数理解析研究所講究録
第 1586 巻 2008 年 74-83

Discontinuity of straightening maps

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December 10, 2007

1 Introduction

The straightening theorem for polynomial-like mappings by Douady and Hubbard [DH] naturally induces a map between a family of renormalizable polynomials and the connectedness locus of polynomials, which we call the straightening map. It is known such a map is always continuous if it is of degree two and this implies the self-similarity of the boundary of the Mandelbrot set. However, Douady and Hubbard also showed that straightening maps for analytic families of polynomial-like mappings of degree greater than two are not continuous in general. In [In3], we gave a much more detailed description how a straightening map becomes discontinuous and gave an example of a discontinuous straightening map for a family of polynomial-like restrictions of polynomials (of greater degree).

Here we show that it is not continuous or at least not a homeomorphism in general and show how generally such a discontinuity occurs. To see this, we study parabolic and Misiurewicz bifurcation to find perturbations which satisfies the assumption of the theorem in [In3] and use Prado-Przytycki-Urbanski theorem [Pr] to show that if the straightening map is continuous (or homeomorphism), then we have local analytic conjugacy between a renormalizable polynomial and the straightening of the renormalization. However, we also show that such a local analytic conjugacy implies global correspondence [In4], which is impossible for renormalizable polynomials.

2 Polynomial-like mappings

A *polynomial-like mapping* is a proper holomorphic map $f : U' \to U$ such that U' and U are topological disks in $\mathbb{C}$ and $U' \Subset U$. We denote the *filled Julia set* by $K(f) = K(f; U', U) = \bigcap_{n\geq 0} f^{-n}(U)$ and the *Julia set* by $J(f) = J(f; U', U) = \partial K(f)$.

For a periodic point $x \in \mathbb{C}$ of period n for a polynomial-like map (or a polynomial) f, let us denote its multiplier by $\mathrm{mult}_f(x)$, i.e.,

$$\mathrm{mult}_f(x) = (f^n)'(x).$$

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Definition. Let $f : U' \to U$ and $g : V' \to V$ are polynomial-like mappings and $\varphi : U \to V$ be a hybrid conjugacy. We say that φ *preserves multipliers* if for any periodic point x for f, we have

$$|\operatorname{mult}_f(x)| = |\operatorname{mult}_g(\varphi(x))|.$$

The following theorem is proved by Prado [Pr] for "off-critically hyperbolic" polynomial-like mappings, and extended to tame rational maps by Przytycki and Urbanski [PU].

Theorem 2.1 (Prado-Przytycki-Urbanski). *Let $f : U' \to U$ and $g : V' \to V$ are tame polynomial-like mappings hybrid equivalent. If there exists a hybrid conjugacy between f and g preserving multipliers, then they are analytically conjugate on neighborhoods of the filled Julia sets.*

We do not give the precise definition of *tame* polynomial-like mappings here, because we need only the fact that polynomial-like mappings hybrid equivalent to $z + z^2$ are tame, which is guaranteed by the following. (See [Ur1], [Ur2] and [PU] for details on tame maps.)

Theorem 2.2. *Every polynomial-like mapping with no recurrent critical points in its Julia set (abbr. NCP) is tame.*

Furthermore, when a given polynomial-like mappings are restrictions of rational maps, we can say more (see [In4] for more details).

Theorem 2.3. *Let f_1 and f_2 be two rational maps. Assume they have polynomial-like restrictions $f_i : U_i' \to U_i$, $i = 1, 2$ analytically conjugate. Then there exist rational maps g, φ_1 and φ_2 such that $\varphi_i \circ g = f_i \circ \varphi_i$ and g has a polynomial-like restriction $g : V' \to V$ analytically conjugate to $f_i : U_i' \to U_i$. In particular, f_1 and f_2 have the same degree.*

Furthermore, if f_1 and f_2 are polynomials, then g, φ_1 and φ_2 are also polynomials.

Outline of proof. By shrinking U_i if necessary, we may assume that there exists an analytic conjugacy $\phi : U_1 \to U_2$ between f_1 and f_2. Let $\Gamma_0 = \{(z, \phi(z));\ z \in U_1\} \subset \hat{\mathbb{C}}^2$ be the graph of ϕ. Define $F : \hat{\mathbb{C}}^2 \to \hat{\mathbb{C}}^2$ by $F(z_1, z_2) = (f_1(z_1), f_2(z_2))$ and let $\Gamma_n = F^n(\Gamma_0)$ for $n \geq 1$. Then it is easy to see that $\Gamma_n \subset \Gamma_{n+1}$. Therefore, $\Gamma = \bigcup_{n \geq 0} \Gamma_n$ is a connected invariant set under F.

Furthermore, we can "desingularize" Γ; namely, there exists a Riemann surface X and holomorphic maps $g : X \to X$ and $\Phi : X \to \mathbb{C}^2$ such that $\Phi(X) = \Gamma$, $F \circ \Phi = \Phi \circ g$, and there exists an open set $\tilde{U}$ such that $\Phi|_U : \tilde{U} \to \Gamma_0$ is a conformal isomorphism and $g : \tilde{U}' \to \tilde{U}$ is a polynomial-like map analytically conjugate to $f_i : U_i' \to U_i$ by $\pi_i \circ \Phi$, where $\pi_i(z_1, z_2) = z_i$ be the natural projection and $\tilde{U}' = (\pi_1 \circ \Phi|_{\tilde{U}})^{-1}(U_1')$.

Since g has a chaotic dynamics (e.g., g has a repelling periodic point), X cannot be hyperbolic. Therefore X is isomorphic to either $\hat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or a torus (and we can show that the case X is a torus cannot happen). Since the degree of F is finite, the degree of g is also finite, thus g is a rational map. It is not difficult to show that $\phi_i = \pi_i \circ \Phi$ is also a rational map. (It is trivial when $X \cong \hat{\mathbb{C}}$. For other cases, see, e.g., [BE]. Note that ϕ_i can be transcendental when $f_i : U_i' \to U_i$ is a polynomial-like mappings of degree *one*, i.e., U_i' and U_i are neighborhoods of a repelling fixed point and ϕ_i is a linearizing coordinate for it.) □

Definition. We say two polynomials f_1 and f_2 are *polynomially semiconjugate up to finite* *cover* if the conclusion of Theorem 2.3 holds.

Combining the theorems above, we have the following.

Theorem 2.4. *Let f_1 and f_2 be polynomials. Assume they have polynomial-like restrictions $f_i : U_i' \to U_i$, $i = 1, 2$ which are hybrid conjugate via a conjugacy preserving multipliers. Then f_1 and f_2 are polynomially semiconjugate up to finite cover.*

3 Analytic families of polynomial-like mappings and straightening maps

In [In3], we have proved a theorem which relates continuity of straightening maps and multipliers of periodic points. Before stating the theorem, we introduce some notations and definitions. Let Poly_d be the set of affine conjugacy classes of polynomials of degree $d(\geq 2)$ and $C_d = \{f \in \mathrm{Poly}_d;\ K(f) \text{ is connected}\}$. By the straightening theorem [DH], for an analytic family of polynomial-like mappings $\mathbf{f} = (f_\lambda : U_\lambda' \to U_\lambda)_{\lambda\in\Lambda}$ (in the sense of [DH]), we can define the *straightening map* $\mathcal{S}_\mathbf{f} : C_\mathbf{f} \to C_d$ as follows. Let $C_\mathbf{f} = \{\lambda \in \Lambda;\ K(f_\lambda) \text{ is connected}\}$ and for $\lambda \in C_\mathbf{f}$, there exists a unique $g_\lambda \in C_d$ hybrid equivalent to f_λ. Let $\mathcal{S}_\mathbf{f}(\lambda) = g_\lambda$.

We say $\mathbf{f} = (f_\lambda : U_\lambda' \to U_\lambda, x_\lambda)_{\lambda\in\Lambda}$ is an *analytic family of polynomial-like mappings with a marked point* if $(f_\lambda : U_\lambda' \to U_\lambda)_{\lambda\in\Lambda}$ is an analytic family of polynomial-like mappings, and $x : \Lambda \to \mathbb{C}$ is a holomorphic map such that $x_\lambda \in U_\lambda$. Also for such a family $\mathbf{f}$ of degree $d \geq 2$, let

$$\mathcal{CK}_\mathbf{f} = \{\lambda \in \Lambda;\ K(f_\lambda) \text{ is connected and } x_\lambda \in K(f_\lambda)\}$$

and we can similarly define the *straightening map*

$$\mathcal{S}_\mathbf{f} : \mathcal{CK}_\mathbf{f} \to \mathcal{CK}_d$$

as follows. Let $\mathcal{CK}_d = \{(f, z);\ f\text{: polynomial of degree } d,\ z \in K(f)\}/\!\sim$, where $(f, z) \sim (g, w)$ if there exists an affine map A such that $A \circ g = f \circ A$ and $A(w) = z$. For each $\lambda \in \mathcal{CK}_\mathbf{f}$, f_λ is hybrid equivalent to some polynomial g_λ of degree d with a hybrid conjugacy ψ_λ. Let us define $\mathcal{S}_\mathbf{f}(\lambda) = (g_\lambda, \psi_\lambda(x_\lambda)) \in \mathcal{CK}_d$, then it is well-defined.

Now let us state the theorem:

Theorem 3.1. *Let $\mathbf{f} = (f_\lambda : U_\lambda' \to U_\lambda, x_\lambda)_{\lambda\in\Lambda}$ be an analytic family of polynomial-like mappings of degree $d \geq 2$ with a marked point. Assume*

(i) *for any $\lambda \in \Lambda$, 0 lies in U_λ' and it is a fixed point for f_λ;*
(ii) *α_λ is a marked repelling periodic point and ω_λ is a marked critical point for f_λ;*
(iii) *for $\lambda = \lambda_0$, 0 is a non-degenerate 1-parabolic fixed point, ω_{λ_0} lies in the basin of 0 for f_{λ_0} and $x_{\lambda_0} = \omega_{\lambda_0}$;*
(iv) *there exist sequences $\lambda_n \to \lambda_0$ and $\lambda_{n,m} \to \lambda_n$ such that*
- *$\lambda_n, \lambda_{n,m} \in \mathcal{CK}_\mathbf{f}$;*
- *$\omega_{\lambda_n} \neq x_{\lambda_n}$ for $n \geq 1$;*
- *0 is a non-degenerate 1-parabolic fixed point for f_{λ_n};*

- $f_{\lambda_{n,m}}$ *geometrically converges to* (f_{λ_n}, g_n) *as* $m \to \infty$ *for some Lavaurs map* g_n *such that* $g(\omega_{\lambda_n}) = \alpha_{\lambda_n}$ *and* $g'(\omega_{\lambda_n}) \neq 0$ *(in particular,* 0 *is no more a parabolic fixed point for* $f_{\lambda_{n,m}}$*).*

(v) $\mathcal{S}_{\mathfrak{f}}(\lambda_{n,m}) \to \mathcal{S}_{\mathfrak{f}}(\lambda_n)$ *as* $m \to \infty$ *and* $\mathcal{S}_{\mathfrak{f}}(\lambda_n) \to \mathcal{S}_{\mathfrak{f}}(\lambda_0)$ *as* $n \to \infty$.

Then

$$|\operatorname{mult}_{f_{\lambda_0}}(\alpha_{\lambda_0})| = |\operatorname{mult}_{g_{\lambda_0}}(\psi_{\lambda_0}(\alpha_{\lambda_0}))|,$$

where $\mathcal{S}(f_\lambda) = (g_\lambda, z_\lambda)$ *and* ψ_λ *is a hybrid conjugacy between* f_λ *and* g_λ.

4 Rational laminations

We briefly review the notion of rational laminations introduced by Thurston [Th] and deeply studied by Kiwi [Ki1]. Let $d \geq 2$ and denote by $m_d : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ the d-fold covering map $t \mapsto dt$. A *(d-invariant) rational lamination* is an equivalence relation λ on $\mathbf{Q}/\mathbf{Z}$ such that

(i) λ is closed (in $\mathbf{Q}/\mathbf{Z} \times \mathbf{Q}/\mathbf{Z}$).
(ii) Each λ-class is finite.
(iii) λ-classes are pairwise unlinked.
(iv) For a λ-class A, $m_d(A)$ is again a λ-class.
(v) $m_d|_A : A \to m_d(A)$ is consecutive-preserving, i.e., if (s,t) is a component of $\mathbf{R}/\mathbf{Z} \setminus A$, then (ds, dt) is a component of $\mathbf{R}/\mathbf{Z} \setminus m_d(A)$.

Here we say two sets $A, B \subset \mathbf{R}/\mathbf{Z}$ are *unlinked* if A is contained in a component of $\mathbf{R}/\mathbf{Z} \setminus B$.

For a polynomial $f \in \mathcal{C}_d$, let $\lambda(f)$ be the *rational lamination of* f, i.e., $s \sim t$ if the external rays $R_f(s)$ and $R_f(t)$ land at the same point. It is easy to see that $\lambda(f)$ is the rational lamination in the above sense. Conversely, Kiwi [Ki1] proved that for any rational lamination λ, there exists a polynomial $f \in \mathcal{C}_d$ such that $\lambda = \lambda(f)$. For a rational lamination λ and a polynomial $f \in \mathcal{C}_d$, we say f *admits* λ if $\lambda \subset \lambda(f)$.

We say two irrational angles s and t are *λ-unlinked* if any λ-class A and $\{s,t\}$ are unlinked, i.e., A is contained in a component of $\mathbf{R}/\mathbf{Z} \setminus \{s,t\}$. We state the basic properties of unlinked relations here (see [Ki1]. See also [In1] and [In2]).

Lemma 4.1. *The λ-unlinked relation is an equivalence relation. For each λ-unlinked class L, we have the following:*

- *For each λ-unlinked class L, $m_d(L)$ is again a λ-unlinked class.*
- *A λ-unlinked class L is finite if and only if it is wandering, i.e.,* $m_d^n(L) \neq m_d^m(L)$ *if* $n \neq m$.
- *If λ is infinite, then $\overline{L}$ is a Cantor set and each component of* $\mathbf{R}/\mathbf{Z} \setminus L$ *has the form* (s,t) *with* $s,t \in \mathbf{Q}/\mathbf{Z}$ *and* $s \sim_\lambda t$. *Furthermore,* $\overline{L}/\lambda$ *is homeomorphic to* $\mathbf{R}/\mathbf{Z}$ *and* $m_d : \overline{L}/\lambda \to m_d(\overline{L})/\lambda$ *is conjugate to* $m_\delta : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ *for some* $\delta = \delta(L) \geq 1$.

We call a λ-unlinked class L is *critical* if $\delta(L) > 1$.

We say a polynomial f is *subparabolic* if each critical point is either eventually periodic or attracted to an attracting or parabolic periodic point.

Lemma 4.2. *Let f be a subparabolic polynomial. Then each infinite $\lambda(f)$-unlinked class L corresponds to some bounded Fatou component Ω, i.e.,*

$$\overline{L} = \{\theta \in \mathbf{R}/\mathbf{Z};\ R_f(\theta) \in \partial\Omega\}.$$

Furthermore, $m_d(L)$ corresponds to $f(\Omega)$ and $\delta(L)$ is equal to the degree of $f : \Omega \to f(\Omega)$.

The lemma follows easily from the fact that the Julia set of f is locally connected in this case, so it is homeomorphic to $(\mathbf{R}/\mathbf{Z})/\hat{\lambda}(f)$, where $\hat{\lambda}(f)$ is the *real lamination* of f. Note that when $J(f)$ is locally connected, then $\hat{\lambda}(f)$ is simply the landing relation on $\mathbf{R}/\mathbf{Z}$ (see [Ki2] for general case).

Definition. We say a rational lamination λ is *combinatorially renormalizable* if there exist a non-trivial rational sublamination λ', a critical infinite λ'-unlinked class L and $n \geq 1$ such that $m_d^n(L) = L$. We call (λ', L) a *combinatorial renormalization* of λ and the smallest such $n \geq 1$ is called the *period* of the renormalization.

We say a combinatorial renormalization (λ', L) is *of capture type* if there exists a critical λ'-unlinked class L' such that

(i) L' is not in the forward orbit of L, i.e., $L' \neq m_d^k(L)$ for every $k \geq 0$.
(ii) L' is *captured* by L: There exists some $N > 0$ such that $m_d^N(L') = L$.
(iii) $\delta(m_d^k(L')) = 1$ for all $0 < k < N$.

We call such a triple (λ', L, L') a *capture renormalization* of f.

Let $f \in C_d$ and λ be a rational sublamination λ of $\lambda(f)$. For an infinite λ-unlinked class L, define a compact set $K(f, L) \subset K(f)$ as follows. By Lemma 4.1, we can write $\mathbf{R}/\mathbf{Z} \setminus L = \bigcup_{k>0}(s_k, t_k)$ for some $s_k, t_k \in \mathbf{Q}/\mathbf{Z}$ with $s_k \sim_{\lambda(f)} t_k$. In particular, the external rays $R_f(s_k), R_f(t_k)$ lands at the same point for each $k > 0$. Let U_k be the component of $\mathbf{C} \setminus \overline{(R_f(s_k) \cup R_f(t_k))}$ containing $R_f(t)$ for $t \in (s_k, t_k)$. Let

$$K(f, L) = K(f) \setminus \left(\bigcup_{k>0} U_k\right).$$

Then we have $K(f, m_d(L)) = f(K, m_d(L))$ and $f : K(f, L) \to K(f, m_d(L))$ is a proper map of degree $\delta(L)$. See [In2] for more details.

By Milnor's thickened puzzle piece argument [Mi, Lemma 1.5], we can relate a combinatorial renormalization of $\lambda(f)$ with an renormalization of f under some assumption.

Lemma 4.3. *Let $f \in C_d$. Assume there exists a combinatorial renormalization (λ, L) of period n such that for each component (s, t) of $\mathbf{R}/\mathbf{Z} \setminus L$, the common landing point x of $R_f(s)$ and $R_f(t)$ is not (pre-)parabolic or if it is (pre-)parabolic, all the basin whose closure contains x is contained in $K_f(L)$. Then there exists a polynomial-like restriction $f^n : U' \to U$ of f^n such that $K(f^n; U', U) = K(f, L)$.*

5 Perturbation

5.1 Definition and statements

We want to use Theorem 2.4 and Theorem 3.1 together to show that a straightening map is discontinuous or not a homeomorphism. To do that, we need to find nice perturbations (satisfying the assumption of Theorem 3.1) for each periodic point of the polynomial-like restriction.

Definition. We say a polynomial f of degree $d \geq 3$ satisfies (C1) if the following hold;

(i) 0 is a non-degenerate 1-parabolic periodic point of f;
(ii) there exists a quadratic-like restriction $f^N : V' \to V$ of f^N hybrid equivalent to $z + z^2$;
(iii) ω, ω' are critical points for f such that $\omega \in V'$, and $f^N(\omega') = \omega$ for some $N > 0$.

Furthermore, we say f satisfies (C2) if f satisfies (i)–(iii) and

(iv) every other critical point is eventually periodic and its forward orbit does not intersect $K(f^N; V', V)$. (In particular, f is subparabolic.)

Definition. We say f satisfies the condition (C3) if f satisfies the condition (C1) and for any periodic point α of $f^N : V' \to V$, there exist convergent sequences

$$f_{n,m} \xrightarrow{m\to\infty} f_n \xrightarrow{n\to\infty} f$$

in C_d such that the following hold; in the following, we denote the continuations of $x = \omega, \omega', \alpha$ for f_n and $f_{n,m}$ by x_n and $x_{n,m}$.

(i) 0 is a periodic point of period N for f_n and $f_{n,m}$. It is a non-degenerate 1-parabolic periodic point for f_n;
(ii) $f_n^N(\omega'_n) \neq \omega_n$ (hence we may assume $f_{n,m}^N(\omega'_{n,m}) \neq \omega_{n,m}$);
(iii) the other critical orbit relations of f are preserved for $f_{n,m}$ (hence also for f_n). More precisely, for any critical point $c \in \mathrm{Crit}(f)$, there exists the continuation $c_{n,m} \in \mathrm{Crit}(f_{n,m})$ for any n, m (i.e., critical points do not bifurcate for this perturbations) such that if $c, c' \in \mathrm{Crit}(f) \setminus \{\omega, \omega'\}$ satisfy $f^k(c) = f^{k'}(c')$, then $f_{n,m}^k(c_{n,m}) = f_{n,m}^{k'}(c'_{n,m})$.
(iv) $f_n^N : V'_n \to V_n$ and $f_{n,m}^N : V'_{n,m} \to V_{n,m}$ are quadratic-like restrictions near 0; In particular, $f_n^N : V'_n \to V_n$ are hybrid equivalent to $z + z^2$;
(v) $f_{n,m}$ geometrically converges to (f_n, g_n) as $m \to \infty$ such that $g_n(\omega_n) = \alpha_n$ and $g'_n(\omega_n) \neq 0$.

The main difficulty to find such perturbations $f_{n,m}$ is that $f_{n,m}$ must be in the connectedness locus. To do this, we start the following polynomial and construct $f_{n,m}$ combinatorially and show the convergence. See § 5.2.

Theorem 5.1. *If any polynomial f of degree $d \geq 3$ satisfy* (C2) *then f satisfies* (C3) *such that $\lambda(f_n) = \lambda(f)$ and $f_{n,m}$ admits $\lambda(f)$ for any n and m.*

The proof is given in § 5.2.

We call a polynomial f *Misiurewicz* if all critical points are preperiodic.

Theorem 5.2. *If a polynomial f_0 of degree $d \geq 3$ is Misiurewicz, then there exists a polynomial f arbitrarily close to f_0 satisfying* (C2).

We prove this theorem in § 5.3.

Therefore, we have the following corollary.

Corollary 5.3. *If a polynomial f_0 of degree $d \geq 3$ is Misiurewicz, then there exists a polynomial f arbitrarily close to f_0 satisfying* (C3).

5.2 Combinatorial construction

Here we give a proof of Theorem 5.1. We first consider perturbations of a quadratic polynomial $Q(z) = z^2 + 1/4$ (which is affinely conjugate to $z + z^2$), and construct $f_{n,m}$ by "tuning" the given polynomial f and them.

Take a repelling periodic point $\alpha(Q)$ of Q and let $\theta \neq 0$ be the landing angle for $\alpha(Q)$. Let c_m be the landing point of the external ray of angle $\theta/2^m$. Then $c_m \to 1/4$ and the critical point 0 is preperiodic for $Q_m(z) = z^2 + c_m$ for sufficiently large m. The point $\alpha(Q_m) = Q_m^m(c_m) = Q_m^{m+1}(0)$ is a landing point for the external ray $R_{Q_m}(\theta)$, hence it is a repelling periodic point for Q_m and $\alpha(Q_m) \to \alpha(Q)$ as $m \to \infty$. Furthermore, Q_m^{m+1} converges to some g on the interior of $K(Q)$ (the parabolic basin) such that $g(0) = \alpha(Q)$, i.e., $Q_m \xrightarrow{\text{geom}} (Q, g)$ and it is easy to see that $g'(0) \neq 0$. Let $\lambda_m = \lambda_{Q_m}$ be the rational lamination for Q_m. For later use, we also fix a sequence $\theta_n \in \mathbb{Q}/\mathbb{Z}$ of periodic angles by the doubling map $m_2 : t \mapsto 2t$ such that $\theta_n \to \theta$. Let $\alpha_n(Q_m)$ be the landing point of the external ray $R_{Q_m}(\theta_n)$. Since Q_m is subhyperbolic, $J(Q_m)$ is locally connected and hence $\alpha_n(Q_m) \to \alpha(Q_m)$ as $n \to \infty$.

We construct a rational lamination $\lambda_{n,m}$ by "combinatorial tuning" [In2], which has a certain desired properties for the rational lamination of $f_{n,m}$. Namely, we construct a lamination $\lambda_{n,m} \supset \lambda(f)$ such that the following hold;

- let L be the λ-unlinked class such that $K(f, L)$ is the closure of the immediate parabolic basin of 0, and let $\phi : \overline{L}/\lambda \to S^1$ be a homeomorphism such that $\phi(m_d^n(t)) = m_2(\phi(t))$. Then $\phi_*(\lambda_{n,m}|_{\overline{L}}) = \lambda(Q_m)$.
- let L' be the λ-unlinked class such that the interior of $K(f, L)$ contains ω', and let $\phi' : \overline{L'}/\lambda \to S^1$ be a homeomorphism such that $\phi(m_d^N(t)) = m_2(\phi'(t))$. (Note that $m_d^N(L') = L$ and $m_d^N : L' \to L$ is 2-to-1.) Then the equivalence relation $\phi'_*(\lambda_{n,m}|_{\overline{L'}})$ is a "lift" of $\lambda(Q_m)$ by m_2 such that there exists a unique critical equivalence class A (i.e., $m_d|_A$ is not one-to-one) with $\#A = 2$ and $m_2(A) = \{\theta_n\}$.

Then by the theorem of Kiwi [Ki1], there exists a postcritically finite polynomial $f_{n,m} \in C_d$ such that $\lambda(f_{n,m}) = \lambda_{n,m}$. Then it has a polynomial-like restriction $f_{n,m}^n : V'_{n,m} \to V_{n,m}$ hybrid equivalent to Q_m by Lemma 4.3. Let $\omega_{n,m}$ and $\omega'_{n,m}$ be the critical points which correspond to the critical $\lambda_{n,m}$-class intersecting $\overline{L}$ and $\overline{L'}$ respectively. Then $\alpha(f_{n,m}) = f_{n,m}^n(\omega_{n,m})$ and $\beta_{n,m} = f_{n,m}^{N+n}(\omega_{n,m})$ are repelling periodic point corresponding to $\alpha(Q_m)$ and $\alpha_n(Q_m)$ by a hybrid conjugacy.

By taking a subsequence, we may assume that $f_{n,m} \xrightarrow{m\to\infty} f_n$ and $f_n \xrightarrow{n\to\infty} \tilde{f}$ for some f_n and $\tilde{f}$. Let us denote $\omega_n = \lim_{m\to\infty} \omega_{n,m}$, $\tilde{\omega} = \lim_{m\to\infty} \omega_n$, $\alpha(f_n) = \lim_{m\to\infty} \alpha(f_{n,m})$ and so on. Then $f_{n,m} \underset{m\to\infty}{\xrightarrow{\text{geom}}} (f_n, g_n)$ with $g_n(\omega_n) = \alpha(f_n)$ and $g_n(f_n^N(\omega_n')) = \beta_n$ are repelling periodic points with $\beta_n - \alpha(f_n) \to 0$ as $n \to \infty$. This implies that $\tilde{f}^N(\tilde{\omega}') - \tilde{\omega} = \lim_{n\to\infty} f_n^N(\omega_n') - \omega_n = 0$. Thus, f and $\tilde{f}$ are subparabolic maps with the same critical relation and combinatorics. Furthermore, since all critical points of f other than ω and ω' are eventually periodic and $f^N(\omega') = \omega$, f is combinatorially rigid, so $\tilde{f}$ must equal to f. This proves the Theorem.

5.3 Misiurewicz bifurcations

The proof of Theorem 5.2 is based on a classical normality argument. We first perturb it to make a critical relation between critical points ω and ω' and make a quadratic-like restriction by use of the result of McMullen [Mc]. Before the proof, we introduce the notion of *passive* and *active* critical points: For a family $(f_\lambda, c_\lambda)_{\lambda\in\Lambda}$ of polynomials with marked critical points, we say c_λ is *passive at* $\lambda_0 \in \Lambda$ if $\{\lambda \mapsto f_\lambda^n(c_\lambda);\ n \geq 0\}$ forms a normal family, and c_λ is *active* otherwise.

In the following, we assume that all of the critical points of f_0 are simple just for simplicity. Choose critical points ω and ω' of f_0 so that no other critical point lies in the backward orbits of them. Let us consider a family

$$(f_\lambda, \omega_\lambda, \omega'_\lambda, c_{1,\lambda}, \dots, c_{d-3,\lambda})$$

of polynomials with all critical points marked. We consider a subfamily where all the critical orbit relations of f_0 except on ω and ω' are preserved. Then since f_0 is Misiurewicz, f_0 is in the bifurcation locus of the family and $c_{j,\lambda}$ is passive at 0. This implies that $\{\lambda \mapsto f_\lambda^n(\omega'_\lambda)\}_{n>0}$ is not a normal family because the family is two-dimensional. Therefore, there exists some $N > 0$ and λ_1 arbitrarily close to 0 such that

$$f_{\lambda_1}^N(\omega'_{\lambda_1}) = \omega_{\lambda_1}. \tag{1}$$

Now consider a further subfamily with the relation (1). Then it is one-parameter family and $\{\lambda \mapsto f_\lambda^n(\omega_\lambda)\}_{n>0}$ is not a normal family. Therefore, by [Mc], we can find λ_2 arbitrarily close to λ_1 such that ω_{λ_2} lies in the immediate basin of a non-degenerate 1-parabolic periodic point. Therefore, we have proved Theorem 5.2.

6 Discontinuity

In this section, we apply all the results above to a family of renormalizable polynomials to show straightening maps are discontinuous, or at least not homeomorphic. Note that all the results in this section are local properties, so Λ should be understood as a small neighborhood of a given parameter (which we denote by 0 in the following) in the original parameter space.

Theorem 6.1. *Let* $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ *be an analytic family of polynomial-like mappings of degree* $d \geq 3$. *Assume*

- *f_0 is a restriction of some polynomial of degree $d' \geq d$;*
- *$g_0 = \mathcal{S}_{\mathbf{f}}(f_0)$ satisfies* (C3);
- *the straightening map $\mathcal{S}_{\mathbf{f}} : \mathcal{C}_{\mathbf{f}} \to \mathcal{S}_f(\mathcal{C}_{\mathbf{f}}) \subset \mathcal{C}_d$ is a homeomorphism into its image, which is a neighborhood of g_0 in $\mathcal{C}_d$.*

Then f_0 and g_0 are affinely conjugate.

It is an easy consequence of Theorem 2.4 and Theorem 3.1.

Furthermore, by Corollary 5.3, we have the following.

Corollary 6.2. *Let $d < d'$ and let $\{f_\lambda\}_{\lambda\in\Lambda} \subset \mathrm{Poly}_d$ be a neighborhood of $f_0 \in \mathcal{C}_d$. Assume*

- *for each $\lambda \in \Lambda$, f_λ has a polynomial-like restriction $f_\lambda : U'_\lambda \to U_\lambda$ of degree d;*
- *f_0 is Misiurewicz.*

Then the straightening map $\mathcal{S}_{\mathbf{f}} : \mathcal{C}_{\mathbf{f}} \to \mathcal{S}_f(\mathcal{C}_{\mathbf{f}}) \subset \mathcal{C}_d$ is not a homeomorphism into its image, or the image does not form a neighborhood of $g_0 = \mathcal{S}(f_0)$.

Let us state another version of discontinuity of straightening maps:

Theorem 6.3. *Let $d < d'$ and let $\{f_\lambda\}_{\lambda\in\Lambda} \subset \mathrm{Poly}_d$ be a neighborhood of $f_0 \in \mathcal{C}_d$. Assume*

- *for each $\lambda \in \Lambda$, f_λ has a polynomial-like restriction $f_\lambda : U'_\lambda \to U_\lambda$ of degree d. Let K_λ be the filled Julia set of the polynomial-like restriction;*
- *f_0 satisfy the condition* (C2) *such that the filled Julia set the quadratic-like restriction hybrid equivalent to $z + z^2$ is contained in K_0; Let ω'_λ be the analytic continuation of ω and ω' in the definition of* (C3);
- *$\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda, f_\lambda^N(\omega'_\lambda))$ is an analytic family of polynomial-like mappings with a marked critical point.*

Then the straightening map $\mathcal{S}_{\mathbf{f}} : \mathcal{C}_{\mathbf{f}} \to \mathcal{S}(\mathcal{C}_{\mathbf{f}}) \subset \mathcal{CK}_d$ is not continuous.

This is also an easy consequence of Theorem 2.4, Theorem 3.1 and Theorem 5.1.

We can similarly prove the following:

Example. Let us consider capture renormalizations of cubic polynomials. The parameter set of renormalizable polynomials for a given combinatorics of capture type can be considered as the connectedness locus of an analytic family of polynomial-like mappings with marked point, which is given by the forward orbit of the captured critical point. In [In2], we proved that for a given combinatorics of a primitive renormalization of capture type for cubic polynomials, the straightening map is bijective. Therefore, the straightening map is not continuous in any neighborhood of any Misiurewicz map.

Note that in the case of disjoint renormalizations of cubic polynomials, straightening maps are always continuous because they only consists of straightening maps for quadratic-like families.

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