数理解析研究所講究録
第 1585 巻 2008 年 115-119

Additive indecomposability of submodular set functions and its generalization*

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1 Introduction

This paper deals with a decomposition of a submodular set function into a sum of submodular set functions on subdomains and its generalization. Submodular set functions have an important role in mathematical programming [3], and a supermodular set function, which is the conjugate of a submodular set function (see Section 2), also is an important concept called a convex game in cooperative game theory [6]. Therefore, the additive decomposition of submodular set functions has broad application possibilities.

This paper is organized as follows. Section 2 explains basic concepts such as inclusion-exclusion family, submodularity, and (weak) k-monotonicity. Section 3 shows the results on additive decompositions of set functions we have obtained so far. Section 4 gives the main results, that is, conditions for additive indecomposability, which provide a foothold for further investigation of additive decompositions.

2 Preliminaries

For a finite set X, the number of elements of X is denoted by $|X|$, the power set of X by 2^X, and, for an integer k such that $0 \leq k \leq |X|$, the family of k-element subsets of X is denoted by $\binom{X}{k}$, i.e,

$$\binom{X}{k} \stackrel{\text{def}}{=} \{Y \in 2^X \mid |Y| = k\}.$$

Throughout this paper, E is assumed to be a finite set.

A family $\mathcal{A}$ of subsets of E is called an *antichain* if A, $A' \in \mathcal{A}$ and $A \subseteq A'$ together imply $A = A'$. For antichains $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \sqsubseteq \mathcal{B}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$; then $\sqsubseteq$ is a partial ordering on the class of all antichains over E, and the class forms a lattice with the following meet:

$$\mathcal{A} \sqcap \mathcal{B} = \mathrm{Max}\{A \cap B \mid A \in \mathcal{A},\ B \in \mathcal{B}\},$$

*This work is partially supported by grants from the Ministry of Education, Culture, Sports, Science and Technology, the 21st Century COE Program "Creation of Agent-Based Social Systems Sciences" and Grant-in-Aid for Scientific Research (C), 19510136, 2007.

where, for a family $\mathcal{F}$ of sets, Max$\mathcal{F}$ is defined by

$$\mathrm{Max}\mathcal{F} \stackrel{\mathrm{def}}{=} \{M \in \mathcal{F} \mid M \text{ is maximal in } \mathcal{F} \text{ with respect to set inclusion } \subseteq\}.$$

A function $f : 2^E \to \mathbb{R}$ satisfying $f(\emptyset) = 0$ is called a *set function* on E. An antichain $\mathcal{A}$ of subsets of E is called an *inclusion-exclusion family*, or an *inclusion-exclusion antichain*, with respect to a set function f on E if ⟨IE⟩ below holds:

$$\langle \mathrm{IE} \rangle: \quad f(X) = \sum_{\mathcal{B} \subseteq \mathcal{A},\ \mathcal{B} \neq \emptyset} (-1)^{|\mathcal{B}|+1} f\left(X \cap \bigcap \mathcal{B}\right) \qquad \text{for all } X \subseteq E.$$

If an antichain $\mathcal{A}$ contains a subset A such that $f(X) = f(X \cap A)$ for every $X \subseteq E$, then $\mathcal{A}$ is an inclusion-exclusion family with respect to f, and $\mathcal{A}$ is called a *trivial* inclusion-exclusion family; for example, $\{E\}$ is a trivial inclusion-exclusion family with respect to any set function on E. For antichains $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A} \sqsubseteq \mathcal{B}$, and if $\mathcal{A}$ is an inclusion-exclusion family with respect to a set function f, then so is $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are inclusion-exclusion antichains with respect to a set function f, then so is $\mathcal{A} \sqcap \mathcal{B}$. Therefore, every set function has its least (with respect to $\sqsubseteq$) inclusion-exclusion antichain.

For a set function f on E, the *sign inversion* $-f$ of f and the *conjugate*, or *dual*, $f^\#$ of f are defined as follows [3]:

$$(-f)(X) \stackrel{\mathrm{def}}{=} -f(X), \qquad f^\#(X) \stackrel{\mathrm{def}}{=} f(E) - f(E \setminus X)$$

for every $X \subseteq E$. For any set function f_A on $A \subseteq E$, we regard f_A as a set function on E by defining $f_A(X) = f_A(X \cap A)$ for every $X \in 2^E \setminus 2^A$. Let $\mathcal{A} \subseteq 2^E$ and $\{f_A\}_{A \in \mathcal{A}}$ be a collection of set functions f_A on $A \in \mathcal{A}$. Then the following holds:

$$f = \sum_{A \in \mathcal{A}} f_A \quad \Longleftrightarrow \quad -f = \sum_{A \in \mathcal{A}} (-f_A) \quad \Longleftrightarrow \quad f^\# = \sum_{A \in \mathcal{A}} f_A^\#;$$

note that, for every set function f_A on $A \subseteq E$, the conjugate $f_A(A) - f_A(A \setminus (\cdot))$ over A coincides, as a set function on E, with the conjugate $f_A(E) - f_A(E \setminus (\cdot))$ over E.

A set function f is said to be *submodular* if the following inequalities hold [3]:

$$f(X \cup Y) + f(X \cap Y) \ \leq \ f(X) + f(Y) \qquad \text{for all } X,\ Y \subseteq E.$$

A set function f is said to be *supermodular* if $f^\#$ is submodular.

The *difference function* $\bigwedge f : 2^E \times \mathbb{N}^{(2^E)} \to \mathbb{R}$ of a set function f on E is defined recursively as follows [2]:

$$\bigwedge f(X, \emptyset) \stackrel{\mathrm{def}}{=} f(X),$$

$$\bigwedge f(X, \mathcal{Y} \uplus \{Y\}) \stackrel{\mathrm{def}}{=} \bigwedge f(X, \mathcal{Y}) - \bigwedge f(X \cap Y, \mathcal{Y}),$$

where $\mathbb{N}$ is the set of nonnegative integers, $\mathcal{Y}$ is a multiset over 2^E——$\mathcal{Y} : 2^E \to \mathbb{N}$ and $\mathcal{Y}(Z) \in \mathbb{N}$ is the multiplicity of $Z \in 2^E$ in $\mathcal{Y}$——, $\uplus$ is the sum of multisets, and it holds that

$$(\mathcal{Y} \uplus \{Y\})(Z) = \begin{cases} \mathcal{Y}(Z) + 1 & \text{if } Z = Y, \\ \mathcal{Y}(Z) & \text{if } Z \neq Y. \end{cases}$$

When $|\mathcal{Y}|$ $(\stackrel{\mathrm{def}}{=} \sum_{Z \in 2^E} \mathcal{Y}(Z)) = k$, we write $\bigwedge f(X, \mathcal{Y})$ as $\bigwedge_k f(X, \mathcal{Y})$ also.

(i) [2] For a positive integer k, a set function f is said to be *k-monotone* if $\bigwedge_k f \geq 0$, i.e., $\bigwedge f(X, \mathcal{Y}) \geq 0$ whenever $X \in 2^E$ and $\mathcal{Y} \in \left(\binom{2^E}{k}\right) \stackrel{\text{def}}{=} \{\mathcal{X} \in \mathbb{N}^{(2^E)} \mid |\mathcal{X}| = k\}$.

(ii) [1] For an integer k greater than 1, a set function f is said to be *weakly k-monotone* if for every $\mathcal{X} \in \left(\binom{2^E}{k}\right)$

$$f\left(\bigcup \mathcal{X}\right) \geq \sum_{\mathcal{Y} \subseteq \mathcal{X}, \mathcal{Y} \neq \emptyset} (-1)^{|\mathcal{Y}|+1} f\left(\bigcap \mathcal{Y}\right),$$

where, for $\mathcal{Z} \in \mathbb{N}^{(2^E)}$, $\bigcup \mathcal{Z} \stackrel{\text{def}}{=} \bigcup_{Z \in \mathcal{Z}} Z = \bigcup(\text{supp}\mathcal{Z})$, $\bigcap \mathcal{Z} \stackrel{\text{def}}{=} \bigcap_{Z \in \mathcal{Z}} Z = \bigcap(\text{supp}\mathcal{Z})$, $Z \in \mathcal{Z}$ means $\mathcal{Z}(Z) > 0$, and $\text{supp}\mathcal{Z}$ is the ordinary set $\{Z \mid \mathcal{Z}(Z) > 0\} \subseteq 2^E$ called the support of $\mathcal{Z}$.

The 1-monotonicity is equivalent to the ordinary monotonicity, i.e., $X \subseteq Y \implies f(X) \leq f(Y)$. The concept of weak 1-additivity is not defined. There are the following relations between submodularity and weak 2-monotonicity:

f is submodular $\iff$ $-f$ is weakly 2-monotone $\iff$ $f^{\#}$ is weakly 2-monotone.

For every integer k greater than 1, a set function f is k-monotone iff f is monotone and weakly k-monotone. If k and k' are integers such that $1 \leq k \leq k'$, and if a set function f is k'-monotone, then f is k-monotone. If k and k' are integers such that $2 \leq k \leq k'$, and if a set function f is weakly k'-monotone, then f is weakly k-monotone.

3 Additive decomposition

This paper deals with the following additive decomposition of a set function f on E with respect to an antichain $\mathcal{A}$ of subsets of E.

⟨AD⟩: A set function f on E is decomposable into a sum of set functions f_A over all $A \in \mathcal{A}$, that is, there exists a collection $\{f_A\}_{A \in \mathcal{A}}$ such that each f_A is a set function on A and

$$f = \sum_{A \in \mathcal{A}} f_A. \tag{1}$$

A necessary and sufficient condition for the additive decomposition ⟨AD⟩ is ⟨IE⟩, that is, $\mathcal{A}$ is an inclusion-exclusion family with respect to f [5].

If f is a submodular set function, and if an antichain $\mathcal{A}$ is an inclusion-exclusion family with respect to f, there does not always exist a collection $\{f_A\}_{A \in \mathcal{A}}$ of *submodular* set functions satisfying Eq. (1), while there always exists a collection $\{f_A\}_{A \in \mathcal{A}}$ of set functions satisying Eq. (1). That is to say, the antichain $\mathcal{A}$ being an inclusion-exclusion family is only a necessary condition and not a sufficient condition for a submodular set function f to be decomposable into a sum of submodular set functions f_A over all $A \in \mathcal{A}$.

So far, the authors have obtained two theorems showing sufficient conditions for the decomposition of submodular set functions into a sum of submodular set functions and their generalizations [4][7]. We show below the two generalized additive decomposition theorems. For an antichain $\mathcal{A}$ of subsets of E, a set function f on E is said to have a *k-monotone* [resp. *weakly k-monotone*] *$\mathcal{A}$-decomposition* if there exists a collection $\{f_A\}_{A\in\mathcal{A}}$ such that each f_A is a k-monotone [resp. weakly k-monotone] set function on A and Eq. (1) holds. The two theorems deal with the following three types of conditions $\bigcap(k,l,\mathcal{A})$, $\mathrm{M}(k',k,\mathcal{A})$, and $\mathrm{wM}(k',k,\mathcal{A})$ on positive integers k, k', and l such that $k \le k'$ and an antichain $\mathcal{A}$:

$\bigcap(k,l,\mathcal{A})$: $|\bigcap\mathcal{B}| \le k$ for any $\mathcal{B} \in \binom{\mathcal{A}}{l}$.

$\mathrm{M}(k',k,\mathcal{A})$: Every k'-monotone set function f with $\mathcal{A}$ as an inclusion-exclusion family has a k-monotone $\mathcal{A}$-decomposition.

$\mathrm{wM}(k',k,\mathcal{A})$: Every weakly k'-monotone set function f with $\mathcal{A}$ as an inclusion-exclusion family has a weakly k-monotone $\mathcal{A}$-decomposition.

Condition $\mathrm{wM}(k',1,\mathcal{A})$ is not considered.

Theorem 1 (Generalized Additive Decomposition Theorem A). *For a positive integer k and an antichain $\mathcal{A}$, the three conditions $\bigcap(k,2,\mathcal{A})$, $\mathrm{M}(k,k,\mathcal{A})$, and $\mathrm{wM}(k,k,\mathcal{A})$ are equivalent to each other.*

Theorem 2 (Generalized Additive Decomposition Theorem B). *Let k and k' be positive integers, $k \le k'$, and $\mathcal{A}$ be an antichain. Then $\bigcap(k-1,k'-k+2,\mathcal{A})$ is a sufficient condition for each of $\mathrm{M}(k',k,\mathcal{A})$ and $\mathrm{wM}(k',k,\mathcal{A})$.*

4 Indecomposability

Our present subject is the unification of Theorems 1 and 2, that is, necessary and sufficient conditions for $\mathrm{M}(k',k,\mathcal{A})$ and $\mathrm{wM}(k',k,\mathcal{A})$. We have found a cue to this subject, and we give it below. Note that, for every integer k greater than 1, a monotone set function f has a k-monotone $\mathcal{A}$-decomposition iff it has a weakly k-monotone $\mathcal{A}$-decomposition.

Proposition 1. *Let k, k', l, n be positive integers such that $k \le k' \le l \le n-3$, and E be an n-element set. If*

$$(n-l)(l-k'+1) - 2(l-k+1) > 0, \tag{2}$$

then there exists a k'-monotone set function f on E with $\binom{E}{l+2}$ as the least inclusion-exclusion family such that f does not have a k-monotone $\binom{E}{l+2}$-decomposition.

If $\mathcal{A}$ is a non-trivial inclusion-exclusion family with respect to a set function f, a k-monotone $\mathcal{A}$-decomposition of f is said to be *non-trivial*.

Corollary 1. *Let k and k' be positive integers such that $k \leq k'$, and E be an n-element set. If*

$$n > 3k' - 2k + 2,$$

then there exists a k'-monotone set function on E with a non-trivial inclusion-exclusion family such that f has no non-trivial k-monotone decomposition.

Corollary 2. *Let k and k' be positive integers such that $k \leq k'$, and E be an n-element set. If*

$$n > k' + 1 + \sqrt{8(k' - k) + 1},$$

then there exist a k'-monotone set function f and an inclusion-exclusion antichain $\mathcal{A}$ with respect to f such that f does not have a k-monotone $\mathcal{A}$-decomposition.

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