

LIL for discrepancies of $\{\theta^n x\}$ — a graphical sketch

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1. INTRODUCTION.

It is said that the sequence $\{x_k\}$ of real numbers is uniformly distributed mod 1 if

$$\frac{1}{N} \#\{k \leq N \mid \langle x_k \rangle \in [a, b)\} \rightarrow b - a \quad (N \rightarrow \infty),$$

for all $[a, b) \subset [0, 1)$, where $\langle x \rangle$ denotes the fractional part of x .

It is well known that $\{k\alpha\}$ is uniformly distributed if and only if α is irrational, and it is easily verified that $\{k!e\}$ is not uniformly distributed. In general it is very difficult to decide that the given concrete sequence is uniformly distributed or not, and various studies are done in this field. For classical results, we refer the reader to Kuipers-Niederreiter [6] and Drmota-Tichy [3].

As a contrast to this difficulty, the next result by Weyl [12] opened the vast possibility of *metric results*.

Theorem 1. *If $n_{k+1} - n_k \geq C > 0$, then $\{n_k x\}$ is uniformly distributed mod 1 for almost every x .*

By considering the example $\{k!e\}$, it is seen that in general it is impossible to expect the above results for all x . By admitting exceptional set of measure zero, one can prove uniform distribution for many sequences. As a corollary of this theorem, we can say that for every real number $\theta > 1$, the sequence $\{\theta^n x\}$ is uniformly distributed mod 1 a.e. In this note, we try to determine the speed of convergence to uniform distribution.

To measure the speed of convergence, we use the following two types of discrepancies D_N and D_N^* :

$$\begin{aligned} D_N(\{x_k\}) &:= \sup_{0 \leq b < a < 1} \left| \frac{1}{N} \sum_{k=1}^N \mathbf{1}_{[b,a)}(\langle x_k \rangle) - (a - b) \right|; \\ &= \sup_{0 \leq b < a < 1} \left| \frac{1}{N} \sum_{k=1}^N f_{b,a}(\langle x_k \rangle) \right|; \end{aligned}$$

where $f_{b,a}(x) = \mathbf{1}_{[b,a)}(\langle x \rangle) - (a - b)$;

$$\begin{aligned} D_N^*(\{x_k\}) &:= \sup_{0 \leq a < 1} \left| \frac{1}{N} \sum_{k=1}^N \mathbf{1}_{[0,a)}(\langle x_k \rangle) - a \right|, \\ &= \sup_{0 \leq a < 1} \left| \frac{1}{N} \sum_{k=1}^N f_{0,a}(\langle x_k \rangle) \right|. \end{aligned}$$

The latter is usually called ‘star discrepancy’. Since $\frac{1}{N} \sum_{k=1}^N \mathbf{1}_{[0,a)}(\langle x_k \rangle)$ can be regarded as $F_N(a)$ where F_N is the distribution function of the empirical measure $\frac{1}{N} \sum_{k=1}^N \delta_{\{x_k\}}$,

star discrepancy measures by the supremum norm the distance between the empirical measure distribution function and uniform distribution function.

Solving the long standing Erdős-Gál conjecture, W. Philipp [9] proved

$$(1) \quad \frac{n_{k+1}}{n_k} > q > 1 \implies \frac{1}{4} < \overline{\lim}_{N \rightarrow \infty} \frac{ND_N(\{n_k x\})}{\sqrt{2N \log \log N}} \leq C_q \quad \text{a.e.}$$

It implies that $D_N = O\left(\sqrt{(\log \log N)/N}\right)$ and that O can't be replaced by o , i.e., the order of speed of convergence is determined. It should be emphasized that it is still not known whether there exists a sequence $\{n_k\}$ for which the limsup above is not constant.

For special sequences, the limsup is constant. W. Philipp [8] studied the sequence generated by mixing measure preserving transformation and proved the exact law of the iterated logarithm:

$$\begin{aligned} \overline{\lim}_{N \rightarrow \infty} \frac{ND_N(\{2^k x\})}{\sqrt{2N \log \log N}} &= \sup_{a < b} \left(\int_0^1 f_{a,b}^2(x) dx + 2 \sum_{k=1}^{\infty} \int_0^1 f_{a,b}(2^k x) f_{a,b}(x) dx \right)^{1/2} \\ &\in [\sqrt{42/9}, \sqrt{2}] \quad \text{a.e.} \end{aligned}$$

Recently, Berkes-Philipp-Tichy [2] stated that "the exact value of the limsup seems to be unknown even in the simplest case $n_k = 2^k$." In this note, we consider the sequence $n_k = \theta^k$ for real number $\theta > 1$ and prove the exact law of the iterated logarithm. We evaluate the value of limsup for most cases, and show that the case $n_k = 2^k$ is a difficult one, and the simplest case is $n_k = 3^k$.

To state our result, let us prepare a notation. When $\theta^r \notin \mathbf{Q}$ ($r \in \mathbf{N}$), put

$$\sigma_{\theta,f}^2 = \int_0^1 f^2(x) dx.$$

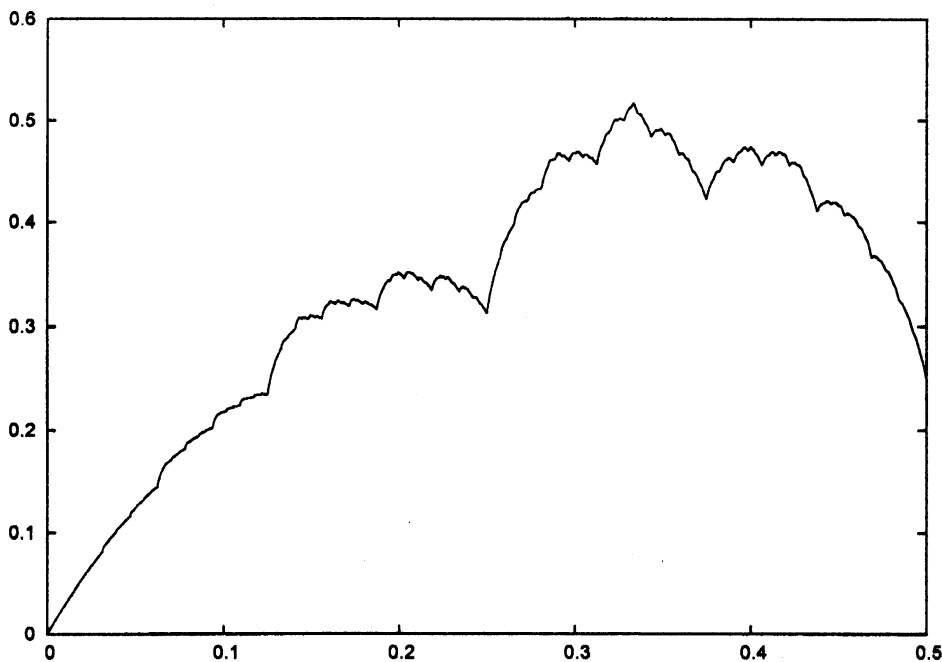
When $\theta^r \in \mathbf{Q}$ for some $r \in \mathbf{N}$, let us take the minimum of such r and denote $\theta = \sqrt[p]{q}$ by using coprime integers p and q . In this case we put

$$\sigma_{\theta,f}^2 = \int_0^1 f^2(x) dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(p^k x) f(q^k x) dx.$$

Theorem 2. *If $\theta > 1$, then*

$$(2) \quad \overline{\lim}_{N \rightarrow \infty} \frac{ND_N(\{\theta^k x\})}{\sqrt{2N \log \log N}} = \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*(\{\theta^k x\})}{\sqrt{2N \log \log N}} = \sup_{0 \leq a < 1} \sigma_{\theta,f_0,a} =: \Sigma_{\theta} \quad \text{a.e. } x$$

The constant Σ_{θ} can be evaluated for most of θ in the following way. By definition, the evaluation is reduced to the calculation of the maximum of $\sigma_{\theta,f_0,a}$ for $0 \leq a < 1$. If we regard $\sigma_{\theta,f_0,a}^2$ as a function of a , it is easily seen that it is symmetric with respect to $a = 1/2$. The following is the the graph of $\sigma_{\theta,f_0,a}^2$ ($0 \leq a \leq 1/2$). It seems that it takes maximum value at $a = 1/3$.



If $\theta^r \notin \mathbb{Q}$ for all $r \in \mathbb{N}$, then

$$\Sigma_{\theta} = \frac{1}{2}.$$

By Chung-Smirnov theorem, discrepancies for uniform i.i.d. obeys the law of the iterated logarithm and the limsup equals to $1/2$. Our result asserts that the behavior of discrepancies imitate that of uniform i.i.d. when θ is not a root of rational number.

Let us consider the case when θ is a root of rational number and denote $\theta = \sqrt[r]{p/q}$ by using p , q , and r satisfying the conditions above. Then we have the estimate

$$(3) \quad \frac{1}{2} \leq \Sigma_{\theta} \leq \frac{1}{2} \sqrt{\frac{pq+1}{pq-1}}.$$

Moreover, if p and q are odd, then we have

$$\Sigma_{\theta} = \frac{1}{2} \sqrt{\frac{pq+1}{pq-1}},$$

i.e., in this case Σ_{θ} attains its upper bound in the above estimate.

Especially, if p is odd and $q = 1$, i.e., in case when θ is a root of odd number p , we have

$$\Sigma_{\theta} = \frac{1}{2} \sqrt{\frac{p+1}{p-1}}.$$

As compared to this, when θ is a root of even number, the expression of Σ_{θ} is completely different. If $p \geq 4$ is even and $q = 1$, then

$$\Sigma_{\theta} = \frac{1}{2} \sqrt{\frac{(p+1)p(p-2)}{(p-1)^3}}.$$

This case does not include the case when $p = 2$. Although we have $\Sigma_{\theta} = 0$ by putting $p = 2$ in this expression, this is a wrong value.

If $p = 2$ and $q = 1$, i.e., $\theta = \sqrt[3]{2}$, we have

$$\Sigma_\theta = \frac{\sqrt{42}}{9}.$$

This case is completely isolated. By using the estimate (3) above, we see that $\sqrt{42}/9$ is the maximum value of Σ_θ , and we can conclude that among the class of sequences $\{\theta^n x\}$, the sequence generate by binary transform is the farthest from the uniform distribution.

When θ is a root of rational number given by the ratio of even and odd numbers, it is difficult to evaluate Σ_θ and we have explicit value only in the following case. If $p = 5$ and $q = 2$, i.e., $\theta = \sqrt[3]{5/2}$, we have

$$\Sigma_\theta = \frac{\sqrt{22}}{9}.$$

2. EXACT LAW OF THE ITERATED LOGARITHM

The reason why Σ_θ appears in the law of the iterated logarithm for discrepancies originated to the limiting behavior of Riesz-Raikov sums $\sum f(\theta^n x)$, where f is a real valued function satisfying $f(x+1) = f(x)$, $\int_0^1 f = 0$, $\int_0^1 f^2 < \infty$. It imitate the limiting behavior of independent or weakly dependent stationary sequence, and obey the central limit theorem and the law of the iterated logarithm,

$$\frac{1}{\sqrt{N}} \sum_{k=1}^N f(\theta^k x) \xrightarrow{\mathcal{L}} N_{0, \sigma_f^2},$$

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f(\theta^k x) \right| = \sigma_{\theta, f}, \quad \text{a.e.},$$

when the condition $\|f(\cdot + h) - f(\cdot)\|_2 = O(h^\alpha)$ is satisfied for $\alpha > 0$. (Cf. Berkes [1], Petit [7], Fukuyama, [4, 5]). In these theorems, $\sigma_{\theta, f}^2$ appears as the limiting variance. By the last limit theorem and a heuristic argument, we have the law of the iterated logarithm for discrepancies:

$$\begin{aligned} \overline{\lim}_{N \rightarrow \infty} \frac{ND_N(\{\theta^k x\})}{\sqrt{2N \log \log N}} &= \overline{\lim}_{N \rightarrow \infty} \sup_{0 \leq b < a < 1} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{b,a}(\langle \theta^k x \rangle) \right| \\ &= \sup_{0 \leq b < a < 1} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{b,a}(\langle \theta^k x \rangle) \right| \\ &= \sup_{0 \leq b < a < 1} \sigma_{\theta, f_{b,a}}, \\ \overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*(\{\theta^k x\})}{\sqrt{2N \log \log N}} &= \overline{\lim}_{N \rightarrow \infty} \sup_{0 \leq a < 1} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{0,a}(\langle \theta^k x \rangle) \right| \\ &= \sup_{0 \leq a < 1} \overline{\lim}_{N \rightarrow \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{0,a}(\langle \theta^k x \rangle) \right| \\ &= \sup_{0 \leq a < 1} \sigma_{\theta, f_{0,a}}. \end{aligned}$$

To make the argument rigorous, we must justify the changes of the orders of sup and lim sup appearing above. It is done by the approximation by so-called discrete discrepancies:

First we note

$$\left| \sup_{a < 1} \sum_{k=1}^N f_{0,a}(\theta^k x) - \max_{I=1}^{2^L-1} \sum_{k=1}^N f_{0,2^{-L}I}(\theta^k x) \right| \leq \max_{I=1}^{2^L-1} \sup_{a < 2^{-L}} \left| \sum_{k=1}^N f_{2^{-L}I, 2^{-L}I+a}(\theta^k x) \right|.$$

For fixed $I < 2^L$, we have the following estimate in the same way as the proof of (1) by Philipp [8]:

$$\limsup_{N \rightarrow \infty} \sup_{a < 2^{-L}} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{2^{-L}I, 2^{-L}I+a}(\theta^k x) \right| \leq C 2^{-L/8}, \quad \text{a.e.}$$

Taking maximum for $I = 1, \dots, 2^L - 1$, we have

$$\limsup_{N \rightarrow \infty} \max_{I=1}^{2^L-1} \sup_{a < 2^{-L}} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^N f_{2^{-L}I, 2^{-L}I+a}(\theta^k x) \right| \leq C 2^{-L/8}, \quad \text{a.e.}$$

By applying the law of the iterated logarithm for Riesz-Raikov sum $\sum f_{0,2^{-L}I}$ and by taking maximum for $I = 1, \dots, 2^L - 1$, we have

$$\limsup_{N \rightarrow \infty} \max_{I=1}^{2^L-1} \left| \frac{1}{\sqrt{2N \log \log N}} \sum_{k=1}^N f_{0,2^{-L}I}(\theta^k x) \right| = \max_{I=1}^{2^L-1} \sigma^2(\theta, f_{0,2^{-L}I}), \quad \text{a.e.}$$

Combining these two asymptotics and letting $L \rightarrow \infty$, we have

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N^*(\{\theta^k x\})}{\sqrt{2N \log \log N}} = \sup_{0 \leq a < 1} \sigma_{\theta, f_{0,a}}, \quad \text{a.e.}$$

In the same way, we can prove

$$\overline{\lim}_{N \rightarrow \infty} \frac{ND_N(\{\theta^k x\})}{\sqrt{2N \log \log N}} = \sup_{0 \leq b < a < 1} \sigma_{\theta, f_{b,a}}, \quad \text{a.e.}$$

If $\theta^r \notin \mathbf{Q}$ ($r \in \mathbf{N}$), we have

$$\sigma_{\theta, f_{b,a}}^2 = (a - b) - (a - b)^2 \leq (1/2) - (1/2)^2 = 1/4,$$

and hence we have

$$\sup_{b,a} \sigma_{\theta, f_{b,a}} = \sup_a \sigma_{\theta, f_{0,a}} = 1/2,$$

which prove the assertion of the theorem in this case.

In case $\theta = \sqrt[p]{p/q}$ we need a bit of consideration. We can express $\sigma_{\theta, f_{b,a}}^2$ by means of the following series:

$$\sigma_{\theta, f_{b,a}}^2 = \tilde{V}(b, a, b, a) + 2 \sum_{k=1}^{\infty} \frac{\tilde{V}(\langle p^k b \rangle, \langle p^k a \rangle, \langle q^k b \rangle, \langle q^k a \rangle)}{p^k q^k},$$

where

$$V(x, \xi) = x \wedge \xi - x\xi \quad \text{and} \quad \tilde{V}(x, y, \xi, \eta) = V(x, \xi) + V(y, \eta) - V(x, \eta) - V(y, \xi).$$

This expression is proved by the following formula.

$$(4) \quad \int_0^1 f_{b,a}(\nu x) f_{b,a}(\mu x) dx = \frac{\tilde{V}(\langle \nu b \rangle, \langle \nu a \rangle, \langle \mu b \rangle, \langle \mu a \rangle)}{\mu\nu}, \quad (b < a, \gcd(\mu, \nu) = 1).$$

Although this is proved by complicated direct calculation in the original paper [5], we present here an elegant new proof given by Tokuzo Shiga [11]. Putting $I_{s,t} = \mathbf{1}_{[0,t]} - \mathbf{1}_{[0,s]}$ ($0 \leq s, t \leq 1$) and $e_n(x) = e^{2\pi i n x}$, by we have the Fourier expansion

$$I_{s,t} = t - s + \sum_{n \neq 0} (e_n, I_{s,t}) e_n,$$

and hence

$$B_t - B_s = \int_0^1 I_{s,t}(\tau) dB_\tau = (t - s)B_1 + \sum_{n \neq 0} (e_n, I_{s,t})(e_n, \dot{B}),$$

where B denotes the standard Brownian motion. By the definition of \tilde{V} , we have for $0 \leq s, t, \xi, \eta \leq 1$,

$$\begin{aligned} \tilde{V}(s, t, \xi, \eta) &= E((B_s - B_t)(B_\xi - B_\eta)) - (s - t)(\xi - \eta) \\ &= \sum_{n \neq 0} (e_n, I_{s,t})(\bar{e}_n, I_{\xi,\eta}). \end{aligned}$$

By noting the relation $(e_{\nu k}, I_{b,a}) = (e_k, I_{\langle \nu b \rangle, \langle \nu a \rangle}) / \nu$, we can prove (4) as follows:

$$\begin{aligned} \int_0^1 f_{b,a}(\mu x) f_{b,a}(\nu x) &= \int_0^1 I_{b,a}(\mu x) I_{b,a}(\nu x) - (a - b)^2 \\ &= \sum_{m, n \neq 0: n\mu = m\nu} (e_n, I_{b,a})(\bar{e}_m, I_{b,a}) \\ &= \sum_{k \neq 0} (e_{\nu k}, I_{b,a})(\bar{e}_{\mu k}, I_{b,a}) \\ &= \frac{1}{\mu\nu} \sum_{k \neq 0} (e_k, I_{\langle \nu b \rangle, \langle \nu a \rangle})(\bar{e}_k, I_{\langle \mu b \rangle, \langle \mu a \rangle}) \\ &= \frac{\tilde{V}(\langle \nu b \rangle, \langle \nu a \rangle, \langle \mu b \rangle, \langle \mu a \rangle)}{\mu\nu} \end{aligned}$$

By using this expression, we can prove $\sup_{b < a} \sigma_{\theta, f_{b,a}} = \sup_a \sigma_{\theta, f_{0,a}}$. Inequality \geq is trivial and we must prove \leq . It is proved by using

$$(5) \quad \tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) \leq V(\langle y - x \rangle, \langle \eta - \xi \rangle) = \tilde{V}(0, \langle y - x \rangle, 0, \langle \eta - \xi \rangle).$$

Actually we have

$$\begin{aligned} \sigma_{\theta, f_{b,a}}^2 &= \tilde{V}(b, a, b, a) + 2 \sum_{k=1}^{\infty} \frac{\tilde{V}(\langle p^k b \rangle, \langle p^k a \rangle, \langle q^k b \rangle, \langle q^k a \rangle)}{p^k q^k} \\ &\leq \tilde{V}(0, a - b, 0, a - b) + 2 \sum_{k=1}^{\infty} \frac{\tilde{V}(0, \langle p^k(a - b) \rangle, 0, \langle q^k(a - b) \rangle)}{p^k q^k} \\ &= \sigma_{\theta, f_{0, a-b}}^2, \end{aligned}$$

and by taking supremum, we have \leq .

Although the inequality (5) is proved by direct calculation in the original paper [5], we here again present an elegant proof by Tokuzo Shiga [11]. Note that we have $\langle -x \rangle = 1 - \langle x \rangle$

and that the relation $\langle y \rangle \geq \langle x \rangle$ implies $\langle y \rangle - \langle x \rangle = \langle y - x \rangle$. If $\langle y \rangle \geq \langle x \rangle$ and $\langle \eta \rangle \geq \langle \xi \rangle$ then we have

$$\begin{aligned}\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) &= |[\langle x \rangle, \langle y \rangle] \cap [\langle \xi \rangle, \langle \eta \rangle]| - (\langle y - x \rangle)(\langle \eta - \xi \rangle) \\ &\leq |[\langle x \rangle, \langle y \rangle]| \wedge |[\langle \xi \rangle, \langle \eta \rangle]| - (\langle y - x \rangle)(\langle \eta - \xi \rangle) \\ &= V(\langle y - x \rangle, \langle \eta - \xi \rangle),\end{aligned}$$

where $|[a, b]|$ denotes the length $b - a$ of interval $[a, b]$. Moreover we have

$$\begin{aligned}\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) &= |[\langle x \rangle, \langle y \rangle]| - |[\langle x \rangle, \langle y \rangle] \cap [\langle \xi \rangle, \langle \eta \rangle]^c| - (\langle y - x \rangle)(\langle \eta - \xi \rangle) \\ &\geq |[\langle x \rangle, \langle y \rangle]| - |[\langle x \rangle, \langle y \rangle]| \wedge |[\langle \xi \rangle, \langle \eta \rangle]^c| - (\langle y - x \rangle)(\langle \eta - \xi \rangle) \\ &= -V(\langle y - x \rangle, \langle \xi - \eta \rangle).\end{aligned}$$

Hence if $\langle y \rangle \geq \langle x \rangle$ and $\langle \eta \rangle < \langle \xi \rangle$ are satisfied, we have

$$\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) = -\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \eta \rangle, \langle \xi \rangle) \leq V(\langle y - x \rangle, \langle \eta - \xi \rangle).$$

The rests are reduced to the above cases by using $\tilde{V}(\langle x \rangle, \langle y \rangle, \langle \xi \rangle, \langle \eta \rangle) = \tilde{V}(\langle y \rangle, \langle x \rangle, \langle \eta \rangle, \langle \xi \rangle)$.

3. EVALUATION OF Σ_θ

In this section, we explain the methods how we evaluate Σ_θ when θ is a root of some rational number. The expression of $\sigma_{\theta, f}$ does not depend on r , we may assume $\theta = p/q$ where p and q are relatively prime. In this case we have the expression

$$(6) \quad \sigma_{\theta, f_0, a}^2 = V(a, a) + 2 \sum_{n=1}^{\infty} \frac{V(\langle p^n a \rangle, \langle q^n a \rangle)}{p^n q^n}.$$

Since

$$V(x, \xi) = \begin{cases} x(1 - \xi) & (0 \leq x \leq \xi \leq 1), \\ (1 - x)\xi & (0 \leq \xi \leq x \leq 1), \end{cases}$$

regarding as a function of x , $V(x, \xi)$ is increasing for $x < \xi$, decreasing for $x > \xi$, and take maximum at $x = \xi$. It implies

$$0 \leq V(x, \xi) = x \wedge \xi - x\xi \leq V(\xi, \xi) = \xi - \xi^2 \leq \frac{1}{4},$$

where the equality holds at $x = \xi$. By this estimate, we have

$$(7) \quad \sigma_{\theta, f_0, a}^2 \leq \frac{1}{4} + 2 \sum_{n=1}^{\infty} \frac{1}{4p^n q^n} = \frac{pq + 1}{4(pq - 1)}.$$

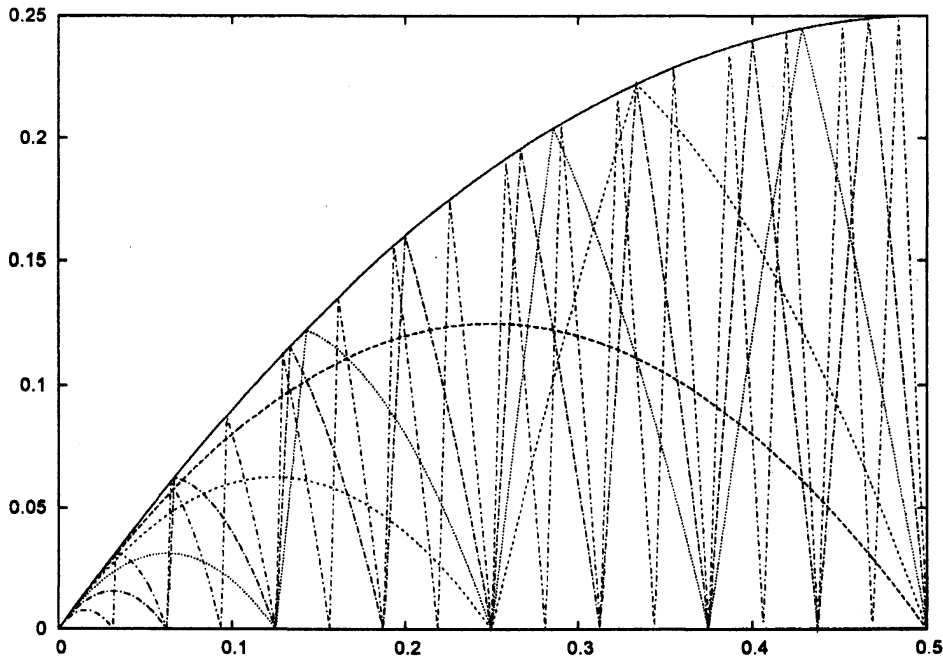
On the other hand, since all the summands of expression are positive, we have

$$\sigma_{\theta, f_0, 1/2}^2 \geq V(1/2, 1/2) = \frac{1}{4}.$$

and hence we have the estimate (3).

Since $V(\langle p^n a \rangle, \langle q^n a \rangle)$ is a minimum of $\langle p^n a \rangle(1 - \langle q^n a \rangle)$ and $\langle q^n a \rangle(1 - \langle p^n a \rangle)$, it appears to be a piecewise parabolic function and takes its local maximal value at points satisfying $\langle p^n a \rangle = \langle q^n a \rangle$.

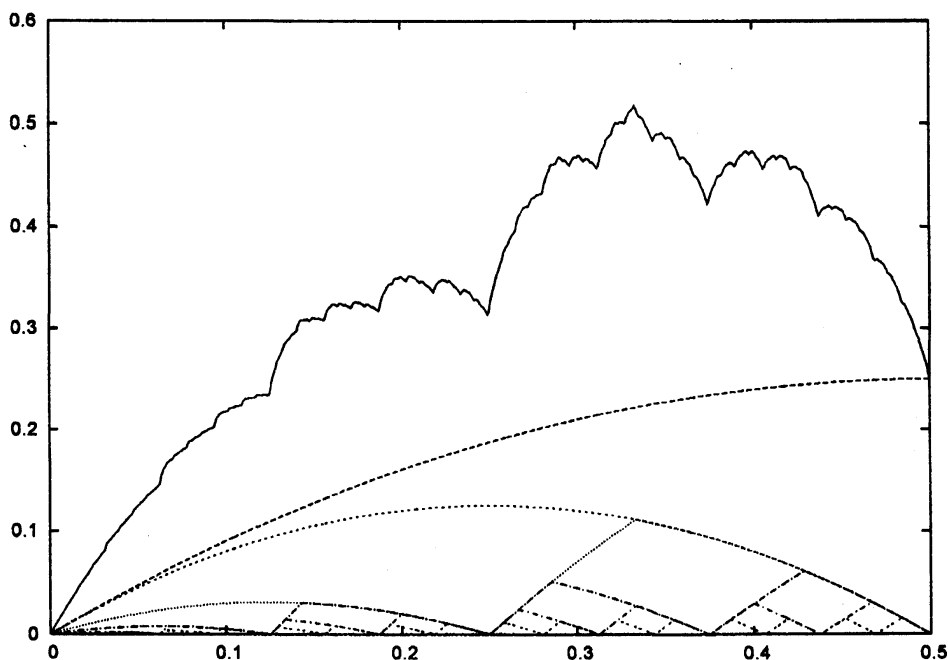
Here we present the graphs of summands $V(\langle 2^n a \rangle, a)$ of expression of $\sigma_{2,f_0,a}^2$.



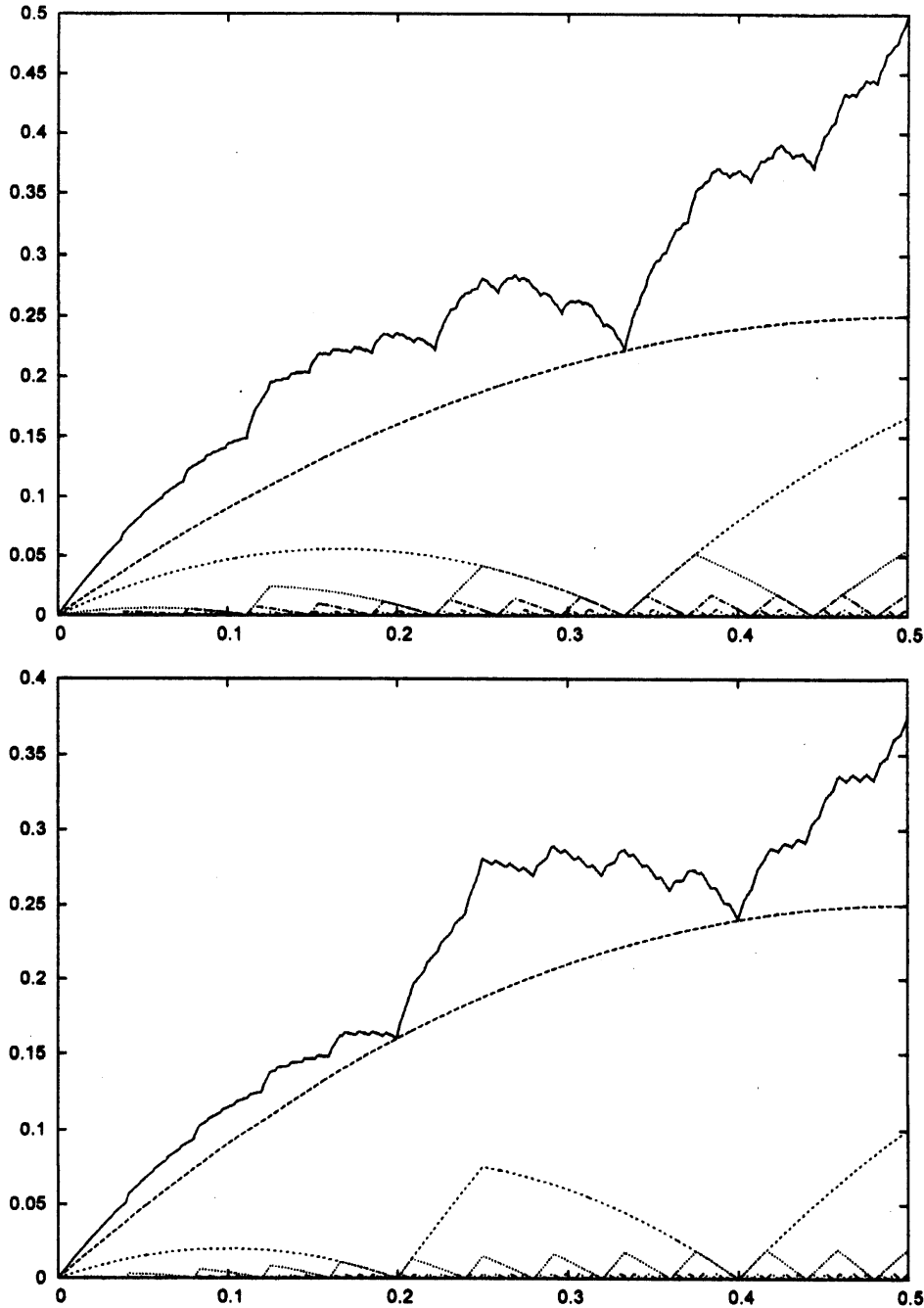
Since $V(\langle 2^n a \rangle, a) \leq a - a^2$, we have the common enveloping parabolic curve for all $V(\langle 2^n a \rangle, a)$. At any local maximal point, graph of $V(\langle 2^n a \rangle, a)$ touch the enveloping parabola from below. As we have note previously, $\sigma_{2,f_0,a}^2$ seems to take maximal value at $a = 1/3$. By this graph, each $V(\langle 2^n a \rangle, a)$ takes local maximal value at $a = 1/3$ if $n = 2, 4, \dots$, but does not if $n = 0, 1, 3, 5, \dots$. That why the case $\theta = 2$ is difficult to investigate. We will return to this case later.

The next figure presents the graph of $\sigma_{2,f_0,a}^2$ and the graphs of each summands of series

$$\sigma_{2,f_0,a}^2 = V(a, a) + V(\langle 2a \rangle, a) + V(\langle 2^2 a \rangle, a)/2 + V(\langle 2^3 a \rangle, a)/2^2 + V(\langle 2^4 a \rangle, a)/2^3 + \dots$$



As compared to this, the case θ is an odd number seems easy to handle. The next two figures present the graphs $\sigma_{3,f_0,a}^2$ and $\sigma_{5,f_0,a}^2$. Every summands take its maximal value at $a = 1/2$, and that why $\sigma_{3,f_0,a}^2$ and $\sigma_{5,f_0,a}^2$ are maximal at $a = 1/2$.

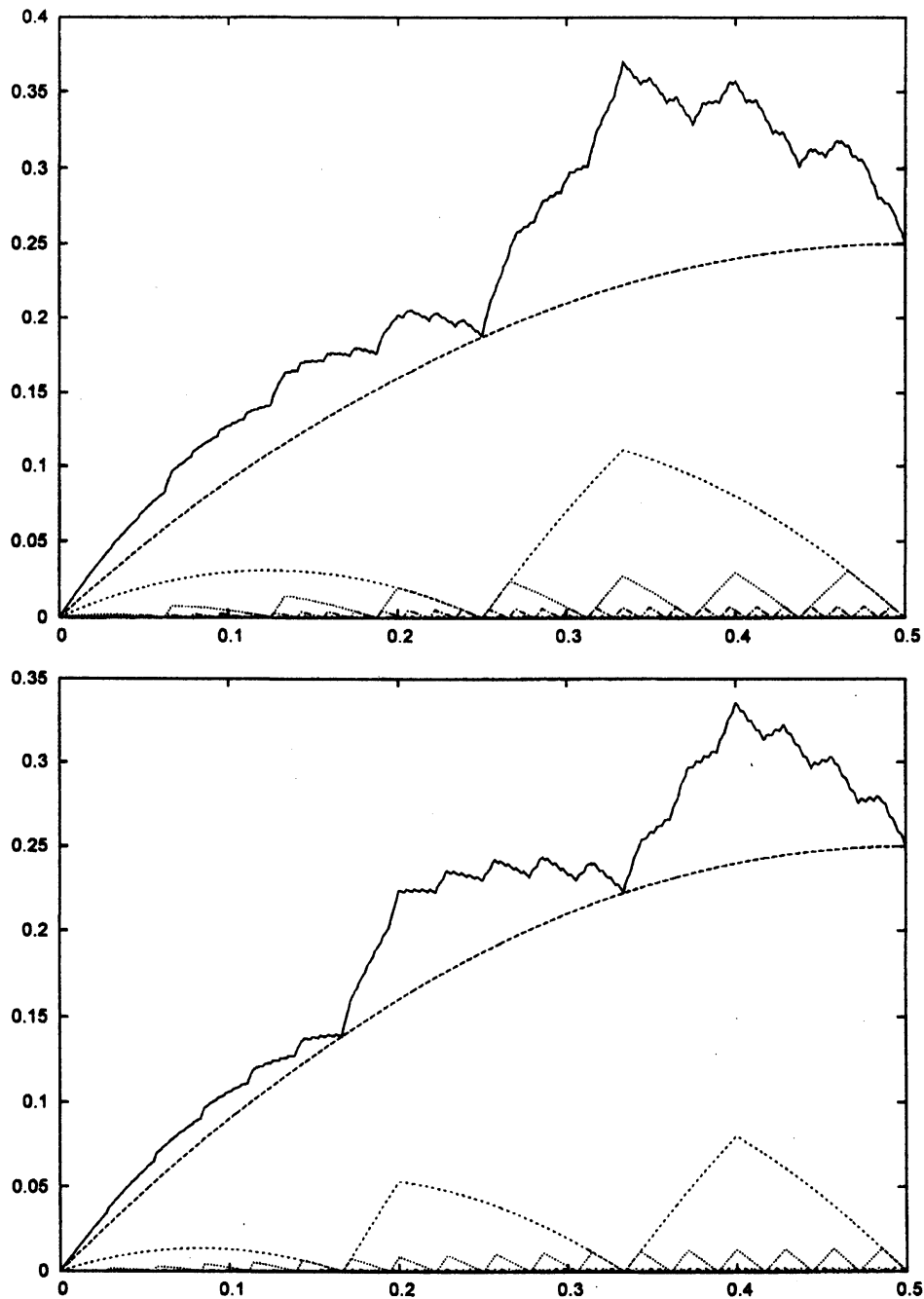


We can verify the above graphical reasoning as follows. When p and q are odd, $1/2$ is a fixed point of transformation $a \mapsto pa \pmod 1$ and $a \mapsto qa \pmod 1$, or $\langle p^n/2 \rangle = \langle q^n/2 \rangle = 1/2$. That why we have

$$\sigma_{\theta,f_0,1/2}^2 = V(1/2, 1/2) + 2 \sum_{n=1}^{\infty} \frac{V(1/2, 1/2)}{p^n q^n} = \frac{pq + 1}{4(pq - 1)} \geq \sigma_{\theta,f_0,a}^2,$$

which proves $\sup_a \sigma_{\theta,f_0,a}^2 = \sigma_{\theta,f_0,1/2}^2 = \frac{pq + 1}{4(pq - 1)}$.

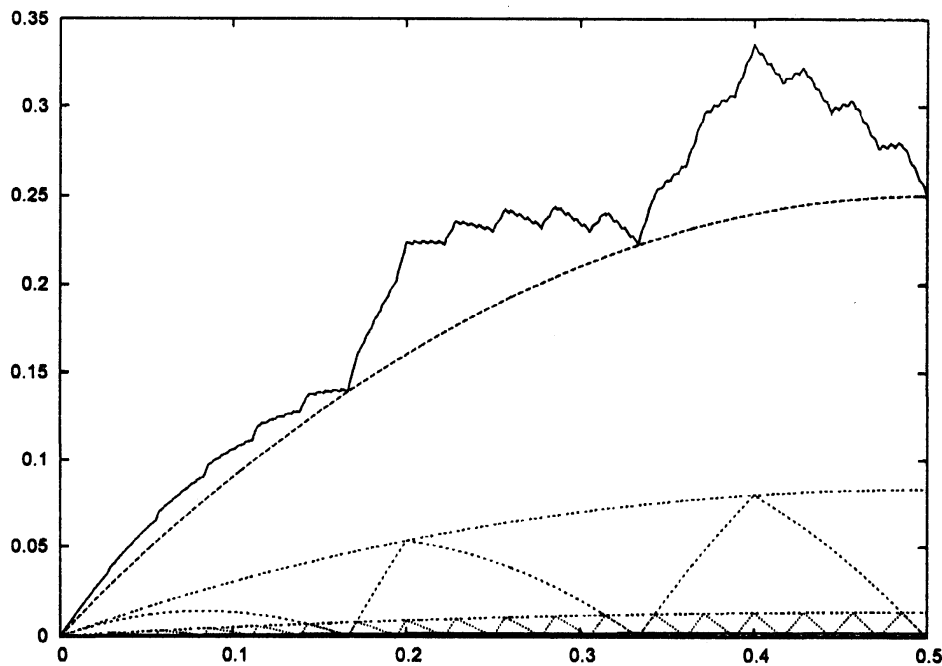
From now on, we assume $q = 1$ and evaluate Σ_p when $p \geq 4$ is an even integer. The graphs of $\sigma_{4,f_0,a}^2$ and $\sigma_{6,f_0,a}^2$ are as below:



$\sigma_{4,f_0,a}^2$ seems to take maximum at $a = 1/3$, and $\sigma_{6,f_0,a}^2$ at $a = 2/5$. In general we expect that $\sigma_{p,f_0,a}^2$ takes its maximum at $a = a_p = (p/2 - 1)/(p - 1)$, which is a maximal fixed point of $a \mapsto pa \bmod 1$ in $[0, 1/2]$, i.e. $\langle pa_p \rangle = a_p$. Therefore we are to prove

$$\sup_{a < 1} \sigma_{p,f_0,a}^2 = \sigma_{p,f_0,a_p}^2 = \frac{(p+1)p(p-2)}{4(p-1)^3}.$$

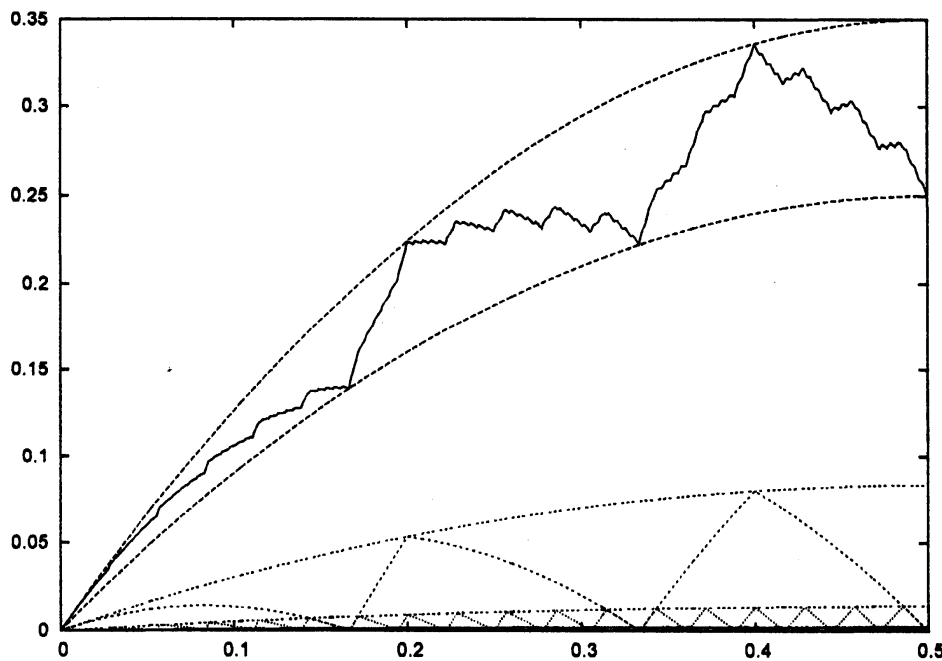
We have the estimate $V(\langle p^n a \rangle, a) \leq V(a, a)$ where equality holds at $a = a_p$. We present the figure of this estimate in case $p = 6$.



Together with these estimates and series expansion (6) of $\sigma_{p,f_0,a}^2$, we have

$$\sigma_{p,f_0,a}^2 \leq V(a, a)(p+1)/(p-1),$$

where the equality holds at $a = a_p$.



By noting that $V(a, a)$ is increasing in $[0, 1/2]$, this estimate proves

$$\sigma_{p,f_0,a}^2 \leq V(a, a)(p+1)/(p-1) \leq V(a_p, a_p)(p+1)/(p-1) = \sigma_{p,f_0,a_p}^2, \quad (a \leq a_p).$$

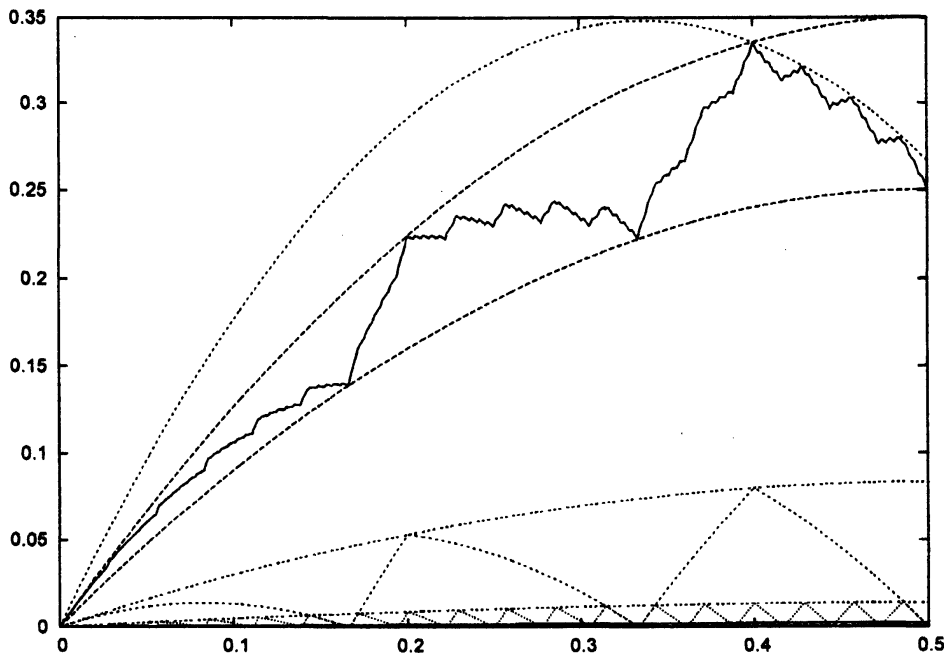
Thus we must consider the case when $a_p < a < 1/2$. By using the series expansion (6) and the estimate $V(\langle p^n a \rangle, a) \leq V(a, a)$, and by using the explicit formula of $V(\langle pa \rangle, a)$ in this interval, we have

$$\begin{aligned} \sigma_{p,f_0,a}^2 &= V(a, a) + \frac{2}{p}V(\langle pa \rangle, a) + 2 \sum_{n=2}^{\infty} \frac{V(\langle p^n a \rangle, a)}{p^n} \\ &\leq V(a, a) + \frac{2}{p}(a - a(pa - p/2 + 1)) + 2 \sum_{n=2}^{\infty} \frac{V(a, a)}{p^n}, \quad (a_p \leq a \leq 1/2), \end{aligned}$$

where the equality holds at $a = a_p$. We can prove easily that the right hand side is a quadratic function decreases in this interval if $p \geq 6$, and as before we can prove

$$\begin{aligned} \sigma_{p,f_0,a}^2 &\leq V(a, a) + \frac{2}{p}(a - a(pa - p/2 + 1)) + 2 \sum_{n=2}^{\infty} \frac{V(a, a)}{p^n} \\ &\leq V(a_p, a_p) + \frac{2}{p}(a_p - a_p(pa_p - p/2 + 1)) + 2 \sum_{n=2}^{\infty} \frac{V(a_p, a_p)}{p^n} \\ &= \sigma_{p,f_0,a_p}^2. \end{aligned}$$

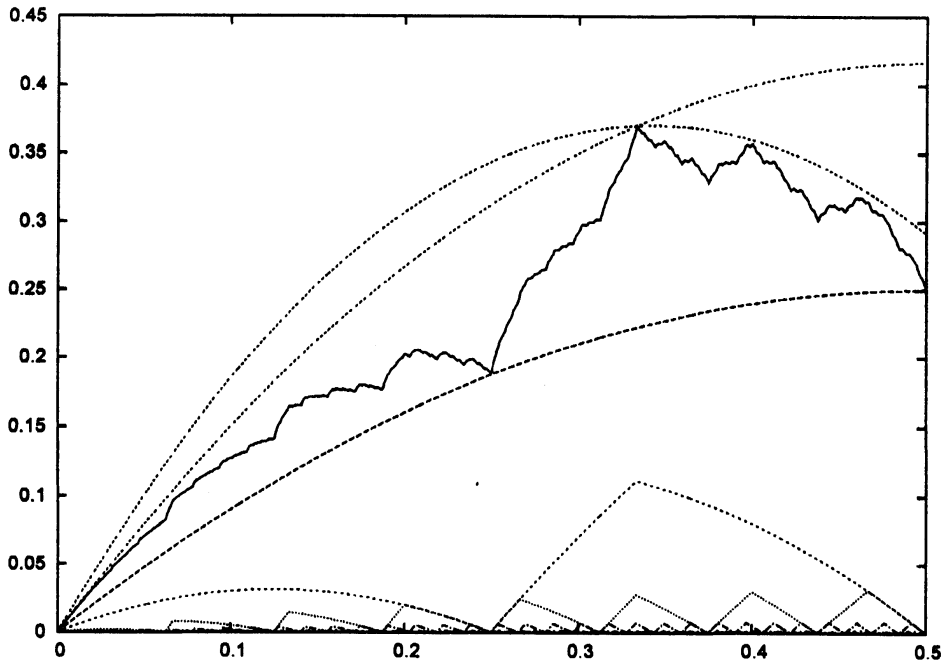
The graphical expression is as follows.



In this way we have proved

$$\sigma_{p,f_0,a}^2 \leq \sigma_{p,f_0,a_p}^2 = \frac{(p+1)p(p-2)}{4(p-1)^3}.$$

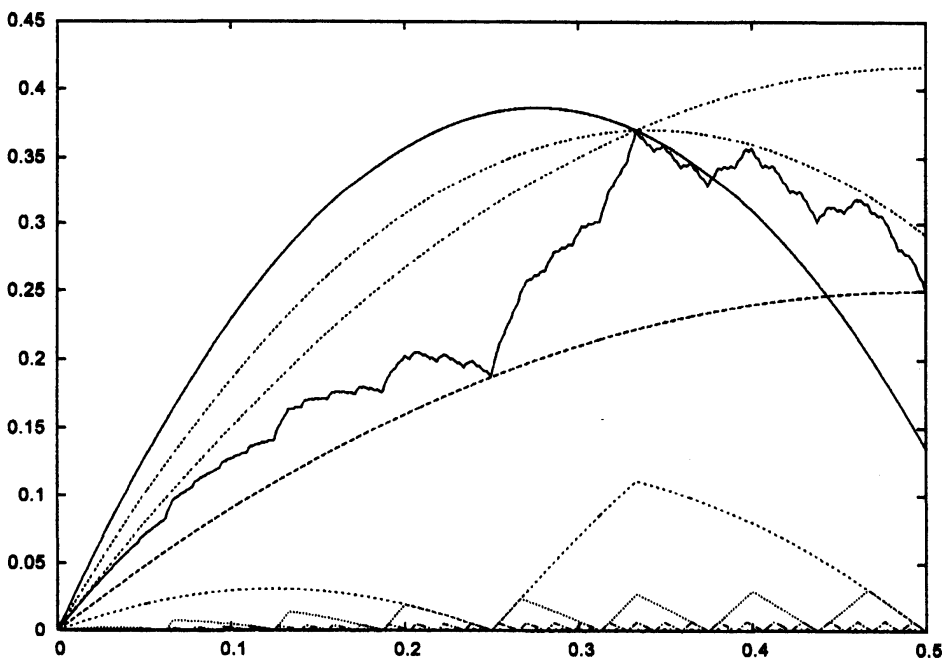
Unfortunately, the second estimate is not sufficient in case when $p = 4$. As the graph below, axis of the second estimating parabola located just right to the point a_4 , and is not decreasing in the interval $[a_4, 1/2]$.



By expressing $V(\langle 4a \rangle, a)$ and $V(\langle 4a \rangle, a)$ explicitly just right of a_4 , we have the estimate

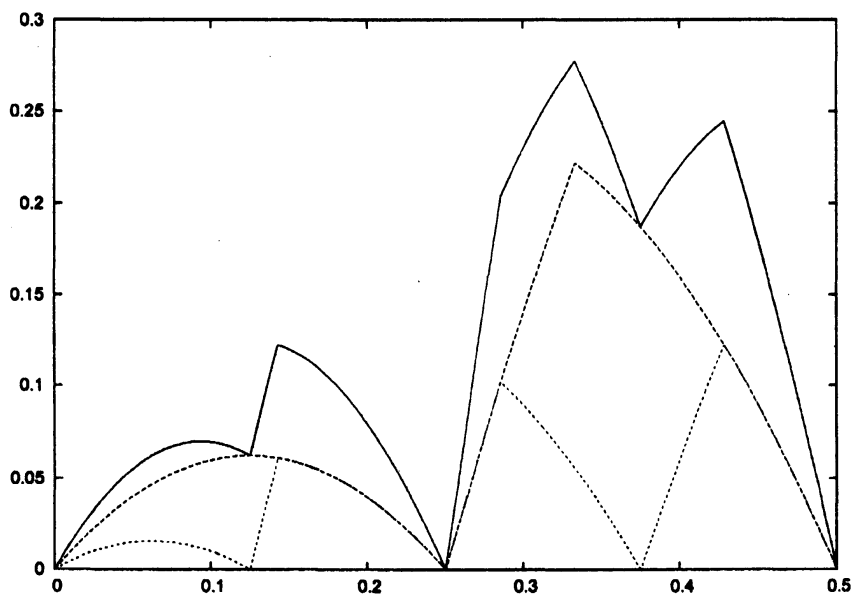
$$\sigma_{4,f_0,a}^2 \leq V(a, a) + \frac{2}{4}(a - a(4a - 1)) + \frac{2}{4^2}(1 - a(16a - 5))2 \sum_{n=3}^{\infty} \frac{V(a, a)}{4^n},$$

where the equality holds for $a = a_4$. By using this estimate, we have the graph below and prove that the maximum is take at $a = a_4$.

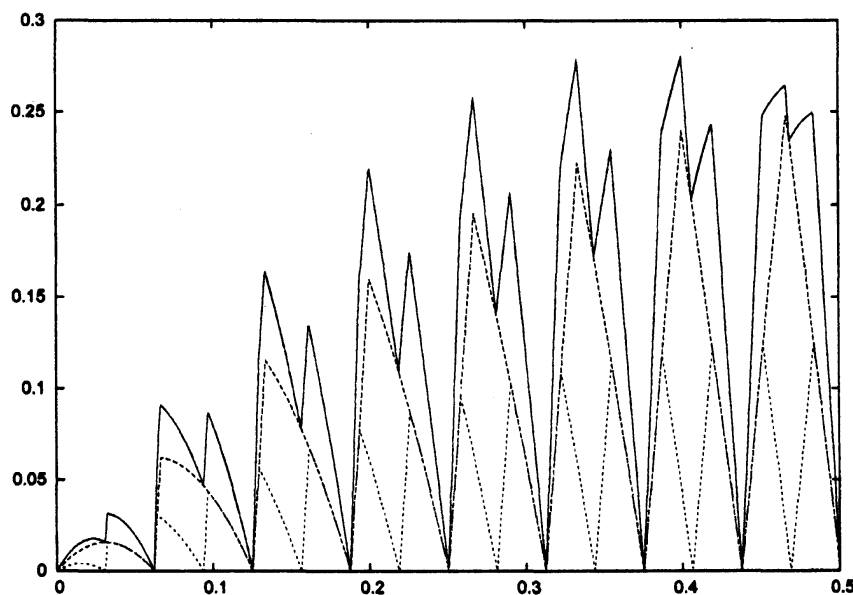


So far, we have evaluated Σ_p for integers $p \geq 3$, and the evaluation of Σ_2 remains. Since there is no fixed point of binary transformation $a \mapsto 2a$ other than $a = 0$, and that why this case is difficult. Our candidate of maximal point is $a = 1/3$. This point has period 2 with respect to the binary transformation, i.e., $\langle 2^{2n}/3 \rangle = 1/3$ and $\langle 2^{2n-1}/3 \rangle = 2/3$. As we explained before, among the summands of the series expansion of $\sigma_{2,f_0,a}^2$, the terms $V(\langle 2^2 a \rangle, a)/2$, $V(\langle 2^4 a \rangle, a)/2^3$, $V(\langle 2^6 a \rangle, a)/2^5$, ... takes local maximum at $a = 1/3$, but consecutive $V(\langle 2^3 a \rangle, a)/2^2$, $V(\langle 2^5 a \rangle, a)/2^4$, $V(\langle 2^7 a \rangle, a)/2^6$, ... does not. We try to make groups consists of $V(\langle 2^2 a \rangle, a)/2$ and $V(\langle 2^3 a \rangle, a)/2^2$, $V(\langle 2^4 a \rangle, a)/2^3$ and $V(\langle 2^5 a \rangle, a)/2^4$, and so on to compensate by the goodness of the big term the badness of small term.

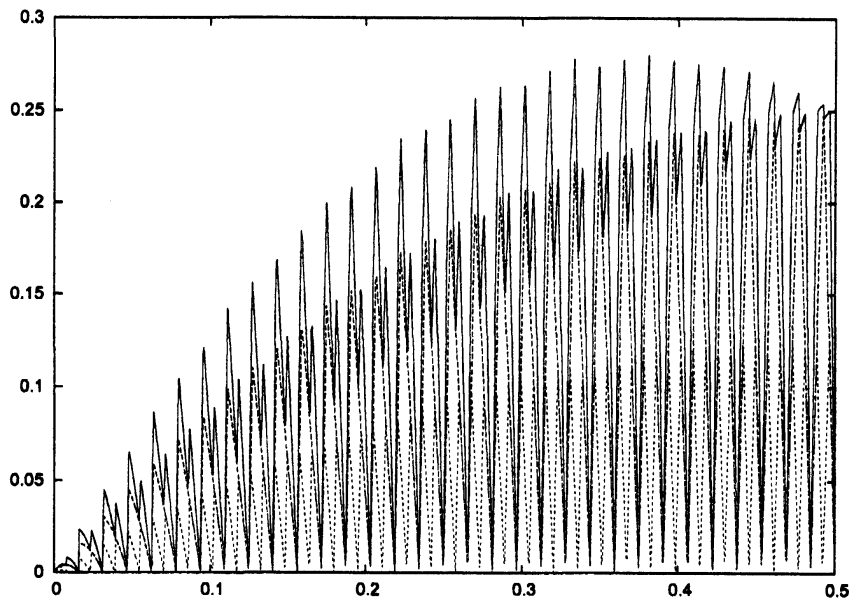
The next is the graphs of $V(\langle 2^2 a \rangle, a)$, $V(\langle 2^3 a \rangle, a)/2$, and $V(\langle 2^2 a \rangle, a) + V(\langle 2^3 a \rangle, a)/2$. It seems that the last one take the maximum at $a = 1/3$ and the badness of $V(\langle 2^3 a \rangle, a)/2$ is compensated.



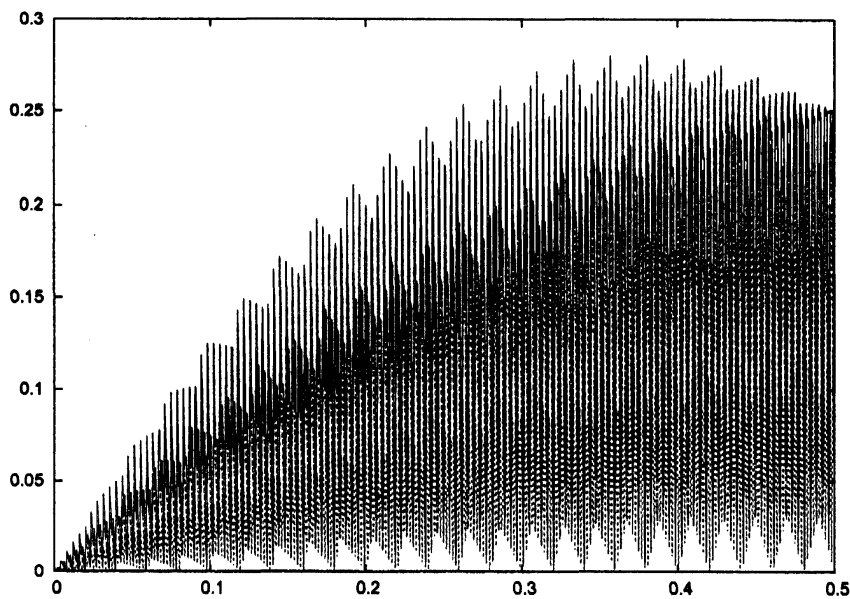
The graphs of $V(\langle 2^4 a \rangle, a)$, $V(\langle 2^5 a \rangle, a)/2$, and $V(\langle 2^4 a \rangle, a) + V(\langle 2^5 a \rangle, a)/2$.



The graphs of $V(\langle 2^6 a \rangle, a)$, $V(\langle 2^7 a \rangle, a)/2$, and $V(\langle 2^6 a \rangle, a) + V(\langle 2^7 a \rangle, a)/2$.



The graphs of $V(\langle 2^8 a \rangle, a)$, $V(\langle 2^9 a \rangle, a)/2$, and $V(\langle 2^8 a \rangle, a) + V(\langle 2^9 a \rangle, a)/2$.



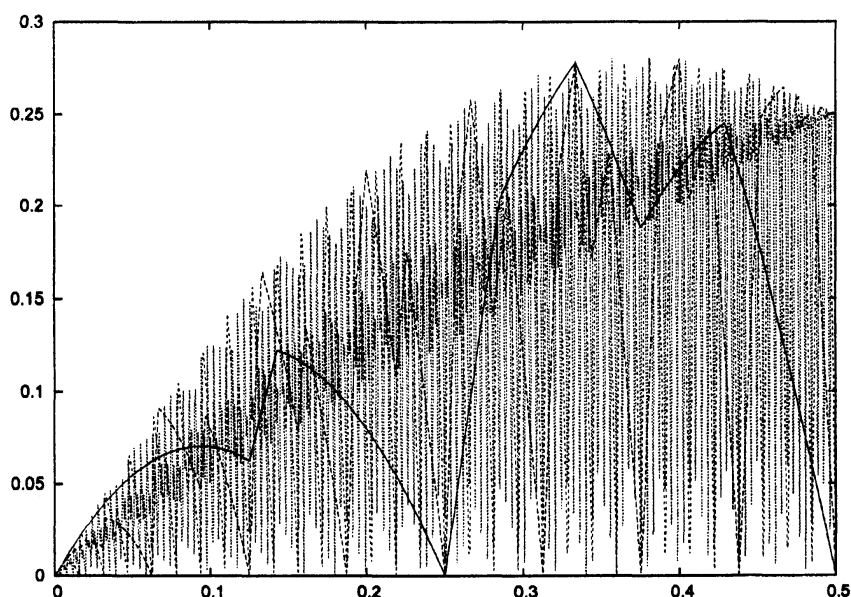
By viewing these figures, our method of compensation seems to be successful. Moreover, we feel that there is a common enveloping curve of these graphs. To see this, we draw these graphs in one display, i.e., we draw

$$V(\langle 2^2 a \rangle, a) + V(\langle 2^3 a \rangle, a)/2,$$

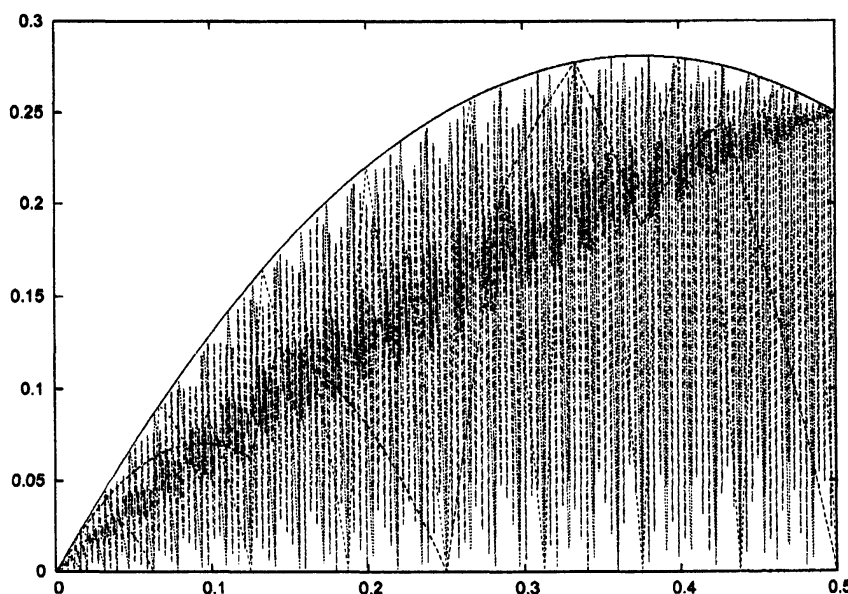
$$V(\langle 2^4 a \rangle, a) + V(\langle 2^5 a \rangle, a)/2,$$

$$V(\langle 2^6 a \rangle, a) + V(\langle 2^7 a \rangle, a)/2,$$

$$V(\langle 2^8 a \rangle, a) + V(\langle 2^9 a \rangle, a)/2.$$



The figure ensures us the existence of the enveloping curve. In this note, everything is estimated by quadratic functions, and hence we hope that our new enveloping curve is also a parabola. Any parabola is determined by three points. Trivially $(0,0)$ is on our parabola. Since $a = 1/3$ should be the local maximal point, by calculating the values at $a = 1/3$, we see that $(1/3, 5/18)$ is on this curve. By viewing the graph, it seems that $(1/2, 1/4)$ is also on the parabola, and hence we can determine the parabola to be $a(3 - 4a)/2$. By drawing the graph of this parabola together with the previous ones, we have the figure below. It seems that the estimate by this parabola seems to be valid.



To verify these arguments rigorously, we are to prove the following inequality. Let us put $V^*(b, a) = V(b, a) + V(\langle 2b \rangle, a)/2$ for $0 \leq a \leq 1/2$ and $0 \leq b \leq 1$. Note that $V^*(a, a) = a(3 - 4a)/2$. We have

$$V^*(b, a) \leq V^*(a, a) \leq V^*(1/3, 1/3),$$

where the equalities hold for $a = b = 1/3$.

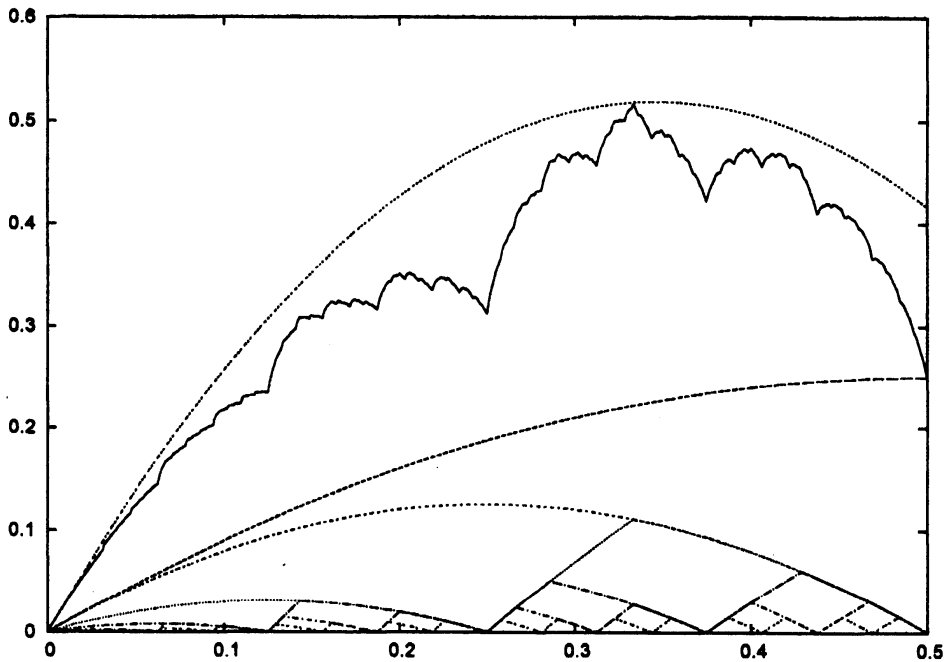
Since $V(b, a)$ and $V(\langle 2b \rangle, a)$ are concrete piecewise quadratic form, we can prove the above inequality elementarily.

By applying the inequality above, we have

$$\begin{aligned}\sigma_{2, f_0, a}^2 &= V(a, a) + V(2a, a) + 2 \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \left(V(\langle 2^{2n} a \rangle, a) + \frac{1}{2} V(\langle 2^{2n+1} a \rangle, a) \right) \\ &= V(a, a) + V(2a, a) + 2 \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \frac{V^*(\langle 2^{2n} a \rangle, a)}{4^n} \\ &\leq V(a, a) + V(2a, a) + 2 \sum_{n=1}^{\infty} \frac{V^*(a, a)}{4^n} \\ &= a(9 - 13a)/3,\end{aligned}$$

where the equality holds for $a = 1/3$. Since the right hand side is increasing for $a \leq 1/3$, we see $\sigma_{2, f_0, a}^2 \leq \sigma_{2, f_0, 1/3}^2 = 42/9^2$ for $a \leq 1/3$. The right hand side is decreasing for $a \geq 3/8$ and its value on $3/8$ is less than $42/9^2$, we have $\sigma_{2, f_0, a}^2 \leq \sigma_{2, f_0, 1/3}^2 = 42/9^2$ for $a \geq 3/8$.

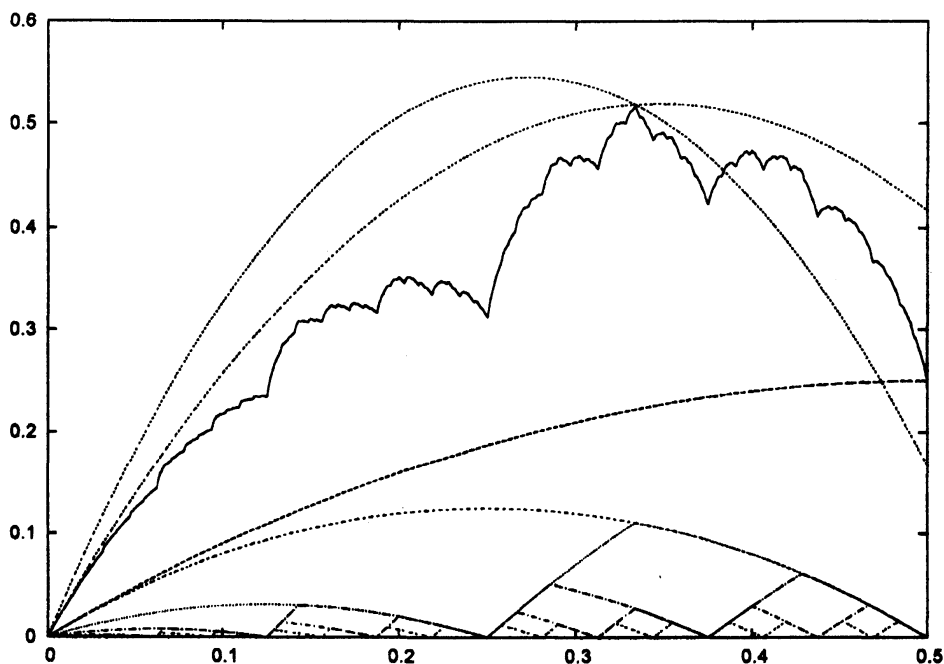
The graphs are as follows:



If $3/1 < a \leq 3/8$, we have

$$\begin{aligned}\sigma_{2, f_0, a}^2 &= V(a, a) + V(2a, a) + V(4a - 1, a)/2 + V(8a - 2, a)/4 + 2 \sum_{n=2}^{\infty} \frac{1}{2^{2n}} \frac{V^*(\langle 2^{2n} a \rangle, a)}{4^n} \\ &\leq V(a, a) + V(2a, a) + V(4a - 1, a)/2 + V(8a - 2, a)/4 + 2 \sum_{n=2}^{\infty} \frac{1}{2^{2n}} \frac{V^*(a, a)}{4^n} \\ &= 2a(6 - 11a)/3, \quad (1/3 < a < 3/8).\end{aligned}$$

Since the last quadratic function decreases for $1/3 < a < 3/8$, we have $\sigma_{2, f_0, a}^2 \leq \sigma_{2, f_0, 1/3}^2 = 42/9^2$ for $1/3 < a < 3/8$. This completes the proof. The graphs are as follows.



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