.1, we first obtain the following result.

Lemma 5.1. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to R satisfying (A1)–(A4), let $r > 0$ and let $x \in E$. Then, there exists a unique $z_r \in C$ such that*

$$f(z_r, y) + \frac{1}{r}\langle y - z_r, J(z_r - x)\rangle \geq 0 \quad \text{for all } y \in C.$$

Proof. Fix $x \in C$. Then, we difine $g : (C - x) \times (C - x) \to R$ as follows:

$$g(z, y) = f(z + x, y + x) + \frac{1}{r}\langle y - z, Jx\rangle. \tag{5.1}$$

From the properties of f, it is easy to prove that g satisfies the following conditions;

(A1) $g(z, z) = 0$ for all $z \in C - x$;

(A2) g is monotone with respect to $C-x$;
(A3) for all $z \in C-x$, $g(z,\cdot)$ is convex and lower semicontinuous;
(A4) g is upper hemicontinuous with respect to $C-x$.

Hence, from Lemma 3.1 there exists a unique element z_r such that

$$f(z_r+x, y+x) + \frac{1}{r}\langle y-z_r, Jx\rangle + \frac{1}{r}\langle y-z_r, Jz_r - Jx\rangle \geq 0$$

for all $y \in C-x$. This implies that

$$f(z_r+x, y+x) + \frac{1}{r}\langle y-z_r, Jz_r\rangle \geq 0$$

for all $y \in C-x$. Putting $u_r = z_r + x$ and $v = y+x$, we have

$$f(u_r, v) + \frac{1}{r}\langle v-u_r, J(u_r - x)\rangle \geq 0$$

for all $v \in C$. This completes the proof. □

Under the conditions in Theorem 5.1, for every $r>0$ we may define a single-valued mapping $F_r : E \to C$ by

$$F_r x = \{z \in C : 0 \leq f(z,y) + \frac{1}{r}\langle y-z, J(z-x)\rangle, \quad y \in C\} \tag{5.2}$$

for $x \in E$, which is called the metric resolvent of f for $r>0$. Also, we can define the Yosida approximation as follows:

$$A_r x = \frac{1}{r}J(x - F_r x). \tag{5.3}$$

As in Takahashi[52, pp.163-165], we can prove the following theorem for Yosida approximations. Before proving it, we need the following lemma; see, for instance, Takahashi[52, Problem 4.5.4].

Lemma 5.2. *Let E be a Banach space. Assume that $u_n \rightharpoonup v$, $v_n \overset{*}{\rightharpoonup} v^*$ and*

$$\lim_{m,n\to\infty}\langle u_n - u_m, v_n - v_m\rangle = 0.$$

Then, $\lim_{n\to\infty}\langle u_n, v_n\rangle = \langle u, v^*\rangle$.

Lemma 5.3. *Assume $r>0$. Then, $A_r : E \to E^*$ is monotone and demicontinuous. Further, if $D \subset E$ is bounded, then $A_r D \subset E^*$ is bounded.*

To show a necessary and sufficient condition for the existence of solutions of the euquilibrium problem, we need the following lemma [51, Theorem 7.1.8]; see also [4].

Lemma 5.4 ([51, 4]). *Let E be a reflexive Banach space and let K be a bounded closed convex subset of E. Suppose A is a monotone and demicontinuous operator. Then there exists $u_0 \in K$ such that*

$$\langle y-u_0, Au_0\rangle \geq 0 \quad \text{for all } y \in K.$$

Using Lemma 5.4, we obtain the following lemma.

Lemma 5.5. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let f be a bifunction $C \times C$ to R satisfying (A1)–(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = F_{r_n} x_n, \\ C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases}$$

where $0 < r_n < \infty$ and P_{C_n} is the metric projection of E onto C_n. Then $\{x_n\}$ is well-defined.

We also have the following lemma.

Lemma 5.6. *If $EP(f) \neq \emptyset$, then $EP(f) \subset C_n$ for all $n \in N$.*

Proof. It is ovbious that $EP(f) \subset C_1 = C$. Suppose $EP(f) \subset C_n$ for some $n \in N$. Let $z \in EP(f)$. From $y_n = F_{r_n} x_n$ and the monotonicity of f, we have

$$\langle y_n - y, \frac{1}{r_n} J(x_n - y_n)\rangle \geq f(y, y_n)$$

for all $y \in C$. Put $y = z$. Then we have

$$\langle y_n - z, \frac{1}{r_n} J(x_n - y_n)\rangle \geq f(z, y_n) \geq 0.$$

Therefore, $z \in C_{n+1}$. By the mathematical inducition, we obtain $z \in C_n$ for all $n \in N$. □

Now, we obtain a necessary and sufficient condition for the existence of solutions of the euquilibrium problem in a Banach space.

Theorem 5.7 (Takahashi and Takahashi [47]). *Let E be a smooth and uniformly convex Banach space and let f be a bifunction $C \times C$ to R satisfying (A1)–(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = F_{r_n} x_n, \\ C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases}$$

where $\liminf_{n\to\infty} r_n > 0$ and P_{C_n} is the metric projection of E onto C_n. Then $\{x_n\}$ is bounded if and only if $EP(f) \neq \emptyset$.

Finally, we can prove a strong convergence theorem for finding a solution of the equilibrium problem by using the shrinking projection method.

Theorem 5.8 (Takahashi and Takahashi [47]). *Let E be a smooth and uniformly convex Banach space and let C a nonempty closed convex subset of E. Let f be a bifunction $C \times C$ to R satisfying (A1)–(A4). For $C_1 = C$ and $x_1 = x \in E$, define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = F_{r_n} x_n, \\ C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases}$$

where $\liminf_{n\to\infty} r_n > 0$ and P_{C_n} is the metric projection of E onto C_n. If $EP(f) \neq \emptyset$, then $\{x_n\}$ converges strongly to the element $P_{EP(f)}(x_1)$, where $P_{EP(f)}$ is the metric projection of E onto $EP(f)$.

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