Blow-up at space infinity
for nonlinear heat equations

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1 Introduction and main theorems

In this paper we gather the papers [5], [6] and [12] for our talk at Kyoto University. In particular we make the proofs of theorems in [5] easier by using the methods in [12] and other.

We consider solutions of the initial value problem for the equation

\[
\begin{cases}
    u_t = \Delta u + f(u), & x \in \mathbb{R}^n, t > 0, \\
    u(x, 0) = u_0(x), & x \in \mathbb{R}^n.
\end{cases}
\]

(1)

The nonlinear term \( f \in C^1(\mathbb{R}_+) \) satisfies that

\[
\int_{C}^{\infty} \frac{d\xi}{f(\xi)} < \infty \text{ with some } C \geq 0,
\]

(2)

and

\[
\begin{cases}
    \text{there exists a function } \Phi \in C^2(\mathbb{R}_+) \text{ such that } \\
    \Phi(v) > 0, \Phi'(v) > 0 \text{ and } \Phi''(v) \geq 0 \text{ for } v > 0, \\
    \int_{1}^{\infty} \frac{d\xi}{\Phi(\xi)} < \infty, \\
    \text{and } f'(v)\Phi(v) - f(v)\Phi'(v) \geq c\Phi(v)\Phi'(v) \text{ for } v > b \\
    \text{with some } b \geq 0 \text{ and } c \geq 0.
\end{cases}
\]

(3)

Remark. The conditions (2) and (3) were used in [12]. They are weaker than the conditions used in [5] and [6]:

\[
f(\delta b) \leq \delta^p f(b)
\]

for all \( b \geq b_0 \) and for all \( \delta \in (\delta_0, 1) \) with some \( b_0 > 0 \), some \( \delta_0 \in (0, 1) \) and some \( p > 1 \).
The initial data $u_0$ is assumed to be a measureable function in $\mathbb{R}^n$ satisfying

$$0 \leq u_0(x) \leq M \text{ a.e. in } \mathbb{R}^n$$

for some positive $M$. We are interested in initial data such that $u_0 \to M$ as $|x| \to \infty$ for $x$ in some sector of $\mathbb{R}^n$. We assume that there exists a sequence $\{x\}_{m=1}^\infty \subset \mathbb{R}^n$ such that

$$\lim_{m \to \infty} u_0(x + x_m) = M \text{ a.e. in } \mathbb{R}^n.$$  \hfill (5)

**Remark.** The condition (5) was given in [12]. This condition is equivalent to the condition in [5] with [6]:

$$\text{essinf}_{x \in \tilde{B}_m}(u_0(x) - M_m(x - x_m)) \geq 0 \text{ for } m = 1, 2, \ldots,$$

where $\tilde{B}_m = B_{r_m}(x_m)$ with a sequence $\{r_m\}_{m=1}^\infty$, a sequence of functions $\{M_m(x)\}_{m=1}^\infty$ satisfying

$$\lim_{m \to \infty} r_m = \infty, \quad M_m(x) \leq M_{m+1}(x) \text{ for } m \geq 1$$

and some sequence of vectors $\{x_m\}_{m=1}^\infty$. Here $B_r(x)$ denotes the opened ball of radius $r$ centered at $x$.

Problem (1) has a unique bounded solution at least locally in time. However, the solution may blow up in finite time. For a given initial value $u_0$ and nonlinear term $f$ let $T^* = T^*(u_0, f)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, we say that the solution blows up in finite time. It is well known that

$$\limsup_{t \to T^*} \|u(\cdot, t)\|_\infty = \infty,$$  \hfill (6)

where $\| \cdot \|_\infty$ denotes the $L^\infty$-norm in space variables.

In this paper we are interested in behavior of a blowing up solution near space infinity as well as location of blow-up directions defined below. A point $x_{BU} \in \mathbb{R}^n$ is called a blow-up point if there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ such that

$$t_m \uparrow T^*, \quad x_m \to x_{BU} \quad \text{and} \quad u(x_m, t_m) \to \infty \quad \text{as} \quad m \to \infty.$$
If there exists a sequence \( \{(x_m, t_m)\}_{m=1}^{\infty} \) such that
\[
t_m \uparrow T^*, \quad |x_m| \to \infty \quad \text{and} \quad u(x_m, t_m) \to \infty \quad \text{as} \quad m \to \infty,
\]
then we say that the solution blows up to at space infinity.

A direction \( \psi \in S^{n-1} \) is called a blow-up direction if there exists a sequence \( \{(x_m, t_m)\}_{m=1}^{\infty} \) with \( x_m \in \mathbb{R}^n \) and \( t_m \in (0, T^*) \) such that \( u(x_m, t_m) \to \infty \) as \( m \to \infty \) and
\[
\frac{x_m}{|x_m|} \to \psi \quad \text{as} \quad m \to \infty.
\] (7)

We consider the solution \( v(t) \) of an ordinary differential equation
\[
\begin{cases}
    v_t = f(v), & t > 0, \\
    v(0) = M.
\end{cases}
\] (8)

Let \( T_v = T^*(M, f) \) be the maximal existence time of solutions of (8), i. e.,
\[
T_v = \int_{M}^{\infty} \frac{ds}{f(s)}.
\]

We are now in position to state our main results.

**Theorem 1.** Assume that \( f \in C^1(\mathbb{R}_+) \) is nondecreasing function and locally Lipschitz in \( \mathbb{R}_+ \). Let \( u_0 \) be a continuous function satisfying (4) and (5). Then there exists a subsequence of \( \{x_m\}_{m=1}^{\infty} \), independent of \( t \) such that
\[
\lim_{m \to \infty} u(x + x_m, t) = v(t) \quad \text{in} \quad \mathbb{R}^n.
\] (9)

The convergence is uniform in every compact subset of \( \mathbb{R}^n \times [0, T_0) \). Moreover, the solution blows up at \( T_v \).

For this theorem we should introduce the results of Gladkov [7]. In his paper there is the result [7, Theorem 1] relative to our first theorem. He considered the initial-boundary value problem:
\[
\begin{cases}
    u_t = u_{xx} + f(x, t, u), & x > 0, 0 < t < T_0, \\
    u(x, 0) = u_0(x), & x > 0, \\
    u(0, t) = \mu(t), & 0 < t < T_0,
\end{cases}
\]
and the ordinary differential equation
\[
\begin{cases}
    v_t = \tilde{f}(t, u), & 0 < t < T_0, \\
    v(0) = M,
\end{cases}
\]
where $T_0 \in (0, \infty]$, $0 \leq f(x, t, u) \leq \tilde{f}(t, u)$, \( \lim_{x \to \infty} f(x, t, u) = \tilde{f}(t, u), \) $0 \leq u_0 \leq M$ and \( \lim_{x \to \infty} u_0(x) = M \). For the equations he had $u(x, t) \to v(t)$ as $x \to \infty$ uniformly for $[0, T]$ with $T < T_0$. For the proof of this result, he used the fundamental solution of the heat equation.

In [5] the expression (9) was the weak sense:

$$\lim_{n \to \infty} u(x_m, t) = v(t).$$

(10)

After [5], (9) was used in [12]. However, for proving Theorems 2 and 3, we can select even the expression (10).

Our second main result is on the location of blow-up points.

**Theorem 2.** Assume the same hypotheses of Theorem 1 and that $f$ satisfies (2) and (3). Let $u_0 \neq M$ a.e. in $\mathbb{R}^n$. Then the solution of (1) has no blow-up points with $\infty$ in $\mathbb{R}^n$. (It blows up only at space infinity.)

There is a huge literature on location of blow-up points since the work of Weissler [15] and Friedman-McLeod [1]. (We do not intend to list references exhaustively in this paper.) However, most results consider either bounded domains or solutions decaying at space infinity; such a solution does not blow up at space infinity [2].

As far as the authors know, before the result of [4] the only paper discussing blow-up at space infinity is the work of Lacey [8]. He considered the Dirichlet problem in a half line. He studied various nonlinear terms and proved that a solution blows up only at space infinity. His method is based on construction of suitable subsolutions and supersolutions. However, the construction heavily depends on the Dirichlet condition at $x = 0$ and does not apply to the Cauchy problem even for the case $n = 1$.

As previously described, the Giga-Umeda [4] proved the statement of Theorems 1 and 2 assuming that \( \lim_{|x| \to \infty} u_0(x) = M \) for positive solutions of \( u_t = \Delta u + u^p \). Later, Simojö[13] had the same results as in [4] by relaxing the assumptions of initial data $u_0 \geq 0$ which is similar to that in the present paper. His approach is a construction of a suitable supersolution which implies that $a \in \mathbb{R}^n$ is not a blow-up point. Although he restricted himself for \( f(s) = s^p \), his idea works our $f$ under slightly strong assumption on $u_0$. Here we give a different approach.

By Simojö's results[13] it is natural to consider a problem of "blow-up direction" defined in (7). We next study this "blow-up direction" for the value $\infty$.

**Theorem 3.** Assume the same hypotheses of Theorem 1. Let a direction $\psi \in S^{n-1}$. If and only if there exists sequences $\{y_m\}_{m=1}^{\infty}$ and satisfying
\[ \lim_{m \to \infty} \frac{y_m}{|y_m|} = \psi \text{ such that} \]
\[ \lim_{m \to \infty} u_0(x + y_m) = M \; \text{a.e. in} \; \mathbb{R}^n, \]

then \( \psi \) is a blow-up direction.

After [5] there are some results in this field. Shimojō had the result of the upperbound and the lowerbound:
\[ v(t - \eta(x, t)) \leq u(x, t) \leq v(t - c\eta(x, t)) \]
with some function \( \eta \) and \( c \in (0, 1) \). Moreover, he proved the complete blow-up of the solution. Seki-Suzuki-Umeda [12] and Seki [11] improved the results of [5] for the quasilinear parabolic equation:
\[ u_t = \Delta \varphi(u) + f(u). \]

In particular they had more results for more general case. In [3] some of the proofs of theorems in [5] were corrected.

This paper is organized as follows. In section 2 we prove Theorem 1 by using the fundamental solution of the heat equation. The proof of Theorem 2 is given in section 3 by using the argument used in [12]. In section 4 we show Theorem 3 using Theorem 1 and Lemma 3.2.

2 Behavior at space infinity

In this section we prove Theorem 1. We give proof of Theorem 1 which is inspired in private communication with Y. Seki and M. Shimojō.

Proof of Theorem 1. Put \( w = v - x \). Then, we have for \( t \in (0, T_0] \) with \( T_0 \in (0, T(M)) \),
\[ w_t = \Delta w + f(v(t)) - f(u(\cdot, t)) \leq \Delta w + C(v - u), \]
where
\[ C = \sup_{t \in [0, T_0]} \left\| \int_0^1 f'(\theta v(t) + (1 - \theta)u(\cdot, t))d\theta \right\|_\infty. \]

Then, by comparison we obtain
\[ w(x, t) \leq e^{Ct_0}e^{\Delta t}(M - u_0(x)) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t}(M - u_0(y))dy. \]
From (5) we have

$$\lim_{m \to \infty} u(x + x_m, t) = v(t) \quad \text{in } \mathbb{R}^n.$$  \hspace{1cm} (12)

It remains to prove that $u$ blows up at $t = T_v$. For this purpose it suffices to prove that $\lim_{m \to \infty} u(x_m, t_m) = \infty$ for some sequence $t_m \to T_v$. We argue by contradiction. Suppose that $\lim_{m \to \infty} u(x_m, t_m) \leq C$ for some $C \in [M, \infty)$. Then we could take $t_0 \in (0, T_v)$ satisfying $v(t_0) \geq C$ and $v_t(t) > 0$ for $t \geq t_0$. By (12) we have

$$\lim_{m \to \infty} u \left( x_m, \frac{t_0 + T_v}{2} \right) = v \left( \frac{t_0 + T_v}{2} \right) > C,$$

which yields a contradiction. We thus proved that $\lim_{m \to \infty} u(x_m, t_m) = \infty$, so that $u(x, t)$ blows up at $T_v$. \hfill \Box

### 3 No blow-up point in $\mathbb{R}^n$

In this section we prove Theorem 2. We use three lemmas for proving the theorem.

**Lemma 3.1.** Assume the same hypothesis of Theorem 1. Let $u$ and $v$ be solutions of (1) and (8) with $u_0$, $M$ and $f$ satisfying (2), (3) and (4). Then there exist $\delta = \delta(a, T_0, u_0, f) \in (0, 1)$ such that for $(x, t) \in B_1(a) \times [t_0, T_v)$,

$$u(x, t) \leq \delta v(t)$$

with $t_0 \in [0, T_v)$.

**Proof.** By (2) there exist $M_f = M_f(f) > M$ and $\delta_f = \delta_f(f) \in (0, 1)$ satisfying for $r \geq M_f$ and $\delta \in (\delta_f, 1)$,

$$f(\delta r) \leq \delta f(r).$$  \hspace{1cm} (13)

Let $T_0 = T_0(u_0, f) \in (0, T_v)$ such that $v(T_0) = M_f$. Since $u_0 \leq M$ and $u_0 \not\equiv M$ a.e. in $\mathbb{R}^n$, we have $u(x, T_0) < v(T_0)$. Note that $u(x, t) < v(t)$ for $t \in (0, T_0]$. Let $w$ be the solution of

$$\begin{cases}
  w_t = \Delta w, \\
  w(x, T_0) = \max\{u(x, T_0)/v(T_0), \delta_f\},
\end{cases} \quad x \in \mathbb{R}^n, t \in (T_0, T^*),$$

Put $\bar{u} = vw$. Then we have

$$\begin{cases}
  \bar{u}_t = \Delta \bar{u} + w f(v), \\
  \bar{u}(x, T_0) = \max\{u(x, T_0), \delta_f v(T_0)\},
\end{cases} \quad x \in \mathbb{R}^n, t \in (T_0, T^*).$$
Since \( w(x, t) \in [\delta_f, 1) \) and \( v(t) \geq M_f \), we have

\[
wf(v) \geq f(wv) = f(\bar{u})
\]

by (13). This \( \bar{u} \) is supersolution of (1).

Since for any \( x \in \mathbb{R}^n \), \( \sup_{t \in [T_0, T^*]} w(x, t) < 1 \), we can take \( \delta = \delta(a, T_0, u_0, f) \in (0, 1) \) satisfying \( w(x, t) \leq \delta \) for \( (x, t) \in B_1(a) \times [T_0, T_v) \). Thus, we obtain

\[
u(x, t) \leq \bar{u}(x, t) = w(x, t)v(t) \leq \delta v(t)
\]

and Lemma 3.1 is proved.

For any \( a \in \mathbb{R}^n \), we consider the solution \( \phi = \phi_a \) of the equation:

\[
\begin{cases}
\phi_t = \Delta \phi + f(\phi), & x \in B_1, t \in (t_1, T_v), \\
\phi(x, 0) = \phi_0(x), & x \in B_1, \\
\phi(x, t) = v(t), & x \in \partial B_1, t \in (t_1, T_v),
\end{cases}
\]

(14)

where \( \phi_0(x) = v(t_1)(1 - \varepsilon \cos \frac{\pi|x|}{2}) \) with \( \varepsilon = \varepsilon(u_0, f, a) > 0 \) sufficiently small satisfying

\[
\phi_0(x) \geq u(x+a, t_1)
\]

and \( B_1 \) denotes the open ball of radius 1 and centered at 0. It is easily seen that

\[
\Delta \phi_0(x) + f(\phi_0(x)) \geq 0.
\]

By the maximum principle [10] we have

\[
\phi(x, t) \geq u(x+a, t) \quad \text{and} \quad \phi_t \geq 0 \quad \text{for} \quad x \in \bar{B}_1, \ t \in [t_1, T_v).
\]

(16)

If \( w \) has no blow-up point in \( \mathbb{R}^n \), the \( u \) has no blow-up point in \( \mathbb{R}^n \), neither. We should show that \( w \) has no blow-up point.

**Lemma 3.2.** Assume the same hypotheses of Lemma 3.1. Let \( \Omega \subset B_1 \) be a domain. If \( \partial_t \phi(x, t) \geq 0 \) in \( \Omega \times (t_1, T_v) \) and there exist \( \nu \in S^{n-1} \) and \( \delta > 0 \), such that

\[
\nu \cdot \nabla \phi(x, t) \leq -\delta |\nabla \phi(x, t)| < 0 \quad \text{in} \quad \Omega \times (t_1, T_v),
\]

then \( \phi \) does not uniformly blow-up in \( \Omega \):

\[
\inf_{x \in \Omega} \phi(x, t) \leq L < \infty \quad \text{for} \quad t \in (t_1, T_v).
\]
Proof of Lemma 3.2. This lemma is proved in [9] (See [9, Lemma 4.1]).

Proof of Theorem 2. Put \( r \in (0, 1) \). Define
\[
\mu(x, t) = \phi(2r - x_1, x', t) - \phi(x_1, x', t),
\]
where \( x = (x_1, x') \) with \( x' = (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1} \). Then, we obtain
\[
\begin{cases}
\mu_t \geq \Delta \mu + C(x, t) \mu, & x \in D_r, t \in (t_1, T_v), \\
\mu(x, 0) = \phi_0(2r - x_1, x') - \phi_0(x_1, x') \geq 0, & x \in D_r, \\
\mu(x, t) \geq 0, & x \in \partial D_r, t \in (t_1, T_v),
\end{cases}
\]
where
\[
C(x, t) = \int_0^1 \{\theta \phi(2r - x_1, x', t) + (1 - \theta) \phi(x_1, x', t)\} d\theta,
\]
\[
D_r = \{x : x_1 < r\} \cap \{x : (x - 2r)^2 < 1\}.
\]

Thus, by the maximum principle [10] we have
\[
\mu \geq 0 \quad \text{in} \quad D \times [t_1, T_v)
\]
and
\[
\phi(2r - x_1, x', t) \geq \phi(x_1, x', t) \quad \text{in} \quad D \times [t_1, T_v).
\]
Since \( r \in (0, 1) \) is arbitrary, we obtain that \( \phi_{x_1} \geq 0 \) for \( x \in \{x|x_1 > 0\} \) and
\[
-e_1 \cdot \nabla \phi \leq -\phi_{x_1} \leq -\frac{\delta x_1}{|x|} |\nabla \phi|, \quad \text{in} \quad D \cup \{x|x_1 \geq 0\}
\]
with some \( \delta > 0 \), where \( e_1 = (1, 0, 0, \ldots, 0) \). Since \( \phi_t \geq 0 \) and \( \inf_{x \in B_1} \phi(x, t) = \phi(0, t) \), by Lemma 3.2 we have
\[
\lim_{t \to T_v} \phi(0, t) \leq L \quad \text{with some} \quad L < \infty.
\]
Thus
\[
\lim_{t \to T_v} u(a, t) \leq L \quad \text{with same} \quad L.
\]
Since \( a \in \mathbb{R}^n \) is arbitrary, \( u \) does not blow up at \( t = T_v \) in \( \mathbb{R}^n \).
4 On blow-up direction

We shall prove Theorem 3 which gives a condition for blow-up direction.

Proof of Theorem 3. We first prove that if $u_0$ satisfies (11), then $\psi$ is a blow-up direction. By assumption we obtain that $u_0(x)$ satisfies (5) with some sequences $\{x_m\}_{m=1}^{\infty}$ satisfying $\lim_{m \to \infty} x_m/|x_m| = \psi$. Then, from the proof of Theorem 1 it follows that

$$\lim_{m \to \infty} u(x_m, t_m) = \infty$$

with the sequence $\{t_m\}_{m=1}^{\infty}$ satisfying $\lim_{m \to \infty} t_m = T_v$. Since $\lim_{m \to \infty} x_m/|x_m| = \psi$ by the assumption we obtain that $\psi$ is a blow-up direction.

We next show that if $\psi$ is a blow-up direction, then there exist $\{x_m\}_{m=0}^{\infty} \subset \mathbb{R}^n$ such that $x_m/|x_m| \to \psi$, $t_m \to T_v$ and $u(x_m, t_m) \to \infty$ as $m \to \infty$. In contrary it says that if for any sequences $\{x_m\}_{m=1}^{\infty} \subset \mathbb{R}^n$ satisfying $\lim_{m \to \infty} x_m/|x_m| = \psi$, $u_0$ does not satisfy (11), then $\psi$ is not a blow-up direction.

Since $\lim_{m \to \infty} u_0(x + x_m) = M$ a.e. in $\mathbb{R}^n$, we have

$$\lim_{m \to \infty} \sup_{x \in B_3(x_m)} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-(x-y)^2/4t} u_0(y) dy < M \quad (17)$$

for $t > 0$. Since the solution of (1) satisfies the integral equation

$$u(x, t) = e^{\Delta t} u_0(x) + \int_0^t e^{\Delta(t-s)} f(u(x, s)) ds,$$

we have

$$u(x, t) \leq e^{\Delta t} u_0(x) + \int_0^t f(v(s)) ds = v(t) - M + e^{\Delta t} u_0(x)$$

for $(x, t) \in \mathbb{R}^n \times [0, T^*)$.

Let $M_f$, $\delta_f$ and $T_0$ be the same as proof of Lemma 3.1. We consider the solution $w$ of

$$\begin{cases}
\frac{\partial w}{\partial t} = \Delta w, \\
\w(x, T_0) = \max\{v(T_0) - M + e^{\Delta T_0} u_0(x)\}/v(T_0), \delta_f, \quad x \in \mathbb{R}^n, t \in (T_0, T_v),
\end{cases}$$

We now introduce $\tilde{u} = vw$. From the proof of Lemma 3.1, it follows that $\tilde{u} \geq u$ for $(x, t) \in \mathbb{R}^n \times [T_0, T^*)$. Then we have

$$u(x, t) \leq v(t) e^{\Delta(t-T_0)} \max\{v(T_0) - M + e^{\Delta T_0} u_0(x)\}/v(T_0), \delta_f\}$$
for \((x, t) \in \mathbb{R}^n \times [T_0, T_v)\).

Put \(U_m = \sup_{x \in B_2(x_m)} e^{T_0} u(x)\). From (17), there exists \(M_0 \in (0, M)\) such that

\[
\lim_{m \to \infty} U_m \leq M_0(<M).
\]

There exists a sequence \(\{V_k\}_{k=1}^{\infty}\) such that \(V_k = (M_0+M)/2\), \(\lim_{k \to \infty} V_k = M_0\) \(V_{k+1} \leq V_k\) and \(V_k \geq U_{m_k}\) with a sequence \(\{m_k\}_{k=1}^{\infty}\) satisfying \(u_{k+1} > u_k\) for \(k \in \mathbb{N}\). Thus, since \((x - y)^2 \leq 2x^2 + 2y^2\), we obtain

\[
\begin{align*}
\sup_{x \in B_1(\tilde{x}_k)} w(x, t) & \leq W_k(t) \\
& = e^{\Delta(t-T_0)} \max \left\{ \frac{v(T_0) - (M - V_k)e^{-\|x\|^2/2t} \int_{|y| < 2} e^{-\|y\|^2/2t} u_0(y) dy}{(4\pi T_0)^{-n/2} v(T_0)}, \delta_f \right\} < 1
\end{align*}
\]

for \(t \in [T_0, T_v)\), where \(\tilde{x}_k = x_{m_k}\). By comparison we have \(W_{k+1}(t) \leq W_k(t)\) for \(t \in [T_0, T_v)\) and \(k \in \mathbb{N}\). From Lemma 3.2 and comparison it follows that there exist the sequence \(\{\eta_k\}_{k=1}^{\infty}\) satisfying \(0 < \eta_{k+1} \leq \eta_k < \infty\) such that

\[
\lim_{t \to T_v} u(x_{m_k}, t) \leq \eta_k.
\]

Since the sequence \(\{x_m\}_{m=1}^{\infty}\) is arbitrary, we obtain that \(\psi\) is not blow-up direction. \(\square\)

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