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ON LOCALIZED AUTOMORPHISMS OF THE CUNTZ ALGEBRAS WHICH PRESERVE THE DIAGONAL SUBALGEBRA

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ABSTRACT. In 1978, Cuntz raised the problem of classifying automorphisms of $\mathcal{O}_n$ which leave both the diagonal and the core UHF subalgebra invariant. In this note, we start developing a machinery that might be useful towards this goal. In particular, we give a practical criterion of invertibility of endomorphisms of $\mathcal{O}_n$ corresponding to unitaries in the normalizer of the diagonal inside the UHF subalgebra. We also analyze the action of such localized automorphisms on the spectrum of the diagonal thus obtaining criteria of outerness.

If n is an integer greater than 1, then the Cuntz algebra $\mathcal{O}_n$ is a unital, simple C^*-algebra generated by n isometries $S_1,\ldots,S_n$ satisfying $\sum_{i=1}^n S_iS_i^*=1$ [5]. As in [5], we denote by W_n^k the set of k-tuples $\alpha=(\alpha^1,\ldots,\alpha^k)$ with $\alpha^m\in\{1,\ldots,n\}$, and we denote by W_n the union $\cup_{k=0}^\infty W_n^k$, where $W_n^0=\{0\}$. Elements of W_n are called multi-indices and if $\alpha\in W_n^k$ then $l(\alpha)=k$, the length of α. If $\alpha=(\alpha^1,\ldots,\alpha^k)\in W_n^k$, then $S_\alpha=S_{\alpha^1}\cdots S_{\alpha^k}$, with $S_0=1$ by convention. Each S_α is an isometry and its range projection is $S_\alpha S_\alpha^*$. Every word in $\{S_i,S_i^*:i=1,\ldots,n\}$ can be uniquely expressed as $S_\alpha S_\beta^*$ for some $\alpha,\beta\in W_n$ [5, Lemma 1.3].

The C^*-subalgebra of $\mathcal{O}_n$ generated by $\{S_\alpha S_\beta^*:l(\alpha)=l(\beta)\}$ is isomorphic to $M_{n^k}(\mathbb{C})$ and denoted $\mathcal{F}_n^k$. The norm closure of the union $\cup_{k=0}^\infty\mathcal{F}_n^k$ is a UHF-algebra of type n^∞ and is denoted $\mathcal{F}_n$. It is called the core UHF-subalgebra of $\mathcal{O}_n$. There exists a faithful conditional expectation from $\mathcal{O}_n$ onto $\mathcal{F}_n$ [5]. The C^*-subalgebra of $\mathcal{O}_n$ generated by all projections $S_\alpha S_\alpha^*$, $\alpha\in W_n$, is denoted $\mathcal{D}_n$ and called the diagonal subalgebra of $\mathcal{O}_n$. It is a maximal abelian subalgebra of $\mathcal{O}_n$, regular both in $\mathcal{F}_n$ and in $\mathcal{O}_n$ [8]. The spectrum of $\mathcal{D}_n$ is naturally identified with X_n, the collection of infinite words on the alphabet $\{1,\ldots,n\}$ [8]. With the product topology, X_n is homeomorphic to the Cantor set. There exists a faithful conditional expectation from $\mathcal{F}_n$ onto $\mathcal{D}_n$ and whence from $\mathcal{O}_n$ onto $\mathcal{D}_n$ as well. We denote $\mathcal{D}_n^k=\mathcal{D}_n\cap\mathcal{F}_n^k$.

Let $\mathrm{End}(\mathcal{O}_n)$ be the semigroup (with composition) of endomorphisms of $\mathcal{O}_n$, that is unital $*$-homomorphisms of $\mathcal{O}_n$ into itself. Since $\mathcal{O}_n$ is simple, each endomorphism is injective and it is invertible (automorphism) if and only if it is surjective. Let $\mathcal{U}(\mathcal{O}_n)$ be the group of all unitaries in $\mathcal{O}_n$. As shown in [6], there is a bijective map $\lambda:\mathcal{U}(\mathcal{O}_n)\to\mathrm{End}(\mathcal{O}_n)$ determined by

$$\lambda_u(S_i)=u^*S_i,\quad i=1,\ldots,n. \tag{1}$$

The inverse of λ is the map $\psi\mapsto\sum_{i=1}^n S_i\psi(S_i^*)$. The map λ becomes a semigroup isomorphism once $\mathcal{U}(\mathcal{O}_n)$ is equipped with the convolution multiplication

$$u*w=u\lambda_u(w). \tag{2}$$

Endomorphisms of Cuntz algebras have been studied extensively by many authors and in variety of contexts. In particular, they appear in connection with Jones index theory for subfactors. We would only like to mention papers [7, 9, 12, 2, 10, 11] which are closest in spirit to the present note. In these and other works, a prominent role is played by

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localized endomorphisms, that is those of the form λ_u with u in $\cup_{k=0}^{\infty}\mathcal{F}_n^k$. Analysis of such endomorphisms and related structures often reduces to clever algebraic manipulations.

In the present paper (and the follow-up [3], in preparation), our focus is on localized automorphisms which preserve the diagonal subalgebra of $\mathcal{O}_n$. Interest in such automorphisms goes back to [1], where outerness of the flip-flop of $\mathcal{O}_2$ is shown. But the real motivation behind our work is the ground breaking paper [6] of Cuntz. Among more recent contributions to this subject of particular note is paper [13] of Matsumoto.

As observed in [6], endomorphism λ_u is invertible if and only if u^* belongs to its range. Indeed, if $u^* = \lambda_u(w)$ then λ_w is the inverse of λ_u. Unfortunately, this condition is difficult to check in general. Our Theorem 7, below, gives a convenient criterion of invertibility of a special class of localized endomorphisms. Let $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) = \{\psi \in \mathrm{Aut}(\mathcal{O}_n) : \psi(\mathcal{D}_n) = \mathcal{D}_n\}$, and let $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \{\psi \in \mathrm{Aut}(\mathcal{O}_n) : \psi|_{\mathcal{D}_n} = \mathrm{id}\}$. Cuntz showed in [6] that $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) = \{\lambda_w \in \mathrm{Aut}(\mathcal{O}_n) : w \in \mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)\}$ (with $\mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)$ denoting the normalizer of $\mathcal{D}_n$ in $\mathcal{O}_n$), and $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \{\lambda_t \in \mathrm{End}(\mathcal{O}_n) : t \in \mathcal{U}(\mathcal{D}_n)\}$. More recently, Power determined in [15][1] the structure of $\mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)$. Namely, every $w \in \mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)$ has a unique decomposition as $w = tu$ with $t \in \mathcal{U}(\mathcal{D}_n)$ and u a finite sum of words. That is, u is a unitary such that $u = \sum_{j=1}^m S_{\alpha_j}S_{\beta_j}^*$ for some $\alpha_j, \beta_j \in W_n$. Clearly, such unitaries form a group, which we denote $\mathcal{S}_n$, and this group acts on $\mathcal{U}(\mathcal{D}_n)$ by conjugation. Thus, Power's result says that $\mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)$ has the structure of semi-direct product $\mathcal{U}(\mathcal{D}_n) \rtimes \mathcal{S}_n$. We denote

$$\lambda(\mathcal{S}_n)^{-1} = \{\lambda_w \in \mathrm{Aut}(\mathcal{O}_n) : w \in \mathcal{S}_n\}. \tag{3}$$

Combining the results of [6] and [15] we obtain the following Theorem 1. This results has been obtained earlier by Matsumoto [13, Theorem 6.5] through a different argument.

Theorem 1. $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cong \mathcal{U}(\mathcal{D}_n) \rtimes \lambda(\mathcal{S}_n)^{-1}$. *In particular,* $\lambda(\mathcal{S}_n)^{-1}$ *is a subgroup of* $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$.

Proof. Let $u \in \mathcal{S}_n$ and let λ_u be invertible. Then $\lambda_u^{-1} \in \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ and thus $\lambda_u^{-1} = \lambda_z$ with $z \in \mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)$. Thus, there are $v \in \mathcal{S}_n$ and $y \in \mathcal{U}(\mathcal{D}_n)$ such that $z = vy$. We have $\mathrm{id} = \lambda_u\lambda_{vy}$ and hence $1 = u * vy = u\lambda_u(v)\lambda_u(y)$. Thus $\mathcal{S}_n \ni u\lambda_u(v) = \lambda_u(y^*) \in \mathcal{U}(\mathcal{D}_n)$. Therefore $y = 1$ and consequently $\lambda_u^{-1} = \lambda_v$. It follows that $\lambda(\mathcal{S}_n)^{-1}$ is a subgroup of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$. Clearly, $\lambda(\mathcal{S}_n)^{-1}$ acts on $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \lambda(\mathcal{U}(\mathcal{D}_n))$ by conjugation.

Let $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$. Then $w \in \mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)$ and hence there are $u \in \mathcal{S}_n$ and $t \in \mathcal{U}(\mathcal{D}_n)$ such that $w = ut$. Since $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n)$ there is $s \in \mathcal{U}(\mathcal{D}_n)$ such that $\lambda_w(s) = t^*$. We have $\lambda_w\lambda_s = \lambda_{ut}\lambda_s = \lambda_{ut\lambda_{ut}(s)} = \lambda_u$. Since $\lambda_s^{-1} = \lambda_{s^*}$ we get $\lambda_w = \lambda_u\lambda_{s^*}$. As both λ_w and λ_{s^*} are invertible, so is λ_u. Thus every element of $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ can be written as a product of two elements from $\lambda(\mathcal{S}_n)^{-1}$ and $\lambda(\mathcal{U}_n)$. Clearly, such a factorization is unique. Finally, as shown in [6], λ is an isomorphism from $\mathcal{U}(\mathcal{D}_n)$ onto $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$. □

We now turn to the main focus of this note, automorphisms which preserve both the diagonal and the UHF subalgebra[2]. It is shown in [6] that if λ_w is invertible then $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ if and only if $w \in \mathcal{F}_n$. Thus, if λ_w is an automorphism then $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ and $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ if and only if $w \in \mathcal{N}_{\mathcal{D}_n}(\mathcal{F}_n)$. This can be further strengthen as follows.

Lemma 2 (R. Conti). *If* $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ *and* $w \in \mathcal{F}_n$ *then* $\lambda_w(\mathcal{F}_n) = \mathcal{F}_n$.

[1] I am indebted to Roberto Conti for bringing this paper to my attention and for thorough discussion of Power's argument.

[2] It is worth mentioning that a formula similar to that describing restriction of such automorphisms to $\mathcal{F}_n$ [6] appeared already in [4] in construction of examples of periodic automorphisms of the hyperfinite II_1 factor.

Proof. Let γ be the standard gauge action of the circle group on $\mathcal{O}_n$ [5], for which $\mathcal{F}_n$ is the fixed-point algebra [5]. Then for each $z \in U(1)$ we have $\lambda_w\gamma_z = \gamma_z\lambda_w$. Thus, also $\lambda_w^{-1}\gamma_z = \gamma_z\lambda_w^{-1}$ and consequently λ_w^{-1} preserves the fixed-point algebra of γ. That is $\lambda_w^{-1}(\mathcal{F}_n) \subseteq \mathcal{F}_n$, as required. □

Since $\mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n) = \mathcal{U}(\mathcal{D}_n) \rtimes \mathcal{S}_n$ by [15], it easily follows that $\mathcal{N}_{\mathcal{D}_n}(\mathcal{F}_n) = \mathcal{U}(\mathcal{D}_n) \rtimes \mathcal{P}_n$, where $\mathcal{P}_n = \mathcal{S}_n \cap \mathcal{F}_n$. We see that $\mathcal{P}_n$ is contained in the algebraic part $\cup_{k=0}^{\infty}\mathcal{F}_n^k$ of $\mathcal{F}_n$, and write $\mathcal{P}_n^k = \mathcal{P}_n \cap \mathcal{F}_n^k$. It is not difficult to see that unitaries in $\mathcal{P}_n$ are related to permutations of multi-indices, as follows. Let $\mathbb{P}_n^k$ denote the set of permutations of W_n^k, and let $\mathbb{P}_n = \cup_{k=0}^{\infty}\mathbb{P}_n^k$. Then for each unitary $w \in \mathcal{P}_n^k$ there exists a permutation $\sigma \in \mathbb{P}_n^k$ such that

$$w = \sum_{\alpha \in W_n^k} S_{\sigma(\alpha)}S_\alpha^*. \tag{4}$$

In that case we write $w \sim \sigma$ and $\lambda_w = \lambda_\sigma$. We denote

$$\lambda(\mathcal{P}_n)^{-1} = \{\lambda_w \in \mathrm{Aut}(\mathcal{O}_n) : w \in \mathcal{P}_n\}. \tag{5}$$

Let $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \{\psi \in \mathrm{Aut}(\mathcal{O}_n) : \psi(\mathcal{F}_n) = \mathcal{F}_n\}$. With help of Lemma 2 one proves the following proposition by an argument similar to that from Theorem 1.

Theorem 3. $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \cong \mathcal{U}(\mathcal{D}_n) \rtimes \lambda(\mathcal{P}_n)^{-1}$. *In particular,* $\lambda(\mathcal{P}_n)^{-1}$ *is a subgroup of* $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$.

If $u \in \mathcal{U}(\mathcal{O}_n)$ then $\mathrm{Ad}(u) = \lambda_{\Phi(u)u^*}$ is the inner automorphism of $\mathcal{O}_n$ determined by u. We denote by $\mathrm{Inn}(\mathcal{O}_n)$ the group of inner automorphisms of $\mathcal{O}_n$.

Lemma 4. *Let* $w \in \mathcal{P}_n$. *If* $\lambda_w \in \mathrm{Inn}(\mathcal{O}_n)$ *then there exists* $u \in \mathcal{P}_n$ *such that* $w = \Phi(u)u^*$.

Proof. Let $w \in \mathcal{P}_n$ and let $\lambda_w = \mathrm{Ad}(z)$. Since $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$, the unitary z belongs to $\mathcal{N}_{\mathcal{D}_n}(\mathcal{O}_n)$ and thus there are $u \in \mathcal{S}_n$ and $t \in \mathcal{U}(D_n)$ such that $z = ut$. For each $x \in \mathcal{D}_n$ we have $\lambda_w(x) = zxz^* = uxu^*$. Therefore $\lambda_w^{-1}\mathrm{Ad}(u) = \lambda_w^{-1}\lambda_{\Phi(u)u^*}$ belongs to $\lambda(\mathcal{S}_n)^{-1}$ (by Theorem 1) and acts trivially on $\mathcal{D}_n$. Hence $\lambda_w = \mathrm{Ad}(u)$.

Since $w \in \mathcal{F}_n$, automorphism λ_w commutes with the gauge action γ of the circle group. Thus $u\gamma_c(a)u^* = \gamma_c(uau^*)$ for all $a \in \mathcal{O}_n$ and $c \in U(1)$. Substituting $a = S_i$, $i = 1, \dots, n$, we see that the unitary $\gamma_c(u)^*u$ commutes with the generators of $\mathcal{O}_n$ and hence it is a scalar. Now applying the Fourier series decomposition of $u \in \mathcal{O}_n$ from [5] we conclude that $u \in \mathcal{F}_n$. Hence $u \in \mathcal{P}_n$. □

Suppose $u \in \mathcal{P}_n^k$, $\sigma \in \mathbb{P}_n^k$, and $u \sim \sigma$. Then the natural inclusion $u \in \mathcal{F}_n^k \subseteq \mathcal{F}_n^{k+m}$ corresponds to the embedding $\mathbb{P}_n^k \hookrightarrow \mathbb{P}_n^{k+m}$ such that $\sigma \mapsto \sigma \times \mathrm{id}_m$, where id_m is the identity on W_n^m and we identify $W_n^{k+m} = W_n^k \times W_n^m$. On the other hand, the embedding $\mathcal{P}_n^k \hookrightarrow \mathcal{P}_n^{k+m}$ given by $u \mapsto \Phi^m(u)$ corresponds to the embedding $\mathbb{P}_n^k \hookrightarrow \mathbb{P}_n^{k+m}$ such that $\sigma \mapsto \mathrm{id}_m \times \sigma$. For $r = 2, 3, \dots$ we define $\sigma^{(r)} \in \mathbb{P}_n^{k+r-1}$ as

$$\sigma^{(r)} = (\mathrm{id}_{r-1} \times \sigma)(\mathrm{id}_{r-2} \times \sigma \times \mathrm{id}_1) \cdots (\sigma \times \mathrm{id}_{r-1}). \tag{6}$$

If $r = 1$ then simply $\sigma^{(1)} = \sigma$. Note that if λ_u is inner and $u = \Phi(w)w^*$ with $w \in \mathcal{P}_n^{k-1}$, then with $w \sim \psi$ we have $\psi = (\mathrm{id}_1 \times \sigma)(\sigma^{-1} \times \mathrm{id}_1)$ and $\psi^{(r)} = (\mathrm{id}_r \times \sigma)(\sigma^{-1} \times \mathrm{id}_r)$. In particular, $\psi^{(k)} = \phi^{-1} \times \psi$. With this notation one notes that the convolution multiplication $\mathcal{P}_n^k \times \mathcal{P}_n^r \to \mathcal{P}_n^{k+r-1}$, $(u, w) \mapsto u * w = u\lambda_u(w)$, corresponds on the permutation level to the map $\mathbb{P}_n^k \times \mathbb{P}_n^r \to \mathbb{P}_n^{k+r-1}$ such that

$$(\alpha, \beta) \mapsto \alpha * \beta = (\alpha \times \mathrm{id}_{r-1})(\alpha^{(r)})^{-1}(\beta \times \mathrm{id}_{k-1})\alpha^{(r)}. \tag{7}$$

If permutation σ is $*$-invertible then we denote its inverse by $\overline{\sigma}$.

If $w \in \mathcal{P}_n$ and λ_w is invertible then $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ and hence there exists a homeomorphism $h_u : X_n \to X_n$ such that

$$(\lambda_w^{-1} f)(x) = f(h_w(x))), \quad x \in X_n, \ f \in \mathcal{D}_n = C(X_n). \tag{8}$$

If $w \sim \sigma$ then we also write $h_w = h_\sigma$. For $x = (x_i) \in X_n$ and $m = 0, 1, \dots$ we denote by x_{+m} the sequence in X_m whose i^{th} term is x_{i+m}. For integers $k \leqq r$ we denote $\pi_{k,r}(x) = (x_k, x_{k+1}, \dots, x_r)$. If $k = r$ then we simply write $\pi_k = \pi_{k,k}$. If $\sigma \in \mathbb{P}_n^k$ then we write $\sigma(x) = \sigma(\pi_{1,k}(x))$.

The following lemma gives a description of the homeomorphisms of X_n corresponding to elements of $\lambda(\mathcal{P}_n)^{-1}$. It turns out that these homeomorphisms are also localized in the sense that the value of the m^{th} coordinate depends only on finitely many neighbouring coordinates in an eventually periodic fashion. The lemma also gives a convenient practical way of deciding if an automorphism from $\lambda(\mathcal{P}_n)^{-1}$ is inner or not.

Lemma 5. *If $w \in \mathcal{P}_n^k$, $w \sim \sigma$, and λ_w is invertible, then $h_w(x) = y$ where*

$$(y_1, \dots, y_{k-1}) = \pi_{1,k-1}(\overline{\sigma}^{(k-1)}(x)), \tag{9}$$

$$y_{k+m} = \pi_k(\overline{\sigma}^{(k)}(x_{+m})). \tag{10}$$

Furthermore, if $\pi_k(\overline{\sigma}^{(k)}(x)) = x_k$ for all $x \in X_n$ then $\lambda_w \in \mathrm{Inn}(\mathcal{O}_n)$. Conversely, if $\lambda_w = \mathrm{Ad}(u)$ with $u \in \mathcal{P}_n^{k-1}$ then $\pi_k(\overline{\sigma}^{(k)}(x)) = x_k$ for all $x \in X_n$.

Proof. At first one checks that $h_w(x) = y$ with y_j the unique element of $\{1, \dots, n\}$ such that $(\lambda_w^{-1}(\Phi^{j-1}(S_{y_j} S_{y_j}^*)))(x) = 1$. Thies yields formulae (9) and (10) with $m = 0$. However, if $t \in \{1, \dots, n\}$ and $j \geq k+1$ then

$$\lambda_w^{-1}(\Phi^{j-1}(S_t S_t^*)) = \lim_{m\to\infty} \mathrm{Ad}(\Phi^m(w) \cdots w)(\Phi^{j-1}(S_t S_t^*)) =$$

$$= \lim_{m\to\infty} \mathrm{Ad}(\Phi^m(w) \cdots \Phi^{j-k}(w))(\Phi^{j-1}(S_t S_t^*)) =$$

$$= \Phi^{j-k}(\lim_{m\to\infty} \mathrm{Ad}(\Phi^m(w) \cdots w)(\Phi^{k-1}(S_t S_t^*))) = \Phi^{j-k}(\lambda_w^{-1}(\Phi^{k-1}(S_t S_t^*))).$$

This implies (10) with $m = 0, 1, \dots$.

If $\pi_k(\overline{\sigma}^{(k)}(x)) = x_k$ for all $x \in X_n$ then there exists $\psi \in \mathbb{P}_n^{k-1}$ such that $h_w(x) = \psi(x)$. This implies $\lambda_w = \mathrm{Ad}(u^*)$ with $u \sim \psi$. Finally, if λ_w is inner and $\sigma = (\mathrm{id}_1 \times \psi)(\psi^{-1} \times \mathrm{id}_1)$ for some $\psi \in \mathbb{P}_n^{k-1}$, then $\overline{\sigma}^{(k)} = \psi \times \mathrm{id}_1 \times \psi^{-1}$ and hence $\pi_k(\overline{\sigma}^{(k)}(x)) = x_k$. □

Theorem 6. *If $u \in \mathcal{P}_n$ and λ_u is invertible then the following conditions are equivalent.*

(1) Automorphism λ_u has infinite order.
(2) The $\mathbb{Z}$ action on $\mathcal{O}_n$ generated by λ_u is outer.
(3) The $\mathbb{Z}$ action on X_n generated by h_u is topologically free.

Proof. (1)⇒(2) This follows from the fact that (by Lemma 4) if $\lambda_u \in \mathrm{Inn}(\mathcal{O}_n)$ then λ_u has finite order.

(2)⇒(3) If the action is not topologically free then for some m the set of fixed points of h_u^m has a non-empty interior. Thus there exists $(x_1, \dots, x_r)$ such that h_u^m fixes each sequence (y_i) whose initial segment coincides with $(x_1, \dots, x_r)$. But then λ_u^m is inner by Lemma 5.

(3)⇒(1) This is obvious. □

We now give a practical criterion of invertibility of endomorphisms corresponding to permutations. First recall that $\mathrm{End}(\mathcal{O}_n)$ contains a distinguished endomorphism Φ, called shift, such that

$$\Phi(a) = \sum_{i=1}^{n} S_i a S_i^*. \tag{11}$$

Let $w \in \mathcal{P}_n^k$. If $k \geq 2$ then we define

$$B_w = \{w, \Phi(w), \ldots, \Phi^{k-2}(w)\}' \cap \mathcal{F}_n^{k-1}. \tag{12}$$

Here prime denotes the commutant. If $k \leqq 1$ then we set $B_w = \mathbb{C}1$. One checks that $b \in \mathcal{F}_n^{k-1}$ belongs to B_w if and only if for each pair $\alpha, \beta \in W_n^l$, $l \in \{0, 1, \ldots, k-2\}$, $S_\alpha^* b S_\beta$ commutes with w. We define a vector space V_w as the quotient

$$V_w = \mathcal{F}_n^{k-1}/B_w. \tag{13}$$

Now for each pair $i, j \in \{1, \ldots, n\}$ we define a linear map $a_{ij}^w : \mathcal{F}_n^{k-1} \to \mathcal{F}_n^{k-1}$ such that

$$a_{ij}^w(b) = S_i^* w b w^* S_j. \tag{14}$$

One checks that $a_{ij}^w(B_w) \subseteq B_w$ for each i, j. Thus, a_{ij}^w induces a linear map

$$\tilde{a}_{ij}^w : V_w \to V_w. \tag{15}$$

With this preparation we make the following definition:

$$A_w = \text{the subring of } \mathrm{End}(V_w) \text{ generated by } \{\tilde{a}_{ij}^w : i, j = 1, \ldots, n\}. \tag{16}$$

Now we are ready to prove the following.

Theorem 7. *If $w \in \mathcal{P}_n$ then endomorphism λ_w is invertible if and only if the corresponding ring A_w is nilpotent.*

Proof. **Necessity.** Let $w \in \mathcal{P}_n^k$ and suppose that λ_w is invertible. By Proposition 3 there exists $u \in \mathcal{P}_n$ such that $\lambda_w^{-1} = \lambda_u$. Thus there exists positive integer l such that $\lambda_w^{-1}(\mathcal{F}_n^{k-1}) \subseteq \mathcal{F}_n^l$. For each $a \in \mathcal{F}_n^l$ the sequence $\mathrm{Ad}(w^*\Phi(w^*) \cdots \Phi^m(w^*))(a)$ stabilizes from $m = l-1$ at the value $\lambda_w(a)$. Consequently, for each $b \in \mathcal{F}_n^{k-1}$ the sequence $\mathrm{Ad}(\Phi^m(w) \cdots \Phi(w)w)(b)$ stabilizes from $m = l-1$ at the value $\lambda_w^{-1}(b)$. There exist elements $c_{\mu\nu}(b) \in \mathcal{F}_n^{k-1}$, $\mu, \nu \in W_n^l$, such that for each $r \geq 1$ we have

$$\sum_{\mu,\nu \in W_n^l} S_\mu c_{\mu\nu}(b) S_\nu^* = \mathrm{Ad}(\Phi^{l-1}(w) \cdots \Phi(w)w)(b) =$$

$$= \mathrm{Ad}(\Phi^{l-1+r}(w) \cdots \Phi(w)w)(b) = \sum_{\mu,\nu \in W_n^l} S_\mu \,\mathrm{Ad}(\Phi^{r-1}(w))(c_{\mu\nu}(b)) S_\nu^*.$$

Hence $c_{\mu\nu}(b) = \mathrm{Ad}(\Phi^{r-1}(w))(c_{\mu\nu}(b))$. Thus $\mathrm{span}\{c_{\mu\nu}(b) : b \in \mathcal{F}_n^{k-1}, \mu\nu \in W_n^l\} \subseteq B_w$. If $\alpha = (i_1, \ldots, i_l)$ and $\beta = (j_1, \ldots, j_l)$ then let $T_{\alpha,\beta} = a_{i_l j_l}^w \cdots a_{i_1 j_1}^w$. For each $b \in \mathcal{F}_n^{k-1}$ we have $T_{\alpha,\beta}(b) = c_{\alpha\beta}(b)$. Consequently $A_w^l = \{0\}$ and A_w is nilpotent.

Sufficiency. Let $w \in \mathcal{P}_n^k$ and suppose that $A_w^l = \{0\}$. Let $b \in \mathcal{F}_n^{k-1}$ and define $T_{\alpha,\beta}$ as above. By hypothesis, $T_{\alpha,\beta}(b)$ commutes with $\mathrm{Ad}(\Phi^m(w))$ for any m. Hence if $r \geq 1$ then we have

$$\mathrm{Ad}(\Phi^{l-1+r}(w) \cdots \Phi(w)w)(b) = \sum_{\mu,\nu \in W_n^l} S_\mu \,\mathrm{Ad}(\Phi^{r-1}(w))(T_{\mu\nu}(b)) S_\nu^* = \sum_{\mu,\nu \in W_n^l} S_\mu T_{\mu\nu}(b) S_\nu^*.$$

Thus for each $b \in \mathcal{F}_n^{k-1}$ the sequence $\mathrm{Ad}(\Phi^m(w) \cdots \Phi(w)w)(b)$ stabilizes from $m = l-1$. We have $w^* = \sum_{i,j=1}^n S_i b_{ij} S_j^*$ for some $b_{ij} \in \mathcal{F}_n^{k-1}$. It follows from the above argument that the sequence

$$\mathrm{Ad}(\Phi^{l-1+r}(w) \cdots \Phi(w)w)(w^*) = \sum_{i,j} \mathrm{Ad}(\Phi(\Phi^{m-1}(w) \cdots \Phi(w)w))(S_i b_{ij} S_j^*) =$$

$$= \sum_{i,j} S_i \operatorname{Ad}(\Phi^{m-1}(w)\cdots\Phi(w)w)(b_{ij})S_j^*$$

stabilizes from $m = l$ at the value $\lambda_w^{-1}(w^*)$. Consequently λ_w is invertible. □

An easy application of Theorem 7 shows that among $\{\lambda_w : w \in \mathcal{P}_2^2\}$ only 4 elements are invertible: the flip-flop, an inner automorphism of order 2, their product, and the identity. This was observed earlier by a different method by Kawamura [10, 11]. Further discussion of Theorem 7 and its far reaching applications will be presented in [3].

We close this note with the following two examples.

Example 8. Consider a partition $W_n^1 = R_1 \cup \ldots R_r$ of W_n^1 into a union of disjoint subsets. Let $\sigma_i \in \mathbb{P}_n^1$, $i = 1, \ldots, r$, be permutations such that $\sigma_i\sigma_j^{-1}(R_m) = R_m$ for all i, j, m. We define $\psi \in \mathbb{P}_n^2$ as $\psi(\alpha, \beta) = (\alpha, \sigma_i(\beta))$ for $\alpha \in R_i$, $\beta \in W_n^1$. So constructed λ_ψ is invertible and we have $\overline{\psi} \in \mathbb{P}_n^3$ such that $\overline{\psi}(\alpha, \beta, \mu) = (\alpha, \sigma_i^{-1}(\beta), \sigma_j\sigma_k^{-1}(\mu))$ for $\alpha \in R_i$, $\beta \in R_k$, $\sigma_i^{-1}(\beta) \in R_j$. By Lemma 5 we have

$$h_\psi(x_1, x_2, \ldots) = (x_1, \sigma_{\nu_1}^{-1}(x_2), \sigma_{\nu_2}^{-1}(x_3), \ldots), \quad x_i \in R_{\nu_i}. \tag{17}$$

Also by Lemma 5, $\lambda_\psi \in \operatorname{Inn}(\mathcal{O}_n)$ if and only if $\psi = \mathrm{id}$.

If $n = 4$, $R_1 = \{1, 2\}$, $R_2 = \{3, 4\}$, $\sigma_1 = (2\ 3)$, $\sigma_2 = (1\ 2\ 4\ 3)$, ψ is constructed from this data as above and $w \sim \sigma_1$, then $\operatorname{Ad}(w)\lambda_\psi$ is the automorphism of $\mathcal{O}_4$ constructed and discussed by Matsumoto and Tomiyama in [14].

Example 9. Let $n \geq 3$, $\phi = (1\ 2\ 3)$, and let ψ be constructed as in Example 8 from the data: $R_1 = \{1, 2\}$, $R_2 = \{3, \ldots, n\}$, $\sigma_1 = \mathrm{id}$, $\sigma_2 = (1\ 2)$. One checks that λ_ϕ and λ_ψ are outer automorphisms of $\mathcal{O}_n$ of order 3 and 2, respectively. We claim that the group generated by λ_ϕ and λ_ψ is isomorphic to a free product $\mathbb{Z}_3 * \mathbb{Z}_2$. Indeed, let $O_\phi(x)$ be the h_ϕ orbit of $x \in X_n$. If $h_\psi(x) \neq x$ then $O_\phi(h_\psi(x)) \neq O_\phi(x)$. Also, if $O_\phi(x) \neq O_\phi(y)$ then there exists at most one $t \in O_\phi(x)$ such that $h_\psi(t) \in O_\phi(y)$. If $x = (x_i)$ then $h_\psi(y) \neq y$ for all $y \in O_\phi(x)$ if the following condition $C[x]$ is satisfied: for each $s \in \{1, 2, 3\}$ there exists an index j such that $x_j = s$ and $x_{j+1} \in \{1, 2, 3\} \setminus \{s\}$. Let $k \geq 1$ and $x = (x_i)$ be such that for each $s \in \{1, 2, 3\}$ there exists an index j such that $x_j = \ldots = x_{j+k} = s$ and $x_{j+k+1} \in \{1, 2, 3\} \setminus \{s\}$. Then for each θ, a reduced word in $h_\phi, h_\phi^{-1}, h_\psi$ of length less or equal k, condition $C[\theta(x)]$ is satisfied. Consequently, for any such θ we have $\theta(x) \neq x$. It follows that the group generated by λ_ϕ and λ_ψ is $\mathbb{Z}_3 * \mathbb{Z}_2$, as claimed. Since each finite-order element of a free product of cyclic groups is conjugate to a power of one of the generators, it follows from Theorem 6 that all non-trivial elements of the group generated by λ_ϕ and λ_ψ are outer automorphisms of $\mathcal{O}_n$.

As shown in Example 9 above, if $n \geq 3$ then $\lambda(\mathcal{P}_n^2)^{-1}$ contains elements which generate in $\operatorname{Out}(\mathcal{O}_n)$ a group isomorphic to $\mathbb{Z}_3 * \mathbb{Z}_2$. By contrast, $\lambda(\mathcal{P}_2^2)^{-1}$ yields $\mathbb{Z}_2$ group in $\operatorname{Out}(\mathcal{O}_2)$. In the forthcoming paper [3], analysis of the automorphisms from $\lambda(\mathcal{P}_2^k)^{-1}$, $k \geq 3$, is presented.

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