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QUASI-FREE FINITE GROUP ACTIONS ON THE CUNTZ ALGEBRA $\mathcal{O}_\infty$

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1. Introduction

The Cuntz algebra $\mathcal{O}_n$, for $n = 2, 3, \cdots, \infty$, is the universal C^*-algebra generated by isometries $\{S_i\}_{i=1}^n$ with mutually orthogonal ranges, satisfying $\sum_{i=1}^n S_iS_i^* = I$ if n is finite. Let $\mathcal{H}_n$ be the closed linear span of $\{S_i\}_{i=1}^n$, which has a Hilbert space structure with inner product $T^*S = \langle S, T\rangle I$. An action α of a group G on $\mathcal{O}_n$ is said to be *quasi-free* if $\alpha_g(\mathcal{H}_n) = \mathcal{H}_n$ for all $g \in G$. There is a one-to-one correspondence between the quasi-free actions of G and the unitary representation of G in $\mathcal{H}_n$. For a quasi-free action α, we denote by $(\pi_\alpha, \mathcal{H}_n)$ the corresponding unitary representation of G. We assume that G is finite in what follows.

A priori, the conjugacy class of a quasi-free action α depends on the unitary equivalence class of the unitary representation $(\pi_\alpha, \mathcal{H}_n)$. Indeed, it really does when n is finite (see, for example, [2], [3]). However, when $n = \infty$, the pair $(\mathcal{O}_\infty, \alpha)$ is equivariantly KK-equivalent to the pair $(\mathbb{C}, \mathrm{id})$, and there is no way to differentiate quasi-free actions as far as K-theory is concerned. In fact, P. Goldstein announced in his preprint [1] (ten years old by now) that any two non-trivial quasi-free $\mathbb{Z}_2$-actions are mutually conjugate. His idea is to develop a $\mathbb{Z}_2$-equivariant version of Lin-Phillips' argument, which gives a uniqueness theorem for unital homomorphisms from $\mathcal{O}_\infty$ to purely infinite C^*-algebras up to approximately unitary equivalence (see [7, Section 7.2]). However, his proof is based on Evans-Su's unpublished work, which is not available for some reason.

The aim of this short note is to announce that any two faithful quasi-free actions are indeed mutually conjugate for every finite group G. Our strategy is basically the same as Goldstein's. However, while Goldstein compares two quasi-free actions directly using an equivariant version of Elliott's intertwining argument, we use a model action splitting argument, which in fact works for every outer action on the Kirchberg algebras.

2. Main results

A Kirchberg algebra is a purely infinite, simple, nuclear, separable C^*-algebra. The reader is referred to [7] for basic properties and classification results for Kirchberg algebras. The Cuntz algebras are typical examples of Kirchberg algebras. We denote by $\mathbb{K}$ the set of compact operators on a separable infinite dimensional Hilbert space.

We fix a finite group G. By a G-C^*-algebra (A, α), we mean a C^*-algebra A with a fixed G-action α. A G-homomorphism φ from a G-C^*-algebra (A, α) into another G-C^*-algebra (B, β) is a homomorphism from A into B intertwining the two G-actions

α and β. We denote by $\mathrm{Hom}_G(A, B)$ the set of G-homomorphisms from (A, α) into (B, β). Two G-homomorphisms $\varphi, \psi \in \mathrm{Hom}_G(A, B)$ are said to be G-approximately unitarily equivalent if there exists a sequence of unitaries $\{u_n\}$ in the fixed point algebra B^G such that

$$\lim_{n\to\infty} \|\varphi(x) - \mathrm{Ad}u_n \circ \psi(x)\|, \quad \forall x \in A.$$

When A and B are separable, an equivariant version of Elliott's intertwining argument implies the following: if there exist $\varphi \in \mathrm{Hom}_G(A, B)$ and $\psi \in \mathrm{Hom}_G(B, A)$ such that $\psi \circ \varphi$ is G-approximately unitarily equivalent to $\mathrm{id}_{(A,\alpha)}$ and $\varphi \circ \psi$ is G-approximately unitarily equivalent to $\mathrm{id}_{(B,\beta)}$, then the two actions α and β are conjugate.

Let β be a faithful outer action of G on a simple C^*-algebra B, and let $\{\lambda_g\}$ be the implementing unitary representation of G in the crossed product $B \rtimes_\beta G$. Let

$$e_\beta = \frac{1}{\#G} \sum_{g\in G} \lambda_g,$$

which is a projection in $B \rtimes_\beta G$. Then the homomorphism

$$\Phi_\beta : B^G \ni x \mapsto xe_\beta \in B \rtimes_\beta G$$

induces the isomorphism $K_*(\Phi_\beta) : K_*(B^G) \to K_*(B \rtimes_\beta G)$.

Let $\hat{G}$ be the unitary dual of G. The dual coaction $\hat{\beta}$ of the action β induces an action of the representation ring $\mathbb{Z}\hat{G}$ on $K_*(B \rtimes_\beta G)$. For a finite dimensional unitary representation (π, H_π) of G, we denoted by $K_*(\hat{\beta}_\pi)$ the corresponding endomorphism in $\mathrm{End}(K_*(B \rtimes_\beta G))$.

The following theorem is an equivariant version of Rørdam's result (cf. [7, Theorem 5.1.2]):

Theorem 2.1. *Let α be a quasi-free action of G on $\mathcal{O}_n$ with finite n, and let (B, β) be a G-C^*-algebra. We assume that B is unital purely infinite simple, and β is outer. For two unital G-homomorphisms $\varphi, \psi \in \mathrm{Hom}_G(\mathcal{O}_n, B)$, let*

$$u = \sum_{i=1}^{n} \psi(S_i)\varphi(S_i)^*,$$

which is a unitary in B^G. Then the following conditions are equivalent.

(1) *The K_1-class $K_1(\Phi_\beta)([u])$ is in the image of $K_1(\hat{\beta}_{\pi_\alpha}) - 1$.*
(2) *The G-homomorphisms φ and ψ are G-approximately unitarily equivalent.*
(3) *$[\varphi] = [\psi]$ in the equivariant KK-group $KK_G(\mathcal{O}_n, B)$.*

The equivalence of (1) and (2) follows from the Rohlin property of the shift of the UHF algebra M_{n^∞} restricted to the fixed point algebra of a product type G-action.

Let $\mathcal{T}_n$ be the Cuntz-Toeplitz algebra, which is the universal C^*-algebra generated by isometries $\{T_i\}_{i=1}^n$ with mutually orthogonal ranges. Then there exists a homomorphism from $\mathcal{T}_n$ to $\mathcal{O}_n$ sending T_i to S_i for $i = 1, 2, \cdots, n$, which gives a short exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{T}_n \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

Quasi-free actions on $\mathcal{T}_n$ are defined in the same way as in the case of the Cuntz algebras, and this short exact sequence is actually a semi-splitting short exact sequence of G-C^*-algebras. Therefore we get the following 6-term exact sequence of equivariant KK-groups:

$$\begin{array}{ccccc} KK_G(\mathcal{O}_n, B) & \longrightarrow & KK_G(\mathcal{T}_n, B) & \longrightarrow & KK_G(\mathbb{K}, B) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow \\ KK_G^1(\mathbb{K}, B) & \longleftarrow & KK_G^1(\mathcal{T}_n, B) & \longleftarrow & KK_G^1(\mathcal{O}_n, B) \end{array}.$$

Let $\tilde{\alpha}$ be the quasi-free action of G on $\mathcal{T}_n$ that is a lift of α. Since $(\mathcal{T}_n, \tilde{\alpha})$ is equivariantly KK-equivalent to $(\mathbb{C}, \mathrm{id})$, we get an exact sequence

$$\begin{array}{ccccc} KK_G(\mathcal{O}_n, B) & \longrightarrow & K_0(B \rtimes_\beta G) & \xrightarrow{1-K_0(\hat{\beta}_{\pi_\alpha})} & K_0(B \rtimes_\beta G) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow \\ K_1(B \rtimes_\beta G) & \xleftarrow{1-K_1(\hat{\beta}_{\pi_\alpha})} & K_1(B \rtimes_\beta G) & \longleftarrow & KK_G^1(\mathcal{O}_n, B) \end{array}.$$

A tedious computation shows $\delta(K_1(\Phi_\beta)([u])) = [\varphi] - [\psi]$, which shows equivalence of (1) and (3). This argument also shows that there exists a short exact sequence

$$0 \to \mathrm{Coker}(1 - K_{1-*}(\hat{\beta}_{\pi_\alpha})) \to KK_G^*(\mathcal{O}_n, B) \to \mathrm{Ker}(1 - K_*(\hat{\beta}_{\pi_\alpha})) \to 0.$$

As in [7, Proposition 7.2.5], Theorem 2.1 implies

Theorem 2.2. *Let α be a quasi-free action of G on $\mathcal{O}_\infty$, and let (B, β) be a unital G-C^*-algebra. We assume that B is purely infinite simple, and β is outer. Then any two unital G-homomorphisms in* $\mathrm{Hom}_G(\mathcal{O}_\infty, B)$ *are G-approximately unitarily equivalent.*

Thanks to Theorem 2.2, we get a G-equivariant version of Kirchberg-Phillips' $\mathcal{O}_\infty$ theorem (cf. [7, Theorem 7.2.6]).

Theorem 2.3. *Let B be a Kirchberg algebra, and let β be an outer action of G on B. Let $\{\gamma^{(i)}\}_{i=1}^\infty$ be any sequence of quasi-free actions of G on $\mathcal{O}_\infty$. Then (B, β) is conjugate to*

$$(B \otimes \bigotimes_{i=1}^{\infty} \mathcal{O}_\infty, \beta \otimes \bigotimes_{i=1}^{\infty} \gamma^{(i)}).$$

Applying Theorem 2.3 to $B = \mathcal{O}_\infty$ with a faithful quasi-free action β, we obtain

Corollary 2.4. *Any two faithful quasi-free G-actions on $\mathcal{O}_\infty$ are mutually conjugate.*

3. Approximately representable actions

An action α of G on a unital separable C^*-algebra A is said to be approximately representable if there exists a sequence of unitaries $\{u(g)_n\}_{n=1}^\infty$ in A for each $g \in G$ such that

$$\lim_{n\to\infty} \|u(g)_n x u(g)_n^* - \alpha_g(x)\| = 0, \quad \forall x \in A,\ \forall g \in G,$$
$$\lim_{n\to\infty} \|u_n(g) u(h)_n - u(gh)_n\| = 0, \quad \forall g, h \in G,$$

$$\lim_{n\to\infty} \|\alpha_g(u(h)_n) - u(ghg^{-1})_n\| = 0, \quad \forall g, h \in G.$$

When G is abelian, an action α is approximately representable if and only if its dual action has the Rohlin property. When G is a cyclic group of prime power order, approximately representable quasi-free actions on $\mathcal{O}_n$ with finite n are completely characterized in [3], and there exist quasi-free actions that are not approximately representable.

Theorem 3.1. *Every quasi-free G-action on $\mathcal{O}_\infty$ is approximately representable.*

It does not seem to me that one could show Theorem 3.1 directly from the definition of quasi-free actions. Our proof uses the intertwining argument between two model actions; one is obviously quasi-free, and the other is an infinite tensor product action, which is obviously approximately representable.

4. Equivariant Rørdam group

Let A and B be simple C^*-algebras. Following Rørdam [6], we denote by $H(A,B)$ the set of approximately unitary equivalence classes of non-zero homomorphisms from $A \otimes \mathbb{K}$ into $B \otimes \mathbb{K}$. Choosing two isometries S_1 and S_2 satisfying the $\mathcal{O}_2$ relation in the multiplier algebra of $B \otimes \mathbb{K}$, we can define the direct sum $[\varphi] \oplus [\psi]$ of two classes $[\varphi]$ and $[\psi]$ in $H(A,B)$ to be the class of the homomorphism

$$A \otimes \mathbb{K} \ni x \mapsto S_1\varphi(x)S_1^* + S_2\psi(x)S_2^* \in B \otimes \mathbb{K}.$$

This makes $H(A.B)$ a semigroup. When A is a Kirchberg algebra, Rørdam semigroup $H(A,B)$ is in fact a group, which is isomorphic to $KL(A,B)$, a certain quotient of $KK(A,B)$ (see [6]).

When (A,α) (resp. (B,β)) is a unital G-C^*-algebra, we equip $A \otimes \mathbb{K}$ (resp. $B \otimes \mathbb{K}$) with a G-C^*-algebra structure by the diagonal action $\alpha_g \otimes \mathrm{Ad}u(g)$ (resp. $\beta_g \otimes \mathrm{Ad}u(g)$), where $u(g)$ is a countable direct sum of the regular representation of G. Then one can introduce an equivariant version $H_G(A,B)$ of Rørdam's semigroup $H(A,B)$ in an obvious way.

Theorem 4.1. *Let (A,α) and (B,β) be G-C^*-algebras with outer actions α and β. We assume that A and B are Kirchberg algebras. Then $H_G(A,B)$ is a group.*

The proof uses Theorem 2.3 and the fact that $(A \otimes \mathcal{O}_2, \alpha \otimes \mathrm{id})$ is conjugate to $(\mathcal{O}_\infty \otimes \mathcal{O}_2, \gamma \otimes \mathrm{id})$ with a quasi-free action γ (see [2]).

Note that there are two natural homomorphisms

$$\mu : H_G(A,B) \to H(A,B),$$

$$\nu : H_G(A,B) \to H(A \rtimes_\alpha G, B \rtimes_\beta G).$$

Theorem 4.2. *Let the notation be as above.*

(1) *If β has the Rohlin property, then μ is injective.*
(2) *If β is approximately representable, then ν is injective.*

Moreover, when the assumption of (1) (resp. (2)) is satisfied, the image of μ (resp. ν) can be completely characterized, so that $H_G(A,B)$ is computable if the algebras involved satisfy UCT.

We conjecture that if G-approximately unitary equivalence in the definition of $H_G(A,B)$ is replaced by G-asymptotically unitary equivalence, one would get the equivariant KK-group $KK_G(A,B)$, as it is the case for trivial G (see [5]).

References

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