Stationary patterns for a cooperative model with nonlinear diffusion

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1 Introduction

In this article we study positive steady-state solutions of the following strongly coupled reaction-diffusion system:

\[
\begin{aligned}
\begin{cases}
    u_t = \Delta \left[ \left( 1 + \frac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) & \text{in } \Omega \times (0, T), \\
    v_t = \Delta v + v(-b + du - v) & \text{in } \Omega \times (0, T), \\
    \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial \Omega \times (0, T), \\
    u(\cdot, 0) = u_0(\cdot), & v(\cdot, 0) = v_0(\cdot) & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \); \( \partial/\partial n \) denotes the outward normal derivative on \( \partial \Omega \); \( a, b, c, d, \mu \) are all positive constants; \( \alpha \) is a non-negative constant; \( u_0 \) and \( v_0 \) are given non-negative functions which are not identically zero. System (P) is a Lotka-Volterra cooperative model with a density-dependent diffusion term of a fractional type; unknown functions \( u \) and \( v \) represent population densities of two cooperative species, respectively; \( a \) and \( -b \) denote the intrinsic growth rates of the respective species; \( c \) and \( d \) denote interaction coefficients. When \( \alpha = 0 \), (P) is reduced to a classical Lotka-Volterra cooperative model with diffusion. See [6] and [13] for such a cooperative model.

In the first equation of (P), the nonlinear diffusion term \( \alpha \Delta \{u/(\mu + v)\} \) describes a situation where species \( u \) tends to leave low-density areas of species \( v \). This situation is natural because relations between \( u \) and \( v \) are cooperative. A population model with density-dependent diffusion was first proposed by Shigesada, Kawasaki and Teramoto [14] to investigate the habitat segregation phenomena between two competing species. Since their work, many mathematicians have studied population models with density-dependent diffusion. However, population models including
density-dependent diffusion terms of a fractional type have appeared in recent years; for example, see [5], [16] for cooperative models with Dirichlet boundary conditions; [2], [3] for prey-predator models with Dirichlet boundary conditions; [12], [15] for three-species prey-predator models with Neumann boundary conditions. See also the monograph of Okubo and Levin [11] for the biological background.

The stationary problem associated with (P) is

\[
\begin{cases}
\Delta \left[ \left( 1 + \frac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) = 0 \quad \text{in } \Omega, \\
\Delta v + v(-b + du - v) = 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

Our main purpose is to study the existence of stationary patterns (i.e. positive non-constant solutions) for (SP) with the weak cooperative condition

\[
\frac{a}{b} > \frac{1}{d} > c.
\]

From now on, we will always assume (1.1). It is well known that, if \( \alpha = 0 \), then every solution of (P) converges to a unique positive constant steady-state

\[
(u^*, v^*) := \left( \frac{a - bc}{1 - cd}, \frac{ad - b}{1 - cd} \right)
\]

uniformly as \( t \to \infty \); see [6]. This implies the following proposition.

**Proposition 1.1.** Let \( \alpha = 0 \). Then \((u^*, v^*)\) is a unique positive solution of (SP).

Proposition 1.1 means that no stationary pattern exists in the linear diffusion case. However, the presence of density-dependent diffusion enables us to construct stationary patterns of (SP). We focus on \( \alpha \) to show the emergence of stationary patterns for (SP).

Let \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \) denote eigenvalues of \(-\Delta\) with the homogeneous Neumann boundary condition on \( \partial \Omega \) and let \( m_i \) denote the algebraic multiplicity of \( \lambda_i \). Then we have the following theorem.

**Theorem 1.1.** Suppose that \( \{v^*(b - \mu)\}/(\mu + v^*) \in (\lambda_l, \lambda_{l+1}) \) for some \( l \geq 1 \) and that \( \sum_{i=1}^{l} m_i \) is odd. Then there exists a positive constant \( \alpha^* = \alpha^*(a, b, c, d, \mu) \) such that (SP) has at least one positive non-constant solution for each \( \alpha > \alpha^* \).

We are also interested in the limiting patterns of (SP) as \( \alpha \to \infty \). Under the restriction \( N \leq 3 \), we obtain the following limiting system as \( \alpha \to \infty \).
Theorem 1.2. Suppose $N \leq 3$ and $b > \mu$. Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ be any sequence such that $\lim_{i \to \infty} \alpha_i = \infty$ and positive functions $(u_i, v_i)$ satisfy (SP) with $\alpha = \alpha_i$. Then, by passing to a subsequence if necessary, it holds that

$$\lim_{i \to \infty} (u_i, v_i) = (\tau(\mu + \overline{v}), \overline{v})$$

in $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$, where $\tau$ is a positive constant satisfying $1 < d\tau < b/\mu$, $\overline{v}$ is a positive function in $\Omega$ and $(\tau, \overline{v})$ satisfies

$$\left\{ \begin{array}{ll} \Delta \overline{v} + \overline{v}\{-b + d\tau \mu + (d\tau - 1)\overline{v}\} = 0 & \text{in } \Omega, \\
\frac{\partial \overline{v}}{\partial n} = 0 & \text{on } \partial \Omega, \\
\int_{\Omega} (\mu + \overline{v})\{a - \tau \mu + (c - \tau)\overline{v}\} dx = 0. & \end{array} \right.$$

We expect that the limiting system (1.2) may give much information on profiles of stationary patterns of (SP) for large $\alpha$. We will give some remarks about (1.2) in the last section.

Throughout the article, the usual norms of $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are defined by

$$\|\psi\|_p := \left( \int_{\Omega} |\psi(x)|^p dx \right)^{1/p} \quad \text{and} \quad \|\psi\|_\infty := \max_{x \in \Omega} |\psi(x)|,$$

respectively.

2 Stability of the constant solution $(u^*, v^*)$

In this section, we will analyze the linearized stability of the constant stationary solution $(u^*, v^*)$ for (P).

The linearized eigenvalue problem of (P) at $(u^*, v^*)$ is given by

$$\left\{ \begin{array}{ll} -\left( 1 + \frac{\alpha}{\mu + v^*} \right) \Delta h + \frac{\alpha u^*}{(\mu + v^*)^2} \Delta k + u^* h - c u^* k = \eta h & \text{in } \Omega, \\
-\Delta k - d v^* h + v^* k = \eta k & \text{in } \Omega, \\
\frac{\partial h}{\partial n} = \frac{\partial k}{\partial n} = 0 & \text{on } \partial \Omega. \end{array} \right.$$

(2.1)

We know that $(u^*, v^*)$ is linearly stable when $\alpha = 0$. Using the expansions of $h$ and $k$ in terms of eigenfunctions of $-\Delta$, one can see that $\eta$ is an eigenvalue of (2.1) if and only if

$$\det \left( \begin{array}{cc} -\eta + \left( 1 + \frac{\alpha}{\mu + v^*} \right) \lambda_i + u^* & -\frac{\alpha u^*}{(\mu + v^*)^2} \lambda_i - c u^* \\
-dv^* & -\eta + \lambda_i + v^* \end{array} \right) = 0$$
for some $i \geq 0$. In particular, $\eta = 0$ is an eigenvalue of (2.1) if and only if
\[
\frac{\lambda_i}{(\mu + v^*)^2} \left( (\mu + v^*)(\lambda_i + v^*) - du^*v^* \right) \alpha + (\lambda_i + u^*)(\lambda_i + v^*) - cdv^*v^* = 0
\]
for some $i \geq 0$. Note that $(\lambda_i + u^*)(\lambda_i + v^*) - cdv^*v^* > 0$ for all $i \geq 0$ because of (1.1). Thus it is easy to see that the linearized stability of $(u^*, v^*)$ changes as $\alpha$ increases in (P) if and only if
\[
(\mu + v^*)(\lambda_1 + v^*) - du^*v^* = (\mu + v^*)\lambda_1 + v^*(\mu + v^* - du^*)
\]
\[
= (\mu + v^*)\lambda_1 + v^*(\mu - b)
\]
\[
< 0.
\]
Therefore, $b > \mu$ is necessary for the linearized stability of $(u^*, v^*)$ to change (and so we do not discuss the case $b \leq \mu$, especially, $-b \geq 0$). This means that the difference in the intrinsic growth rates between two species $u$ and $v$ contributes to creating stationary patterns in (SP).

3 Proof of Theorem 1.1

3.1 Reduction to the semilinear system

Our method of the proof of Theorem 1.1 will be based on the Leray-Schauder degree theory (see e.g., [9]) and some a priori estimates. We first introduce a new unknown function $U$ by
\[
U = \left( 1 + \frac{\alpha}{\mu + v} \right) u. \tag{3.1}
\]
Clearly, there exists a one-to-one correspondence between $(u, v) > 0$ and $(U, v) > 0$. As far as we discuss positive solutions, (SP) is rewritten in the following equivalent form:
\[
(EP) \begin{cases}
\Delta U + \frac{\mu + v}{\mu + v + \alpha} U \left( a - \frac{\mu + v}{\mu + v + \alpha} U + cv \right) = 0 \quad \text{in } \Omega, \\
\Delta v + v \left( b + d \frac{\mu + v}{\mu + v + \alpha} U - v \right) = 0 \quad \text{in } \Omega, \\
\frac{\partial U}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]
Thus, it is sufficient to show the existence of positive non-constant solutions of (EP).
3.2 A priori estimates

In this subsection, we will give some a priori estimates for positive solutions of (EP). Before stating the a priori estimates, we recall the following maximum principle due to Lou and Ni [7].

Lemma 3.1. Suppose that $g \in C(\overline{\Omega} \times \mathbb{R})$.

(i) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$
\Delta w(x) + g(x, w(x)) \geq 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} \leq 0 \quad \text{on } \partial \Omega,
$$

and $w(x_0) = \max_{\Omega} w$, then $g(x_0, w(x_0)) \geq 0$.

(ii) If $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$
\Delta w(x) + g(x, w(x)) \leq 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} \geq 0 \quad \text{on } \partial \Omega,
$$

and $w(x_0) = \min_{\Omega} w$, then $g(x_0, w(x_0)) \leq 0$.

Now we can derive the following a priori estimates.

Lemma 3.2. Let $\zeta$ be any fixed positive number. Then there exist two positive constants $C_*(\zeta) = C_*(\zeta, a, b, c, d, \mu) < C^*(\zeta) = C^*(\zeta, a, b, c, d, \mu)$ such that, if $\alpha \leq \zeta$, then any positive solution $(U, v)$ of (EP) satisfies

$$
a \leq U(x) \leq C^*(\zeta) \quad \text{and} \quad C_*(\zeta) \leq v(x) \leq C^*(\zeta) \quad \text{for all } x \in \overline{\Omega}.
$$

Proof. Let $U(x_0) = \max_{\Omega} U$ and $v(y_0) = \max_{\Omega} v$ with $x_0, y_0 \in \overline{\Omega}$. Applying Lemma 3.1 to (EP), we have

$$
\max_{\Omega} U \leq \frac{\mu + v(x_0) + \alpha}{\mu + v(x_0)} (a + cv(x_0))
$$

and

$$
\max_{\Omega} v \leq -b + d \frac{\mu + v(y_0)}{\mu + v(y_0) + \alpha} U(y_0) \leq -b + d \max_{\Omega} U. \tag{3.2}
$$

Thus

$$
\max_{\Omega} U \leq a + cv(x_0) + \zeta \frac{a + cv(x_0)}{\mu + v(x_0)}
$$

$$
\leq a + c(-b + d \max_{\Omega} U) + \zeta \max_{\Omega} \left\{ \frac{a}{\mu}, c \right\}.
$$

Therefore, we see

$$
\max_{\Omega} U \leq \frac{a - bc + \zeta \max\{a/\mu, c\}}{1 - cd}. \tag{3.3}
$$
It follows from (3.2) and (3.3) that
\[
\max_{\overline{\Omega}} v \leq -b + \frac{d(a - bc + \zeta \max\{a/\mu, c\})}{1 - cd} = \frac{ad - b + \zeta d \max\{a/\mu, c\}}{1 - cd} \tag{3.4}
\]
Hence we have obtained the desired upper bound of \((U, v)\).

Let \(U(z_0) = \min_{\overline{\Omega}} U\) with some \(z_0 \in \overline{\Omega}\). Using Lemma 3.1 to the first equation of (EP), we get
\[
\min_{\overline{\Omega}} U \geq \frac{\mu + v(z_0) + \alpha}{\mu + v(z_0)} (a + cv(z_0)) \geq a, \tag{3.5}
\]
Thus we have obtained the desired lower bound of \(U\).

Finally, we derive a lower bound of \(v\) by contradiction. Suppose that there exist a certain positive constant \(\zeta_0\) and a sequence \(\{(U_i, v_i, \alpha_i)\}_{i=1}^{\infty}\) such that \(\alpha_i \leq \zeta_0\) for all \(i \in \mathbb{N}\), \(\lim_{i \to \infty} \alpha_i = \alpha_\infty\) for some non-negative constant \(\alpha_\infty\),
\[
\lim_{i \to \infty} \min_{\overline{\Omega}} v_i = 0 \tag{3.6}
\]
and positive functions \((U_i, v_i)\) satisfy
\[
\begin{cases}
\Delta U_i + \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i \left( a - \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i + cv_i \right) = 0 \text{ in } \Omega, \\
\Delta v_i + v_i \left( -b + d \frac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) = 0 \text{ in } \Omega, \\
\frac{\partial U_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 \text{ on } \partial \Omega.
\end{cases} \tag{3.7}
\]
By using the regularity theory for elliptic equations (see e.g., [1]) to the second equation of (3.7), it follows from (3.3) and (3.4) that
\[
\|v_i\|_{W^{2,p}(\Omega)} \leq C(\zeta_0)
\]
with some positive constant \(C(\zeta_0) = C(\zeta_0, a, b, c, d, \mu)\) independent of \(i\). If \(p > N\), then Sobolev's embedding theorem implies \(\{v_i\}_{i=1}^{\infty}\) is compact in \(C^1(\overline{\Omega})\). Consequently, there exists a subsequence, which is still denoted by \(\{v_i\}_{i=1}^{\infty}\), such that
\[
\lim_{i \to \infty} v_i = v_\infty \text{ in } C^1(\overline{\Omega}) \tag{3.8}
\]
with some non-negative function \(v_\infty \in C^1(\overline{\Omega})\). Similarly, there exists a non-negative function \(U_\infty \in C^1(\overline{\Omega})\) such that
\[
\lim_{i \to \infty} U_i = U_\infty \text{ in } C^1(\overline{\Omega}) \tag{3.9}
\]
Therefore, \(v_\infty\) satisfies
\[
\Delta v_\infty + v_\infty \left( -b + d \frac{\mu + v_\infty}{\mu + v_\infty + \alpha_\infty} U_\infty - v_\infty \right) = 0 \text{ in } \Omega, \quad \frac{\partial v_\infty}{\partial n} = 0 \text{ on } \partial \Omega
\]
in a weak sense. By standard elliptic regularity theory we have \( v_\infty \in C^2(\overline{\Omega}) \) and thus \( v_\infty \) is a classical solution of the above equation. Then it follows from (3.6),(3.8) and the strong maximum principle that \( v_\infty \equiv 0 \) in \( \overline{\Omega} \). We can easily see from the above argument that \( U_\infty \) satisfies

\[
\Delta U_\infty + \frac{\mu}{\mu + \alpha_\infty} U_\infty \left( a - \frac{\mu}{\mu + \alpha_\infty} U_\infty \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial U_\infty}{\partial n} = 0 \quad \text{on } \partial \Omega
\]

in the classical sense. Then by the strong maximum principle and Lemma 3.1, either \( U_\infty \equiv a(\mu + \alpha_\infty)/\mu \) or \( U_\infty \equiv 0 \) in \( \overline{\Omega} \). Combining (3.5) and (3.9), we can conclude \( U_\infty \equiv a(\mu + \alpha_\infty)/\mu \) in \( \overline{\Omega} \). Hence

\[
\lim_{i \to \infty} \left( -b + d \frac{-\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) = ad - b > 0 \quad \text{uniformly in } \Omega
\]

by (1.1) and this means

\[
v_i \left( -b + d \frac{-\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) > 0 \quad \text{in } \Omega
\]

for sufficiently large \( i \in \mathbb{N} \) because \( v_i > 0 \) in \( \Omega \). On the other hand, from the second equation of (3.7), we have

\[
\int_{\Omega} v_i \left( -b + d \frac{-\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i \right) dx = - \int_{\Omega} \Delta v_i dx = - \int_{\partial \Omega} \frac{\partial v_i}{\partial n} d\sigma = 0
\]

for all \( i \in \mathbb{N} \). This is a contradiction; thus our proof is complete. \( \square \)

### 3.3 Completion of the proof of Theorem 1.1

Set \( X = C(\overline{\Omega}) \times C(\overline{\Omega}) \). For each \( \alpha \geq 0 \), define an operator \( F_\alpha \) by

\[
F_\alpha \begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} (-\Delta + I)^{-1} \left[ U + \frac{\mu + v}{\mu + v + \alpha} U \left( a - \frac{\mu + v}{\mu + v + \alpha} U + cv \right) \right] \\ (-\Delta + I)^{-1} \left[ v + v \left( -b + d \frac{\mu + v}{\mu + v + \alpha} U - v \right) \right] \end{pmatrix}
\]

where \( I \) is the identity map from \( C(\overline{\Omega}) \) into itself, and \( (-\Delta + I)^{-1} \) is the inverse operator of \(-\Delta + I\) subject to the homogeneous Neumann boundary condition on \( \partial \Omega \). It is easy to see that \( F_\alpha : X \to X \) is well-defined, and that by elliptic regularity theory and Sobolev’s embedding theorem, \( F_\alpha \) is a continuous and compact operator for each \( \alpha \geq 0 \). From these observations, one can define the Leray-Schauder degree of \( I - F_\alpha \) at 0 in a suitable open set. Furthermore, \( (U, v) \) is a positive solution of \( (I - F_\alpha)(U, v) = 0 \) if and only if \( (U, v) \) is a positive solution of (EP).

In view of (3.1), we set

\[
U^*_\alpha = \left( 1 + \frac{\alpha}{\mu + v^*} \right) u^*.
\]
Hence \((U_\alpha^*, v^*)\) is a zero point of \(I-F_\alpha\). Then we can calculate the index of \(I-F_0\) at \((u^*, v^*)\) and the index of \(I-F_\alpha\) at \((U_\alpha^*, v^*)\) for sufficiently large \(\alpha\), which are denoted by \(\text{index}(I-F_0, (u^*, v^*))\) and \(\text{index}(I-F_\alpha, (U_\alpha^*, v^*))\), respectively. We refer to [10] for the proofs of Lemmas 3.3 and 3.4.

**Lemma 3.3.** It holds that \(\text{index}(I-F_0, (u^*, v^*)) = 1\).

**Lemma 3.4.** Suppose that \(\{v^*(b-\mu)/(\mu + v^*)\} \in (\lambda_l, \lambda_{l+1})\) for some \(l \geq 1\). Then there exists a positive constant \(\alpha^* = \alpha^*(a, b, c, d, \mu)\) such that, if \(\alpha > \alpha^*\), then

\[
\text{index}(I-F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^l m_i},
\]

where \(m_i\) denotes the algebraic multiplicity of \(\lambda_i\) defined in Section 1.

By virtue of Lemmas 3.3 and 3.4, we are ready to prove Theorem 1.1. In the proof of Theorem 1.1, we represent \((EP)\) as \((EP)_\alpha\) to indicate the dependence on \(\alpha\).

**Proof of Theorem 1.1.** Fix any \(\alpha > \alpha^*\), where \(\alpha^*\) is a constant given in Lemma 3.4. It follows from Lemma 3.2 that there exist two positive constants \(C_*(\alpha) = C_*(\alpha, a, b, c, d, \mu) < C^*(\alpha) = C^*(\alpha, a, b, c, d, \mu)\) such that

\[
a \leq U(x) \leq C^*(\alpha) \quad \text{and} \quad C_*(\alpha) \leq v(x) \leq C^*(\alpha)
\]

for all \(x \in \overline{\Omega}\).

We define

\[
S = \left\{(U, v) \in X \bigg| \frac{a}{2} \leq U \leq 2C^*(\alpha), \quad \frac{C_*(\alpha)}{2} \leq v \leq 2C^*(\alpha) \quad \text{in} \quad \overline{\Omega} \right\};
\]

so that \(I-F_\nu\) has no zero point on the boundary of \(S\) for any \(\nu \in [0, \alpha]\). Note that \(I-F_0\) has a unique zero point \((u^*, v^*)\) in \(S\). On account of the homotopy invariance of the Leray-Schauder degree and Lemma 3.3, we have

\[
\text{deg}(I-F_\alpha, S, 0) = \text{deg}(I-F_0, S, 0) = \text{index}(I-F_0, (u^*, v^*)) = 1.
\]

Suppose that \((EP)_\alpha\) has no positive non-constant solution, i.e. \(I-F_\alpha\) has a unique zero point \((U_\alpha^*, v^*)\) in \(S\). Then from the assumption \(\sum_{i=1}^l m_i\) being odd and Lemma 3.4, it follows that

\[
\text{deg}(I-F_\alpha, S, 0) = \text{index}(I-F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^l m_i} = -1,
\]

which contradicts (3.10). Thus we complete the proof. \(\square\)
4 Proof of Theorem 1.2

We first state some a priori estimates independent of $\alpha$.

**Lemma 4.1.** Suppose that $N \leq 3$. Then there exists a positive constant $C_0 = C_0(a, b, c, d, \mu)$ independent of $\alpha$ such that any positive solution $(u, v)$ of (SP) satisfies

$$\|u\|_{\infty} \leq C_0 \quad \text{and} \quad \|v\|_{\infty} \leq C_0.$$

Lemma 4.1 can be proved by combining the $L^2$-estimates for positive solutions of (SP) (independent of $\alpha$ and $N$) with Harnack inequality (due to Lin, Ni and Takagi [4], and Lou and Ni [8]). We refer to [10] for the proof of Lemma 4.1.

We are now in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ be any sequence such that $\lim_{i \to \infty} \alpha_i = \infty$ and positive functions $(u_i, v_i)$ satisfy (SP) with $\alpha = \alpha_i$. Set

$$\psi_i = \left(\frac{1}{\alpha_i} + \frac{1}{\mu + v_i}\right) u_i.$$

Note that positive functions $(\psi_i, v_i)$ satisfy

$$\begin{cases}
\Delta \psi_i + \frac{u_i(a - u_i + cv_i)}{\alpha_i} = 0 \quad \text{in} \quad \Omega, \\
\Delta v_i + v_i(-b + du_i - v_i) = 0 \quad \text{in} \quad \Omega, \\
\frac{\partial \psi_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}$$

and that $\{\psi_i\}_{i=1}^{\infty}$ is bounded independently of $i$ by Lemma 4.1. Then by the compactness argument as in the proof of (3.8), there exists a subsequence, which is still denoted by $\{\psi_i\}_{i=1}^{\infty}$, such that

$$\lim_{i \to \infty} \psi_i = \tau \quad \text{in} \quad C^1(\bar{\Omega})$$

for a non-negative function $\tau \in C^1(\bar{\Omega})$. Similarly, we see

$$\lim_{i \to \infty} v_i = \bar{v} \quad \text{in} \quad C^1(\bar{\Omega}) \quad (4.1)$$

for a non-negative function $\bar{v} \in C^1(\bar{\Omega})$. Therefore, we obtain

$$\lim_{i \to \infty} u_i = \lim_{i \to \infty} \frac{\psi_i}{1/\alpha_i + 1/(\mu + v_i)} = \tau(\mu + \bar{v}) \quad \text{in} \quad C^1(\bar{\Omega}). \quad (4.2)$$
We will show that \( \tau \) is a positive constant. Observe that \( \tau \) satisfies

\[
\Delta \tau = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \tau}{\partial n} = 0 \quad \text{on} \quad \partial \Omega
\]

in a weak sense. A standard elliptic regularity theory ensures \( \tau \in C^2(\bar{\Omega}) \); so that \( \tau \) must be a non-negative constant. Let \( v_i(x_i) = \max_{\Omega} v_i \) with some \( x_i \in \bar{\Omega} \). It follows from Lemma 3.1 that

\[
u_i(x_i) \geq \frac{b + v_i(x_i)}{d} > \frac{b}{d} (> 0)
\]

for all \( i \in \mathbb{N} \). This fact, together with (4.2), yields \( \tau > 0 \).

We next prove \((\tau, \bar{v})\) satisfies (1.2). Note that \( \bar{v} \) satisfies

\[
\Delta \bar{v} + \bar{v}\{-b + d \tau \mu + (d \tau - 1) \bar{v}\} = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \bar{v}}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \quad (4.3)
\]

in a weak sense. In the standard manner, one can see that \( \bar{v} \in C^2(\bar{\Omega}) \) and \( \bar{v} \) is a classical nonnegative solution of (4.3). It follows from the strong maximum principle that either \( \bar{v} \equiv 0 \) or \( \bar{v} > 0 \) in \( \Omega \). We show \( \bar{v} > 0 \) in \( \Omega \) by contradiction. Suppose that \( \bar{v} \equiv 0 \) in \( \Omega \). Then it follows from (4.1) and (4.2) that

\[
\lim_{i \to \infty} a - u_i + cv_i = a - \tau \mu \quad \text{and} \quad \lim_{i \to \infty} b + du_i - v_i = b + d \tau \mu
\]

uniformly in \( \Omega \). On the other hand,

\[
\int_{\Omega} u_i(a - u_i + cv_i)dx = \int_{\Omega} v_i(-b + du_i - v_i)dx = 0 \quad (4.4)
\]

for all \( i \in \mathbb{N} \). Consequently, \( a - \tau \mu = -b + d \tau \mu = 0 \) because of \( u_i > 0 \) and \( v_i > 0 \) in \( \Omega \) and thus \( ad - b = 0 \). This contradicts (1.1). Therefore \( \bar{v} > 0 \) in \( \Omega \).

By (4.1), (4.2) and (4.4), it is clear that

\[
\int_{\Omega} (\mu + \bar{v})\{a - \tau \mu + (c - \tau) \bar{v}\}dx = \int_{\Omega} (\mu + \bar{v})\{a - \tau (\mu + \bar{v}) + c \bar{v}\}dx = 0.
\]

Hence it only remains to show \( 1 < d \tau < b/\mu \). By the assumption of Theorem 1.2,

\[
-b + d \tau \mu < -\mu + d \tau \mu = \mu (d \tau - 1).
\]

It thus follows from Lemma 3.1 and (4.3) that if \( d \tau - 1 \leq 0 \), then \( \max_{\Omega} \bar{v} \leq 0 \) and this contradicts \( \bar{v} > 0 \) in \( \Omega \). Therefore, \( d \tau > 1 \). Using Lemma 3.1 and \( \bar{v} > 0 \) in \( \Omega \) again, we obtain \( d \tau < b/\mu \). Hence we complete the proof. \( \square \)
5 Remarks about the limiting system (1.2)

We easily see that \((\tau, \overline{v}) = (u^*/(\mu + v^*), v^*)\) is the only positive constant solution of (1.2). So our concern is about positive non-constant solutions of (1.2). We discuss the differential equations without the integral constraint in (1.2) under the restriction \(N \leq 3\):

\[
\begin{cases}
\Delta \overline{v} + \overline{v}\{-b + d\tau \mu + (d\tau - 1)\overline{v}\} = 0 & \text{in } \Omega, \\
\frac{\partial \overline{v}}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(5.1)

Set

\[w = \frac{d\tau - 1}{b - d\tau \mu} \overline{v},\]

where \(1 < d\tau < b/\mu\). Then (5.1) is rewritten in the following equivalent form:

\[
\begin{cases}
\frac{1}{b - d\tau \mu} \Delta w - w + w^2 = 0 & \text{in } \Omega, \\
\frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(5.2)

We note that, if \((0 <) b - d\tau \mu \ll 1\), then (5.2) has no positive non-constant solution (see [4]). Therefore, \(b \gg 1\) is necessary for (1.2) to have positive non-constant solutions. We will study (1.2) in detail in the future.

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References


