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OPERATIONAL STRUCTURE OF ENTANGLED QUANTUM SYSTEMS

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ABSTRACT. We investigate the quantum interaction between two quantum systems in terms of C*-algebras. We introduce the *correlation* CP-map between the bipartite systems and study the operational and statistical structure of the quantum interaction. We then generalize the informational quantities into this quantum setting, and introduce a new entropy of CP-maps, which vanishes at extreme CP-maps in CP-convexity.

Introduction.

We prepare two quantum states described by density operators ρ and σ on Hilbert spaces H_1 and K_1 respectively, and so initially we have the compound state $\omega_0 = \rho \otimes \sigma$ on the tensor product Hilbert space $H_1 \otimes K_1$. Suppose that there exists an interaction between the two subsystems for some time interval, and that the systems are in equilibrium with the exterior for this time period, then the compound state is changed to a state ω described by

$$\omega = U(\rho \otimes \sigma)U^*$$

where U is a unitary operator representing the quantum evolution of the systems. After the interaction, the states of the subsystems on H_2 and K_2 are described by $\mathrm{Tr}_{K_2}\omega$ and $\mathrm{Tr}_{H_2}\omega$ respectively. Then the *channels* φ^* from $T(H_1)$ to $T(H_2)$, and ϕ^* from $T(K_1)$ to $T(K_2)$, are defined by

$$\varphi^*(\rho) = \mathrm{Tr}_{K_2}\omega = \mathrm{Tr}_{K_2}U(\rho \otimes \sigma)U^* \quad \text{and} \quad \phi^*(\sigma) = \mathrm{Tr}_{H_2}\omega = \mathrm{Tr}_{H_2}U(\rho \otimes \sigma)U^*,$$

The detailed version of this paper will be submitted for publication elsewhere.

which are duals of the *operations* φ from $B(H_2)$ to $B(H_1)$, and ϕ from $B(K_2)$ to $B(K_1)$ respectively, representing the change of observables. In this note, to simplify our arguments, we assume that $H_1 = H_2 = H$ and $K_1 = K_2 = K$, and that φ and ϕ are unital, which guarantee that φ^* and ϕ^* are trace preserving, i.e., $\mathrm{Tr}\varphi^*(\rho) = 1$ and $\mathrm{Tr}\phi^*(\sigma) = 1$. Recall from K. Kraus [10] that the unital operation φ is a completely positive map on $B(H)$ of the form

$$\varphi(a) = \sum_i V_i^* a V_i \quad \text{for } a \in B(H) \text{ with } V_i \in B(H) \text{ such that } \sum_i V_i^* V_i = I_H,$$

and ϕ has a similar representation.

We also note that a normal state ω on the tensor product $B(H) \otimes B(K)$ is represented by a normal completely positive map ψ_ω from $B(K)$ to $T(H)$ with $\mathrm{Tr}\,\psi_\omega(I_K) = 1$, i.e.,

$$\omega(a \otimes b) = \mathrm{Tr}(a\,\psi_\omega({}^t b)) \quad \text{for} \quad a \in B(H) \quad \text{and} \quad b \in B(K),$$

where we can observe that $\psi_\omega(I_K) = \varphi^*(\rho)$ and $\psi_\omega^*(I_H) = {}^t\phi^*(\sigma)$. We then define the *correlation CP-map* ψ from $B(K)$ to $B(H)$ by

$$\psi(b) := \varphi^*(\rho)^{-\frac{1}{2}} \psi_\omega(b) \varphi^*(\rho)^{-\frac{1}{2}} \quad \text{for} \quad b \in B(K),$$

where $\varphi^*(\rho)^{-\frac{1}{2}}$ is defined on the support of $\varphi^*(\rho)$. Then ψ is a unital CP-map from $B(s(\phi^*(\sigma))K)$ to $B(s(\varphi^*(\rho))H)$, where $s(\phi^*(\sigma))$ [resp. $s(\varphi^*(\rho))$] denotes the support projection of $\phi^*(\sigma)$ [resp. $\varphi^*(\rho)$], so that it can be represented as

$$\psi(b) = \sum_j W_j^* \, b \, W_j \quad \text{where} \quad W_j \in B(H, K) \quad \text{with} \quad \sum_j W_j^* W_j = I_{s(\varphi^*(\rho))H}.$$

We thus consider the following diagram of CP-maps:

$$\begin{tikzcd}
\varphi^*(\rho) \arrow[rr, "{}^t\psi^*"] & & \phi^*(\sigma) \\
& \omega = U(\rho \otimes \sigma)U^* \arrow[ul, "\mathrm{Tr}_K"'] \arrow[ur, "\mathrm{Tr}_H"] & \\
\rho \in T(H) \arrow[uu, "\varphi^*"] \arrow[r] & \omega_0 = \rho \otimes \sigma \arrow[u] & \sigma \in T(K) \arrow[l] \arrow[uu, "\phi^*"']
\end{tikzcd}$$

It should be noted here that the CP-map φ depend on σ and U, ϕ depend on ρ and U, and ψ depends on ρ, σ and U. We also note that, when we focus on the operation φ, we can assume that σ is pure without loss of generality, i.e., the system K is closed before the interaction. In fact, we can consider the Hilbert space $\tilde{K} = K \otimes K$ and take a pure state $\tilde{\sigma}$ on $\tilde{K}$ (which we call a *purification* of σ), and a unitary $\tilde{U}$ on $H \otimes \tilde{K}$ such that $\tilde{U}|_{H \otimes K} = U$ and $\mathrm{Tr}_{\tilde{K}} \tilde{U}(\rho \otimes \tilde{\sigma})\tilde{U}^* = \varphi^*(\rho)$. (See [12], for example, for the detailed procedure of the purification.)

Our purpose in this note is to find out the relations between the statistical and informational quantities, such as entropy, mutual entropy, dissemination and equivocation, which we define for the density operators and operations (or channels) in the above diagram. Based on this structure theory, we shall provide a new description of quantum information theory, and define a new entropy of operations, vanishing at extreme CP-maps in CP-convexity, which provides a measure of the complexity of the quantum interaction.

Preliminaries.

1. Notations. In this note, to simplify our arguments, we restrict ourselves on $B(H)$ as the C*-algebra representing the quantum system, the C*-algebra of all bounded linear operators on a Hilbert space H (infinite dimensional in general). We denote by $T(H)$ the set of all trace class operators on H, and by $S(B(H))_n$, which we shall abbreviate by $S(H)$, the normal states of $B(H)$, i.e., the density operators on H. In particular, the pure states of $B(H)$ are one dimensional projections on H, and will be written as $P(H)$. For a density operator $\rho \in S(H)$, we denote by $S(\rho)$ the von Neumann entropy of ρ, i.e., $S(\rho) = -\mathrm{Tr}\rho \ln \rho$.

We use the notation $CP(B(K), B(H))_n$, for example, for the set of all normal CP-maps from $B(K)$ to $B(H)$, and use $S_H(B(K))_n$ for the unital elements. In particular, $S_H(B(H))_n$ denotes the unital operations on H, and $CP(B(K), T(H))_n$ denotes the set of all TCP-maps (trace class operator valued CP-maps) from $B(K)$ to $T(H)$. As we mentioned in Introduction, there exists one-to-one correspondence $\omega \in S(B(H) \otimes B(K))_n \leftrightarrow \psi_\omega \in CP(B(K), S(H))_n$. For $\varphi \in CP(B(K), B(H))$, we denote by $s(\varphi)$ the *support* of φ, i.e., $s(\varphi) = s(\varphi(I_K))$.

2. CP-convexity. The notion of CP-convexity was originally introduced by the author in [3] and studied in [4-6]. We denote by $Q_H(B(H))_n$ the normal contractive CP-maps on $B(H)$, and call the normal *CP-state space* of $B(H)$. Let $\{\varphi_i\} \subset Q_H(B(H))_n$ be a family of normal CP-states, and suppose that $\varphi \in Q_H(B(H))_n$ is expressed as

$$\varphi = \sum_i S_i^* \varphi_i S_i \quad \text{with} \quad S_i \in B(H) \quad \text{and} \quad \sum_i S_i^* S_i = I_{s(\varphi)H},$$

then we say that φ is a *CP-convex combination* of φ_i, and abbreviate it by $\varphi = \text{CP-}\sum_i S_i^* \varphi_i S_i$. (Note that an operation is a CP-convex combination of non-unital CP-states.)

A CP-state is defined to be *CP-extreme* if $\varphi = \text{CP-}\sum_i S_i^* \varphi_i S_i$ implies that each φ_i is unitarily equivalent to φ. It can be shown that $\varphi \in Q_H(B(H))_n$ is CP-extreme iff φ is a unitary transform, i.e., $\varphi = U^* \cdot U$ with a unitary U.

We also define $\varphi \in Q_H(B(H))_n$ to be *conditionally CP-extreme* if $\varphi = \text{CP-}\sum_i S_i^* \varphi_i S_i$ with $S_i \geq 0$ implies that $\varphi_i = \varphi$ for all i. A CP-state $\varphi \in Q_H(B(H))_n$ is conditionally CP-extreme iff φ is a conditional transform, i.e., $\varphi = u^* \cdot u$ with a partial isometry u. (cf. [7] for CP-extreme states for CP-state space of general C*-algebras).

3. Lindblad entropy. Let $\varphi \in S_H(B(H))_n$ be an operation represented by

$$\varphi(a) = \sum_i V_i^* a V_i \quad \text{for} \quad a \in B(H) \quad \text{and} \quad V_i \in B(H), \ \sum_i V_i^* V_i = I_H.$$

We then define the following entropies of φ:

(1) $S_\rho^1(\varphi) := S(M_\rho^L(\varphi))$ where $M_\rho^L(\varphi) := (\operatorname{Tr} V_i \rho V_j^*)$, which does not depend on the decomposition of φ, and we call the *Lindblad matrix* of φ w.r.t. ρ.

(2) $S_\rho^2(\varphi) := \inf\{-\sum_i \lambda_i \ln \lambda_i; \varphi = \sum_i \lambda_i \varphi_i, \ \ \varphi_i \text{ pure with } \|\varphi_i\|_\rho = 1\}$

$$\text{where } \lambda_i = \operatorname{Tr} V_i \rho V_i^*, \ \varphi_i = \lambda_i^{-1} V_i^* \cdot V_i, \ \ \|\varphi\|_\rho = \operatorname{Tr} \rho \varphi(I_H).$$

(3) $S_\rho^3(\varphi) := S(\rho_\varphi)$ with $\rho_\varphi = \sum_i (\ \cdot \ , V_i)_\rho V_i \in S(B(H)_\rho)_n$, where $B(H)_\rho$ is the GNS-representation space of $B(H)$ with ρ, and $(X, Y)_\rho = \operatorname{Tr} \rho Y^* X$.

(4) $S_\rho^4(\varphi) := S(\Omega_\rho^\varphi)$ with $\Omega_\rho^\varphi = \sum_i (\ \cdot \ V_i, V_i)_\rho \in S(B(B(H)_\rho))_n$.

(5) $S_\rho^5(\varphi) := S(\omega_\rho^\varphi)$ where $\omega_\rho^\varphi \in S(B(H) \otimes B(H))_n$

$$\text{with } \omega_\rho^\varphi(a \otimes b) = \operatorname{Tr} \rho^{\frac{1}{2}} \varphi(a) \rho^{\frac{1}{2}}\, {}^t b \quad \text{for } a, b \in B(H).$$

(6) $S_\rho^6(\varphi) := S(\tau_\rho^\varphi)$ with $\tau_\rho^\varphi = \sum_i (V_i \otimes I)\, \tilde{\rho}\, (V_i \otimes I)^*$ where $\tilde{\rho}$ is the purification

of ρ on $H \otimes H$.

The entropies $S^1_\rho(\varphi)$ and $S^6_\rho(\varphi)$ were introduced by G. Lindblad [11, 12] and shown that they are equivalent, and $S^3_\rho(\varphi)$ was studied by R. Alicki [1]. We can show that the above entropies are all equivalent.

Theorem 1. $S^1_\rho(\varphi) = S^2_\rho(\varphi) = \cdots = S^6_\rho(\varphi)$, *which we shall call the Lindblad entropy of* φ *with respect to* ρ, *and denote by* $S^L_\rho(\varphi)$.

In quantum communication theory, this entropy is also called the *entropy exchange* (e.g., [9]), however G. Lindblad [11] was the first to define and study this entropy.

4. Entanglement of formation. Let $\varphi_\rho \in CP(B(K), T(H))$ be a TCP-map which is expressed as

$$\begin{aligned}\varphi_\rho &= \sum_i v_i^* \cdot v_i \quad \text{with} \quad \sum_i v_i^* v_i = \rho \in S(H)\\ &= \sum_i \lambda_i \tilde{v}_i^* \cdot \tilde{v}_i \quad \text{with} \quad \lambda_i = \mathrm{Tr} v_i^* v_i,\ \tilde{v}_i = \lambda_i^{-\frac{1}{2}} v_i,\ \tilde{v}_i^* \tilde{v}_i \in S(H).\end{aligned}$$

Then

$$E(\varphi_\rho) := \inf\{\sum_i \lambda_i S(\tilde{v}_i^* \tilde{v}_i)\, ; \varphi_\rho = \sum_i \lambda_i \tilde{v}_i^* \cdot \tilde{v}_i)\}$$

is called the *entanglement of formation* of φ_ρ, which is the CP-map version of the original definition for compound states (cf. e.g., [2], [14]).

Definition. Let $\varphi \in S_H(B(K))_n$ be a normal unital CP-map from $B(K)$ to $B(H)$, and $\rho \in S(H)$ be a density operator on H, then we define $E_\rho(\varphi) := E(\varphi_\rho)$, with $\varphi_\rho = \rho^{\frac{1}{2}} \varphi \rho^{\frac{1}{2}}$, to be the *entanglement of formation of* φ *with respect to* ρ.

Now, we shall consider the structure of the interaction in 3 steps.

Case I : $\rho \in P(H), \sigma \in P(K)$.

Let $\rho = P_0 \in P(H)$ and $\sigma = Q_0 \in P(K)$. In this case, $\omega_0 = P_0 \otimes Q_0$ and $\omega = U(P_0 \otimes Q_0)U^* \in P(B(H) \otimes B(K))$, and $\omega(a \otimes b) = \mathrm{Tr}\, a\, w^*\, {}^t b w$ $(a \in B(H)$, $b \in B(K))$, so that $w^* w = \varphi^*(P_0)$ and $ww^* = {}^t\phi^*(Q_0)$, and note that

$$\varphi_{P_0} = P_0 \varphi(\cdot) P_0 = \sum_i P_0 V_i^* \cdot V_i P_0 = \varphi^*(P_0)(\cdot)P_0,$$
$$\psi_\omega = w^* \cdot w, \;\; w \in B(H,K), \;\; w^* w = \varphi^*(P_0),$$

so that we can deduce the following equalities.

Theorem 2. $S(\varphi^*(P_0)) = S(\phi^*(Q_0)) = S^L_{P_0}(\varphi) = E_{\varphi^*(P_0)}(\psi)$

This result suggests that there exists a symmetric structure in the interacting system in the simplest case when the initial states are both pure, and this principle will be extended to the general case where the initial states are mixed states (Theorem 4 and Theorem 5).

Case II : $\rho \in S(H), \sigma \in P(K)$.

As we mentioned in Introduction, the general case can be reduced to this case. Let

$$\rho = \sum_j \mu_j P_j \quad \text{with} \quad \mu_j > 0, \; \sum_j \mu_j = 1,$$

where we do not assume that this is the spectral decomposition, if not indicated so. Observe that the situation is the superposition of the cases of the pure states $\omega_j = U(P_j \otimes Q_0)U^*$ considered in Case I with weights μ_j, i.e.,

$$\omega = U(\rho \otimes Q_0)U^* = \sum_j \mu_j U(P_j \otimes Q_0)U^* = \sum_j \mu_j \omega_j,$$

so that we have

$$\varphi_\rho = \rho^{\frac{1}{2}}\varphi(\cdot)\rho^{\frac{1}{2}} = \sum_i v_i^* \cdot v_i \text{ with } v_i = V_i\rho^{\frac{1}{2}},\ \sum_i v_i^* v_i = \rho, \sum_i v_i v_i^* = \varphi^*(\rho),$$

$$\psi_\omega = \sum_j \mu_j w_j^* \cdot w_j,\ \ w_j \in B(H,K),\ \ w_j^* w_j = \varphi^*(P_j),\ \ \sum_j \mu_j w_j^* w_j = \varphi^*(\rho).$$

We now fix CONS's $\{e_j\}$, $\{f_i\}$ of H and K respectively. We can show that the operators w_j are determined by V_i up to unitary equivalence.

Lemma 3. *Let $V_i = (\mathbf{v}_{1i}\mathbf{v}_{2i}\cdots\mathbf{v}_{ji}\cdots)$ where $\mathbf{v}_{ji} = V_i e_j$ is the column vectors. Then, $w_j^* = (\mathbf{v}_{j1}\mathbf{v}_{j2}\cdots\ \mathbf{v}_{ji}\cdots)U_f$ where U_f is an invertible operator on K depending on the CONS $\{f_i\}$.*

Theorem 4. *$\phi^*(Q_0)$ is unitarily equivalent to ${}^tM_\rho^L(\varphi)$ where $M_\rho^L(\varphi)$ is the Lindblad matrix.*

This result has a profound meaning in measurement theory. Suppose now that H is an apparatus and K is an observed system. Then the CP-map $\kappa := \varphi \circ \psi$ represents the measurement of the observables $b \in B(K)$ by the apparatus in the state ρ. Then, by Theorem 4, we can deduce that

$$\mathrm{Tr}\,(\varphi\circ\psi)(b)\rho = \mathrm{Tr}\, b(\psi^*\circ\varphi^*)(\rho) = \mathrm{Tr}\, b\ {}^t\phi^*(Q_0) = \mathrm{Tr}\, b\, \tilde{M}_\rho^L(\varphi),$$

where $\tilde{M}_\rho^L(\varphi)$ is an operator on K unitarily equivalent to the Lindblad matrix. Thus a measurement is nothing but a functional by the trace with the Lindblad matrix of the operation φ in the same Hilbert space K.

Case III : $\rho \in S(H), \sigma \in S(K)$.

In this case, we can show that the both operations φ and ϕ have the same Lindblad entropy, which includes Theorem 4 as a special case.

Theorem 5. $S^L_\rho(\varphi) = S^L_\sigma(\phi)$

The proof of this theorem is reduced to Theorem 2 using the purifications of ρ and σ. This result may be compared to the Newton's third principle in mechanics, and presents a simple and beautiful symmetry in interactions.

We now assume that $\rho = \sum_j \mu_j P_j$ and $\sigma = \sum_k \gamma_k Q_k$ be the spectral decompositions. Let φ^k be defined by Q_k, and ϕ^j be defined by P_j, i.e.,

$$\begin{aligned}
\varphi^k &= \sum_i V_i^{k*} \cdot V_i^k \quad \text{with } V_i^k \in B(H), \ \sum_i V_i^{k*} V_i^k = I_H \\
\phi^j &= \sum_l S_l^{j*} \cdot S_l^j \quad \text{with } S_l^j \in B(K), \ \sum_l S_l^{j*} V_l^j = I_K \\
\psi_\omega^k &= \sum_j \mu_j w_j^{k*} \cdot w_j^k \quad \text{with } \sum_j \mu_j w_j^{k*} w_j^k = \varphi^{k*}(\rho) \\
\psi_\omega^{j*} &= \sum_k \gamma_k w_j^k \cdot w_j^{k*} \quad \text{with } \sum_k \gamma_k w_j^k w_j^{k*} = \phi^{j*}(\sigma).
\end{aligned}$$

In the case that all of the above decompositions are orthogonal decompositions, and assume that all operators are represented in CONS's from the spectral decompositions of density operators ρ in H_1, σ in K_1, $\varphi^*(\rho)$ in H_2, and $\phi^*(\sigma)$ in K_2 in Introduction, then we can show the following equalities between the matrix elements of the CP-coefficients.

Theorem 6. $[V_q^k]_{pj} = [S_p^j]_{qk} = [w_j^{k*}]_{pq} = [U]_{qj}^{kp}$

This result implies that, given ρ and σ, if we know U or the CP-coefficients of the families $\{\varphi^k\}$, $\{\phi^j\}$, $\{\psi_\omega^k\}$ or $\{\psi_\omega^{j*}\}$, then we can determine the others.

Quantum information theory.

We shall extend the notions of the classical information theory in our quantum setting associated with the density operators and channels in the diagram, where we can assume that $\sigma = Q_0$ (Case II) without loss of generality.

Definition. Let φ^* be a channel from $T(H_1)$ to $T(H_2)$ and $\rho \in S(H_1)$. We define the *dissemination* $D_\rho(\varphi^*)$ of the channel φ^* with respect to ρ by

$$D_\rho(\varphi^*) := \inf\{\sum_j \mu_j S(\varphi^*(P_j));\ \rho = \sum_j \mu_j P_j,\ P_j \in P(H), \mu_j > 0, \sum_j \mu_j = 1\}$$

The next result shows that the entanglement of the correlation CP-map between the interacting bipartite systems is nothing but the dissemination of the channel φ^*.

Theorem 7. $E_{\varphi^*(\rho)}(\psi) = D_\rho(\varphi^*)$.

In fact, the correspondence

$$\rho \leftrightarrow \omega_0 = \rho \otimes Q_0 \leftrightarrow \omega = U(\rho \otimes Q_0)U^* \leftrightarrow \psi_\omega$$

is affine isomorphic, so we have

$$\begin{aligned} E_{\varphi^*(\rho)}(\psi) &= \inf\{\sum_j \mu_j S(w_j^* w_j); \psi_\omega = \sum_j \mu_j w_j^* \cdot w_j\} \\ &= \inf\{\sum_j \mu_j S(\varphi^*(P_j)); \rho = \sum_j \mu_j P_j\} = D_\rho(\varphi^*) \end{aligned}$$

Definition. Let φ^* be a channel from $T(H_1)$ to $T(H_2)$ and $\rho \in S(H_1)$. Then we define the *mutual entropy* $I_\rho(\varphi^*)$ of φ^* with respect to ρ by

$$I_\rho(\varphi^*) := S(\varphi^*(\rho)) - D_\rho(\varphi^*)$$

The idea comes from the Holevo bound as the preliminary notion (cf. [9]), or Ohya's pseudo mutual entropy for finite decompositions (cf. [15]). Note that, from Theorem 7, in our diagram

$$I_\rho(\varphi^*) = S(\varphi^*(\rho)) - E_{\varphi^*(\rho)}(\psi).$$

Further, we shall consider the mutual entropy for the correlation CP-map ψ, i.e.,

$$I_{\varphi^*(\rho)}(\psi^*) = S(\phi^*(Q_0)) - D_{\varphi^*(\rho)}(\psi^*) = S^L_\rho(\varphi) - E_\rho(\kappa).$$

We note that the mutual entropy of the correlation CP-map depends on the direction of the map. In fact, the composite state ω can be described by the TCP-map ψ^*_ω, i.e., $\omega(a \otimes b) = \mathrm{Tr}\psi^*_\omega(a)\,{}^t b$ for $a \in B(H)$ and $b \in K(H)$, so by the *conjugate* correlation CP-map

$$\psi^c := {}^t\phi^*(Q_0)^{-\frac{1}{2}}\psi^*_\omega\,{}^t\phi^*(Q_0)^{-\frac{1}{2}}.$$

Then we have

$$I_{\phi^*(Q_0)}(\psi^{c*}) = S(\varphi^*(\rho)) - D_{\phi^*(Q_0)}(\psi^{c*}) = S(\varphi^*(\rho)) - E_\rho(\varphi),$$

which is different from $I_{\varphi^*(\rho)}(\psi^*)$ in general.

Definition. Let φ^* be a channel from $T(H_1)$ to $T(H_2)$ and $\rho \in S(H_1)$. We then define the *equivocation* $V_\rho(\varphi^*)$ of φ^* with respect to ρ by

$$V_\rho(\varphi^*) := S(\rho) - I_\rho(\varphi^*) = S(\rho) + D_\rho(\varphi^*) - S(\varphi^*(\rho)).$$

All of the above defined informational quantities are represented as functions of φ and ρ, and the functional properties of these quantities and their applications are discussed in [8] and [13].

New entropy for CP-maps.

Definition. Let φ be a CP-map on $B(H)$ with $\varphi(I_H) = s(\varphi)$ and $\rho \in S(s(\varphi)H)$. We then define an entropy $S_\rho(\varphi)$ of φ with respect to ρ by

$$S_\rho(\varphi) := S^L_\rho(\varphi) + I_{\phi^*(Q_0)}(\psi^{c*}) = S^L_\rho(\varphi) + S(\varphi^*(\rho)) - E_\rho(\varphi).$$

We note that $S_\rho(\varphi)$ is a concave function with respect to φ. Furthermore, we can show that it vanishes if and only if φ is CP-extreme, i.e.,

$$S_\rho(\varphi) = 0 \text{ for all } \rho \in S(s(\varphi)H) \quad \text{iff} \quad \varphi \text{ is a conditional transform.}$$

If φ is a unital CP-map, then

$$S_\rho(\varphi) = 0 \text{ for all } \rho \in S(H) \quad \text{iff} \quad \varphi \text{ is a unitary transform.}$$

For example, let $\varphi = \sum_i \lambda_i U_i^* \cdot U_i$ with unitary U_i and $\lambda_i > 0$, $\sum_i \lambda_i = 1$. Then $S_\rho(\varphi) \geq S_\rho^L(\varphi) + S(\varphi^*(\rho)) - S(\rho) \geq 0$ (cf. [8] for some other examples).

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