数理解析研究所講究録
第 1591 巻 2008 年 108-118

Interface motions driven by reaction, diffusion and convection

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1. INTRODUCTION

1.1. **Interfaces driven by reaction-diffusion equations.** Interfacial phenomena have been studied in terms of reaction-diffusion equations. In particular the Allen-Cahn equation

$$\text{(A-C)} \qquad \frac{\partial u}{\partial t} = \Delta u + g(u), \quad t > 0,\ x \in \mathbb{R}^N,$$

where g is a bi-stable reaction term, has been a widely used model to describe various physical phenonena. Here, bi-stability of g means that it is the minus of the derivative, $g(u) = -W'(u)$, of a double well potential $W(u)$ with non-degenerate wells located at $u = u_{\pm}$. When $W(u_-) \neq W(u_+)$, a hyperbolic scaling gives rise to

$$\frac{\partial u}{\partial t} = \varepsilon \Delta u + \frac{1}{\varepsilon} g(u), \qquad (\varepsilon > 0 \text{ is a scaling parameter})$$

in which case the interface is driven by the difference of potential values $W(u_-) - W(u_+)$;

$$V = c,$$

where V stands for the normal speed of the interface and c is a constant determined from W, i.e., $c \propto W(u_+) - W(u_-)$.

When the two wells have the same depth, then $c = 0$ and hence the interface equation above does not give any information on the motion of the interface. In this case, we apply a parabolic scaling to (A-C) so that it has the following form;

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2} g(u).$$

The motion law of the interface for this equaton is the so called "mean curvature flow", namely,

$$V = H,$$

where H stands for the sum of principal curvatures of the interface. Such interface evolutions for reaction-diffusion equations have been established by many authors (see [2]).

The purpose of this paper is to investigate what happens to these interface equations when a convection term is added to (A-C).

1.2. **Reaction-diffusion with convection.** We investigate interfacial phenomena for the following equation;

$$\text{(RDC)} \qquad u_t + \operatorname{div} \mathbf{f}(u) = \Delta u + g(u), \quad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

This equation is derived as follows.

When a physical quantity u is carried by a flux $\mathbb{J}$ with source term $g(u)$, then the balance equation is expressed as

$$\text{(BL)} \qquad \frac{\partial u}{\partial t} + \operatorname{div} \mathbb{J} = g(u).$$

If the flux $\mathbb{J}$ is represented by a (vector-valued) function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^N$ and the gradient of u,

$$\text{(Flux)} \qquad \mathbb{J} = -\mu \nabla u + \mathbf{f}(u) \qquad (\mu > 0 \text{ viscosity}),$$

equation (BL) reduces to (RDC). In diffusive flow fields, the flux $\mathbf{f}$ is supposed to originate from fluid flows, and therefore, should be coupled with Navier-Stokes equations governing the flow field. We are considering here a simplified problem without reference to such flow-field equations. From now on, we set the viscosity equal to 1; $\mu = 1$

To describe the dynamics of (RDC) as $t \to \infty$, we perform a hyperbolic spatio-temporal scaling; $(x,t) \to (x/\varepsilon, t/\varepsilon)$ which reduces (RDC) to

$$\text{(1)} \qquad u_t + \operatorname{div} \mathbf{f}(u) = \varepsilon \Delta u + \varepsilon^{-1} g(u),$$

where $\varepsilon > 0$ is a scaling parameter.

Our objective below is to investigate the dynamics of (1) in the singular limit $\varepsilon \to 0$.

> To consider the singular limit $\varepsilon \to 0$ means that we are describing the variation of u over large spatial ranges as time $t \to \infty$ in the original system.

We work throughout under the following hypotheses.

(H1): (i) $g \in C^2(\mathbb{R})$, $g(u_*) = 0$ at $u_* = u_-, 0, u_+$,

$g'(u_-) < 0, \quad g'(0) > 0, \quad g'(u_+) < 0$ (bi-stable reaction term).

(ii) $\mathbf{f} \in C^3(\mathbb{R}, \mathbb{R}^N)$

Well-posedness of initial value problem: Under the hypothesis **(H1)**, the problem (1) with an initial condition

$$u(x,0) = \phi(x) \in \mathrm{BC}^2_{\mathrm{unif}}(\mathbb{R}^N) \tag{2}$$

possesses a unique global (in time) solution living in $\mathrm{BC}^2_{\mathrm{unif}}(\mathbb{R}^N)$. This is proved by a standard way by using abstract theories for evolution equations (see [1], for example).

2. Planar Waves

First thing to do is to study planar traveling wave solutions. The traveling wave solution of (1) in the $\nu \in S^{N-1}$-direction; $u(x,t) = U(\frac{x\cdot\nu - st}{\varepsilon})$ satisfies

$$\begin{gathered} U''(z) + (s - \mathbf{f}'(U(z))\cdot\nu)\, U'(z) + g(U(z)) = 0, \\ z \in \mathbb{R}, \qquad (' = \mathrm{d}/\mathrm{d}z), \\ U(\pm\infty) = u_\pm, \quad U(0) = 0, \end{gathered} \tag{3}$$

where s is the wave speed to be determined together with the wave profile. The following result is obtained by a phase plane analysis.

Proposition 2.1 ([3] and [5]). (i) *For each direction $\nu \in S^{N-1}$, there uniquely exists a wave speed $s = s(\nu)$ for which the problem (3) has a unique heteroclinic orbit connecting $(u_-, 0)$ (at $z = -\infty$) and $(u_+, 0)$ (at $z = +\infty$).*

(ii) *The wave speed $s(\nu)$ depends on ν as smooth as the nonlinear terms $\mathbf{f}'(u)$ and $g(u)$ do on u.*

(iii) *The wave profile $Q(z;\nu)$ with $Q(0;\nu) = 0$ depends on (z,ν) as smooth as the nonlinear terms $\mathbf{f}'(u)$ and $g(u)$ do on u, and it is a (strictly) monotone increasing function of z.*

(iv) *If $\mathbf{f}(u)$ is even and $g(u)$ is odd in u, then $Q(z)$ is an odd function of z and the wave speed satisfies $s(\nu) \equiv 0$.*

An important feature is that the wave speed is orientation (direction) dependent. This anisotropy later gives rise to anisotropic mean curvature flows.

It is interesting to note that the wave speed and the wave profile are related as follows.

$$s(\nu) = \frac{(\mathbf{f}(u_+) - \mathbf{f}(u_-)) \cdot \nu}{u_+ - u_-} + \frac{1}{u_+ - u_-} \int_{-\infty}^{\infty} g(Q(z;\nu))\mathrm{d}z$$

and

$$s(\nu) = \frac{G(u_-) - G(u_+)}{\int_{-\infty}^{\infty} Q_z(z;\nu)^2 \mathrm{d}z} + \frac{\int_{-\infty}^{\infty} Q_z(z;\nu)^2 \mathbf{f}'(Q(z;\nu)) \cdot \nu \mathrm{d}z}{\int_{-\infty}^{\infty} Q_z(z;\nu)^2 \mathrm{d}z},$$

where $G(u)$ is an anti-derivative of $g(u)$. The first formula looks like a generalized Rankin-Hugoniot condition for viscous shocks for conservation laws, while the second expression resembles the wave speed characterization for bistable reaction-diffusion equations, with modification in terms of the entire wave profile. It is also important to note that $s(\nu)$ depends not only on the asymptotic states (which is the case for reation-diffusion equations), but also on the entire viscous wave profile.

Lemma 2.2. *In the nonlinear eigenvalue problem (3), if the nonlinearities are given by*

$$\begin{aligned} g(u) &= -R(u-u_-)u(u-u_+), \\ &\quad \textit{with } u_- < 0 < u_+, \quad R > 0, \\ \mathbf{f}(u) &= \frac{1}{2}u^2\mathbf{a} + u\mathbf{b}, \quad \mathbf{a},\ \mathbf{b} \in \mathbb{R}^N, \end{aligned}$$

then, the wave speed $s(\nu)$ and wave profile $Q(z)$ are explictly represented as follows.

$$\begin{aligned} s(\nu) &= \frac{u_- + u_+}{4}\left\{\mathbf{a}\cdot\nu - \sqrt{(\mathbf{a}\cdot\nu)^2 + 8R}\right\} + \mathbf{b}\cdot\nu, \\ Q(z) &= \frac{-u_+u_- + u_+u_-\mathrm{e}^{-D(u_+-u_-)z}}{-u_- + u_+\mathrm{e}^{-D(u_+-u_-)z}}, \end{aligned}$$

where D is defined by

$$D = \frac{\sqrt{(\mathbf{a}\cdot\nu)^2 + 8R} - \mathbf{a}\cdot\nu}{4}.$$

The proof is the same as the case without convection term (see [6]). One may also substitute these functions into (3) to directly verify the lemma.

∎

Generically, we expect that the wave speed $s(\nu)$ vanishes on a subset in S^{N-1} which has codimension at least one. However, we have been unable to prove or disprove this expectation in general case. The set where $s(\nu)$ vanishes,

$$\mathcal{P} := \{\nu \in S^{N-1} \mid s(\nu) = 0\}$$

is called a set of pinned directions. When the wave speed is given as in Lemma 2.2, then $\mathcal{P}$ is generically of codimension at least one. Althourgh this is the case for the specific cases, we make the following hypothesis.

(H2): The set of pinned directions $\mathcal{P}$ has codimension at least one in S^{N-1}.

3. Results

We now give some of main results. There are two cases, one in which **(H2)** is valid, and the other where $\mathcal{P} = S^{N-1}$.

3.1. When codim $\mathcal{P} \leq 1$.

We call this case a hyperbolic scaling case.

Theorem 3.1. *Under the hypotheses* **(H1)** *and* **(H2)**, *we consider the following Cauchy problem*

$$\left\{ \begin{array}{rcl} u^\varepsilon_t &=& \varepsilon \Delta u^\varepsilon - \mathbf{f}'(u^\varepsilon) \cdot \nabla u^\varepsilon + \varepsilon^{-1} g(u^\varepsilon) \\ u^\varepsilon(x,0) &=& \phi^\varepsilon(x). \end{array} \right. \tag{4}$$

There exist two functions $\underline{u}^\varepsilon_0(x) < \overline{u}^\varepsilon_0(x)$ and a constant $T > 0$ such that the following statement is true: If the initial function satisfies

$$\underline{u}^\varepsilon_0(x) < \phi^\varepsilon(x) < \overline{u}^\varepsilon_0(x),$$

then the solution $u^\varepsilon(x,t)$ converges to a limit $u^0(x,t) = \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ for almost all $(x,t) \in \mathbb{R}^N \times [0,T]$. The limit function $u^0(x,t)$ is a piece-wise constan function, assuming only two values u_- and u_+. The bulk regions

$$\Omega^\pm(t) := \{x \in \mathbb{R}^N;\ u^0(x,t) = u_\pm\}$$

are separated by a hypersurface $\Gamma(t)$, and the hypersurface (interface) evolves according to the motion law

$$V = s(\nu),$$

where V represents the normal velocity of the interface $\Gamma(t)$, and ν is a unit normal vector on $\Gamma(t)$ pointing into the interior of the bulk region $\Omega^+(t)$.

If we define ε-dependent interface $\Gamma^\varepsilon(t)$ by

$$\Gamma^\varepsilon(t) = \{x \in \mathbb{R}^N \mid u^\varepsilon(x,t) = 0\},$$

then its motion law is governed by

$$V^\varepsilon = s(\nu^\varepsilon) + \varepsilon\left\{H^\varepsilon(y,t) + \sum_{p,q=1}^{N} \mathbf{T}^\varepsilon_{pq} K^{pq}_\varepsilon(y,t)\right\} + \mathcal{O}(\varepsilon^2), \tag{5}$$

where ν^ε is the unit normal vector of $\Gamma^\varepsilon(t)$ (pointing into the interior of + region), $s(\nu^\varepsilon)$ is the wave speed evaluated at ν^ε, $H^\varepsilon(y,t)$ is the sum of principal curvatures (mean curvature, for short) of $\Gamma^\varepsilon(t)$ at $y \in \Gamma^\varepsilon(t)$, $(\mathbf{T}^\varepsilon_{pq})$ is a symmetric, positive semi-definite $N \times N$ matrix depending only on $(\mathbf{f}, g, \nu^\varepsilon)$, K^{pq}_ε is a symmetric tensor related to the second fundamental form of $\Gamma^\varepsilon(t)$.

We note that $T > 0$ in the statement above is determined by the time interval where $V = s(\nu)$ has a smooth soluiton.

Now, let us give explicit forms to the quantities appearing in the theorem. For this purpose, we use the travelling wave profile $Q = Q(z;\nu)$. Let $P = P(z;\nu)$ be defined by

$$P(z) = Q_z(z;\nu)\exp\left(\int_0^z [s(\nu) - \mathbf{f}'(Q(\tau;\nu)\cdot\nu]\,\mathrm{d}\tau\right).$$

We also let $\Gamma(t)$ be represented by γ_0 as follows.

$$\gamma_0 : \mathcal{M} \times [0,T] \ni (y,t) \mapsto \gamma_0(y,t) \in \Gamma(t),$$

where $\mathcal{M}$ is a reference manifold.

Then, representing by $g = (g_{ij})$ and $h = (h_{ij})$ the first and second fundamental forms of $\Gamma(t)$, respectively, with $g^{-1} = (g^{ij})$, we have

$$K^{pq} = \frac{\partial(\gamma_0)^p}{\partial y^j}\frac{\partial(\gamma_0)^q}{\partial y^l} g^{jk} h_{ks} g^{sl},$$

$$\mathbf{T} = M_0^{-1}\int_{-\infty}^{\infty} \frac{1}{PQ_z} L(z)\otimes L(z)\,\mathrm{d}z,$$

$$L(z) = \int_{-\infty}^{z} P(z')Q_z(z')\,[\nabla_\nu s(\nu) - \mathbf{f}'(Q(z'))]\,\mathrm{d}z',$$

$$M_0 = \int_{-\infty}^{\infty} P(z)Q_z(z)\,\mathrm{d}z > 0.$$

It is clear from these formula that $\mathbf{T}$ is positive semi-definite. Generically, we expect that the matrix $\mathbf{T}$ is *positive definite,* not only positive *semi*-definite. To see this, let $\mathbf{a} \in \mathbb{R}^N$, then we have

$$\mathbf{a}^\top \mathbf{T}\mathbf{a} = M_0^{-1} \int_{-\infty}^{\infty} \frac{1}{PQ_z} (L(z) \cdot \mathbf{a})^2 \, \mathrm{d}z \geq 0.$$

Therefore, $\mathbf{a}^\top \mathbf{T}\mathbf{a} = 0$ implies $L(z) \cdot \mathbf{a} \equiv 0$, which in turn implies

$$(\nabla_\nu s(\nu) - \mathbf{f}'(Q(z;\nu))) \cdot \mathbf{a} \equiv 0.$$

This is possible for non-zero $\mathbf{a}$ only when the vector $\nabla_\nu s(\nu) - \mathbf{f}'(Q(z;\nu))$ is parallel to a constant vector for all $z \in \mathbb{R}$. Generically, we do not expect that this should happen.

On the other hand, when $\mathbf{f}'(u) = \mathbf{b}$ is a constant vector, then

$$s(\nu) = c + \mathbf{b} \cdot \nu$$

where c is the traveling wave speed of

$$U_{zz} + cU_z + g(U) = 0, \quad z \in \mathbb{R}$$
$$\lim_{z \to \pm\infty} U(z) = u_\pm, \quad U(0) = 0.$$

Therefore, we have $\nabla_\nu s(\nu) - \mathbf{f}'(Q(z;\nu)) \equiv 0$, and hence $\mathbf{T} = 0$.

3.2. **When $s(\nu) \equiv 0$.** We have been unable to give general conditions which imply $s(\nu) \equiv 0$ on S^{N-1}. In this subsection, therefore, we assume the following conditions are fulfilled.

(H3): $\mathbf{f}(u)$ is even and $g(u)$ is odd.

Evidently, Proposition 2.1 says that **(H3)** implies $s(\nu) \equiv 0$.

Theorem 3.2. *Under the hypotheses* **(H1)** *and* **(H3)**, *we consider the following problem;*

$$\text{(6)} \qquad \begin{cases} u_t^\varepsilon &= \Delta u^\varepsilon - \varepsilon^{-1}\mathbf{f}'(u^\varepsilon) \cdot \nabla u^\varepsilon + \varepsilon^{-2} g(u^\varepsilon) \\ u^\varepsilon(x,0) &= \phi^\varepsilon(x). \end{cases}$$

There exist a class of initial functions ϕ^ε and a constant $T > 0$ such that the solution u^ε of (6) converges to a limit for almost all $(x,t) \in \mathbb{R}^N \times [0,T]$;

$$u^0(x,t) := \lim_{\varepsilon \to 0} u^\varepsilon(x,t).$$

The limi function $u^0(x,t)$ is piecewise constant, taking on two values u_- and u_+.

The interface $\Gamma(t)$ separating two bulk regions $\Omega^\pm(t) := \{x \in \mathbb{R}^N;\ u^0(x,t) = u_\pm\}$ evolves according to the following motion law.

$$\text{(7)} \qquad V = H + \sum_{p,q=1}^{N} \mathbf{T}_{pq} K^{pq} = \sum_{p,q=1}^{N} (\delta_{pq} + \mathbf{T}_{pq}) K^{pq},$$

where

$$K^{pq} = \frac{\partial(\gamma_0)^p}{\partial y^j}\frac{\partial(\gamma_0)^q}{\partial y^l} g^{jk} h_{ks} g^{sl},$$

$$\mathbf{T} = M_0^{-1} \int_{-\infty}^{\infty} \frac{1}{PQ_z} L(z) \otimes L(z)\, \mathrm{d}z,$$

$$L(z) = \int_{-\infty}^{z} P(z) Q_z(z) \mathbf{f}'(Q)\, \mathrm{d}z',$$

$$A(z) = -\int_{0}^{z} \mathbf{f}'(Q(\tau;\nu)) \cdot \nu\, \mathrm{d}\tau,$$

$$P(z) = \mathrm{e}^{A(z)} Q_z(z).$$

As before, $\mathbf{T}$ is positive semi-definite, and generically positive definite. It is also of interest to note that $\mathbf{T}$ introduces a kind of Riemannian metric (possibly degenerate) in the ambient space $\mathbb{R}^N$. If $\mathbf{T}$ were the $N \times N$ identity matrix, then $\mathbf{T}K = \mathrm{tr}(hg^{-1})$ would be the sum of principal curvatures of the interface.

Proposition 3.1. *The sum* $\mathbf{T}_{pq}K^{pq}$ *is a weighted sum of principal curvatures:*

$$\mathbf{T}_{pq}K^{pq} = \sum_{i=1}^{N-1} w^i \kappa_i,$$

where κ_i $(i = 1, 2, \ldots, N-1)$ *are principal curvatures of* $\Gamma(t)$ *and*

$$w^i = \sum_{p,q=1}^{N} \sum_{j=1}^{N-1} \mathbf{T}_{pq} \frac{\partial(\gamma_0)^p}{\partial y^i}\frac{\partial(\gamma_0)^q}{\partial y^j} g^{ji}.$$

Therefore, (7) is rewritten as

(7) $$V = \sum_{i=1}^{N-1} (1 + w^i)\kappa_i,$$

namely, the interface $\Gamma(t)$ *is driven by an anisotropic mean curvature flow.*

By inspecting the proof of Theorem 3.2, we obtain the existence result for equilibrium solutions of (6).

Corollary 3.2. *Let* Γ^* *be* anisotropically minimal *in the sense that*

$$0 = \sum_{i=1}^{N-1} (1 + w^i)\kappa_i \qquad on\ \ \Gamma^*.$$

If Γ^* is non-degenerate in the sense that an elliptic linear operator $\mathcal{L}$ defined on Γ^* does not have 0-eigenvalue, then there exists a family of equilibirum solutions $u^\varepsilon(x)$ of (6) for small $\varepsilon > 0$ so that

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = \begin{cases} u_+ & x \in \Omega_+ \\ u_- & x \in \Omega_- \end{cases} .$$

The linear operator $\mathcal{L}$ has the following explicit form.

$$\begin{aligned} \mathcal{L}A =& \Delta^{\Gamma^*} A + \mathbf{T}_{pq}\left[\nabla_{\Gamma^*}(\nabla_{\Gamma^*}A)^p\right]^q \\ &+ \mathbb{F}\cdot\nabla_{\Gamma^*}A + \mathbb{G}\cdot\nabla^{(1)}_{\Gamma^*}A \\ &+ \left[H^{(1)} - \mathbf{T}_{pq}\left(\nabla^{(1)}_{\Gamma^*}\nu^p\right)^q\right]A, \end{aligned}$$

where Δ^{Γ^*} is the Laplace-Beltrami operator on Γ^*,

$$H^{(1)} = \sum_{j=1}^{N-1}(\kappa_j)^2,$$

$$\nabla_{\Gamma^*}A = \sum_{j,k=1}^{N-1}\frac{\partial\gamma^*}{\partial y^j}g^{jk}\frac{\partial A}{\partial y^k},$$

$$\nabla^{(1)}_{\Gamma^*}A = \sum_{j,s,l,k=1}^{N-1}\frac{\partial\gamma^*}{\partial y^j}g^{js}h_{sl}g^{lk}\frac{\partial A}{\partial y^k},$$

and $\mathbb{F}$ and $\mathbb{G}$ are vector fields on Γ^* to which we do not give explicit forms. However, we emphasize that $\mathcal{L}$ is the linearization of the right hand side of (7) around Γ^*, relative to normal variations of hypersuface.

4. Some Examples

4.1. Symmetric nonlinearity. We deal with the following specific nonlinearity;

$$\mathbf{f}(u) = \frac{1}{2}u^2\mathbf{a} \in \mathbb{R}^N, \qquad g(u) = -u(u^2-1)$$

Then we obtain; wave profile : $Q(z) = \tanh(Dz)$ and wave speed: $s(\nu) \equiv 0$. Note that the ν-dependency of the wave profile is only throught the quantity D defined by

$$D = \frac{1}{4}(\sqrt{(\mathbf{a}\cdot\nu)^2+8} - \mathbf{a}\cdot\nu)$$

To describe the interface equation, we need to compute:

$$Q_q := \frac{\partial Q}{\partial \nu^q} = -\frac{D\mathbf{a}_q}{\sqrt{(\mathbf{a}\cdot\nu)^2+8}}\frac{z}{\cosh^2(Dz)},$$
$$P = Q_z e^A = D\,(\cosh(Dz))^{-(\frac{\mathbf{a}\cdot\nu}{D}+2)}$$
$$M_0 = \int_{-\infty}^{\infty} e^{A(z)} Q_z^2(z)\,\mathrm{d}z = D^2 \int_{-\infty}^{\infty} (\cosh(Dz))^{-(\frac{\mathbf{a}\cdot\nu}{D}+4)}\,\mathrm{d}z$$

and

$$\begin{aligned} M_0\mathbf{T}_{pq} &= -\int_{-\infty}^{\infty} e^{A(z)} Q_z Q_q \mathbf{f}_p'(Q)\,dz \\ &= \frac{D^2\mathbf{a}_p\mathbf{a}_q}{\sqrt{(\mathbf{a}\cdot\nu)^2+8}} \int_{-\infty}^{\infty} z\tanh(Dz)\,(\cosh(Dz))^{-(\frac{\mathbf{a}\cdot\nu}{D}+4)}\,\mathrm{d}z \\ &= \frac{D^2\mathbf{a}_p\mathbf{a}_q}{\sqrt{(\mathbf{a}\cdot\nu)^2+8}}\left(\frac{1}{\mathbf{a}\cdot\nu+4D}\right)\int_{-\infty}^{\infty} (\cosh(Dz))^{-(\frac{\mathbf{a}\cdot\nu}{D}+4)}\,\mathrm{d}z. \end{aligned}$$

Therefore, by using $4D + \mathbf{a}\cdot\nu = \sqrt{(\mathbf{a}\cdot\nu)^2+8}$, we obtain

$$\mathbf{T}_{pq} = \frac{\mathbf{a}_p\mathbf{a}_q}{(\mathbf{a}\cdot\nu)^2+8}.$$

On the other hand, a simple computation yields

$$\mathbf{T}_{pq}K^{pq} = \frac{1}{2D^2+1}\mathbf{a}\cdot\nabla_{\Gamma(t)}D,$$

and the interface equation is given by

$$V = H(y,t) + \mathbf{a}\cdot\nabla_{\Gamma(t)}\left(\frac{\operatorname{Arctan}(\sqrt{2}D)}{\sqrt{2}}\right) \tag{8}$$

Here $D > 0$ in $Q(z) = \tanh(Dz)$ means "**steepness**" of the wave profile, and (8) says that the tangential variation of the "steepness"

$$\frac{\operatorname{Arctan}\left(\sqrt{2}D\right)}{\sqrt{2}}$$

of the wave profile is converted to the normal speed of the interface in the singular limit. At a formal level, this kind of observation was first given in [4].

As before, the second term in the right hand side of (8) $\mathbf{T}_{pq}K^{pq}$ is rewritten and (8) reduces to

$$V = \sum_{i=1}^{N-1}(1 + w^i)\kappa_i, \tag{9}$$

where

$$w^i = \frac{1}{(\mathbf{a}\cdot\nu)^2 + 8}\sum_{j=1}^{N-1}\left(\frac{\partial\gamma_0}{\partial y^j}\cdot\mathbf{a}\right)\left(\frac{\partial\gamma_0}{\partial y^i}\cdot\mathbf{a}\right)g^{ji}$$

In other words, (9) is interpreted as a weighted mean curvature flow. Note that w^i in (9) is 0 when $\mathbf{a}$ is parallel to ν.

Although the matrix $\mathbf{T}$ originates from the first order differential operator div $\mathbf{f}$, it exhibits a curvature effect (a second order differential operator) in the singular limit.

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