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Bifurcation Analysis for a Ginzburg-Landau equation

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1 Introduction

We consider a Ginzburg-Landau functional of the superconductivity in a thin film given by

$$E(\psi):=\int_{D}\left\{\frac{1}{2}|(\nabla-iA_{0})\psi|^{2}+\frac{\kappa^{2}}{4}(1-|\psi|^{2})^{2}\right\}a(x,y)dxdy, \tag{1.1}$$

where D is a 2-dimansional domain, ψ is the complex valued order parameter describing a macroscopic superconducting state, κ is the Ginzburg-Landu parameter, $a(x,y)$ is a positive function describing the variable thickness of the film and A_0 is a vector potential for an applied magnetic field. The above energy can be reduced from the full Ginzburg-Landau energy in a thin domain

$$\Omega(\epsilon)=\{(x,y,z)\in\mathbb{R}^{3}:(x,y)\in D,\quad 0<z<\epsilon a(x,y)\},$$

as $\epsilon\rightarrow 0$. Mathematical justification for the reduction is found in [2], [7] and [8].

Throughout the paper we assume

$$A_{0}=h(0,x),\quad D:=\{(x,y):0<x<L_{1},0<y<L_{2}\},$$

and periodic boundary conditions for ψ in y-direction. Thus we also assume $a(x,y)$ is periodic in y.

By the change of variables and parameters such as

$$\begin{array}{l}x=L_{1}x',\quad y=(L_{2}/2\pi)y',\quad h=(2\pi/L_{1}L_{2})h',\quad\lambda:=\kappa^{2}L_{1}^{2},\quad d:=2\pi L_{1}/L_{2},\\ E(\psi)=E'(\psi)/d,\qquad a'(x',y')=a(L_{1}x',(L_{2}/2\pi)y')\end{array}$$

and dropping the primes, we obtain the non-dimensional form

$$E(\psi):=\int_{0}^{2\pi}dy\int_{0}^{1}\left\{\frac{1}{2}|\psi_{x}|^{2}+\frac{d^{2}}{2}|(\partial/\partial y-ihx)\psi|^{2}+\frac{\lambda}{4}(1-|\psi|^{2})^{2}\right\}a(x,y)dx. \tag{1.2}$$

We extend the domain of $\psi(x,y)$ and $a(x,y)$ over $(0,1)\times\mathbb{R}$ so that $\psi(\cdot,y+2\pi)=\psi(\cdot,y)$ and $a(\cdot,y+2\pi)=a(\cdot,y)\ \ (y\in\mathbb{R})$ are satisfied.

In this article we are dealing with the gradient equation for (1.2) given by

$$\begin{cases} \psi_t = \dfrac{1}{a} D_A(aD_A\psi) + \lambda(1-|\psi|^2)\psi, \\ \qquad\qquad\qquad (x,y,t) \in (0,1) \times \mathbb{R}/2\pi \times (0,\infty), \\ \psi_x(0,y,t) = \psi_x(1,y,t) = 0, \qquad (y,t) \in \mathbb{R}/2\pi \times (0,\infty), \\ \psi(x,y,0) = \psi_0(x,y,0), \qquad (x,y) \in [0,1] \times \mathbb{R}/2\pi, \end{cases} \tag{1.3}$$

and discuss a bifurcation of non-trivial solutions around some critical values of the parameters together with the stability of the bifurcating solutions, where we put

$$D_A := (\partial/\partial x, d(\partial/\partial y - ihx)).$$

Note that $\mathbb{C}$ is always identified with $\mathbb{R}^2$. Hence solutions of (1.3) generate a smooth semiflow in the space $X_1 := H^1((0,1) \times \mathbb{R}/2\pi; \mathbb{C})$. As for the constant case of $a(x,y)$, the work of [3] shows that in the parameter space (h,λ) there are bifurcation curve $C_k : \lambda = \sigma_k(h)$ where a non-trivial solution bifurcates from the trivial solution $\psi = 0$. Such a bifurcating solution can be expressed as

$$\psi = \epsilon w^{(k)}(x)\exp(iky) + O(\epsilon)$$

near the bifurcation curve for each $k \geq 0, k \in \mathbb{Z}$. Here $w^{(k)}$ is a positive eigenfunction of the eigenvalue problem

$$-w_{xx} + d^2(hx-k)^2 w = \sigma w, \qquad w_x(0) = w_x(1) = 0,$$

corresponding to the first eigenvalue $\sigma = \sigma_k(h)$. Then any two curves C_k and C_m ($0 \leq k < m$) intersect at $(h_c, \lambda_c) := (k+m, \sigma_k(k+m))$. Therefore a bifurcation analysis in a neighborhood of (h_c,λ_c) tells that the existence of a mixed mode solution, which is written as

$$\psi = \alpha w^{(k)}(x)\exp(iky) + \beta w^{(m)}(x)\exp(imy) + O(\epsilon^2), \quad |\alpha|, |\beta| = O(\epsilon), \tag{1.4}$$

for $(h,\lambda) = (h_c+\xi, \lambda_c+\eta)$, $\xi, \eta = O(\epsilon^2)$. Moreover, there is a parameter region in which this mixed mode solution allows a vortex structure in the sense that it possesses isolated zeros of ψ (see [3]).

The purpose of this article is to drive a normal form of the reduced equation on the center manifold in the presence of the nonconstant $a(x,y)$. Here we assume the heterogeneity of $a(x,y)$ is small so that it can be regarded as a perturbation to the homogeneous one near (h_c,λ_c). Thus we will be able to investigate how the bifurcation structure near the critical point of (h_c,λ_c) is affected by the presence of a nonconstant $a(x,y)$ and the stability of bifurcating solutions.

2 Reduction of the flow on the center manifold

We recall the following property of the bifurcation curve C_k given by [3]:

$$\frac{d\sigma_0}{dh} > 0 \quad (h > 0),$$

and for $k \geq 1$,

$$\frac{d\sigma_k}{dh} < 0 \quad (h \in (0, k]).$$

Thus for $h > 0$, C_0 and C_k ($k \geq 1$) first intersect at $(h, \lambda) = (k, \sigma_0(k))$ transversally. Moreover, we can prove the next lemma by the induction argument:

Lemma 2.1 *Suppose that for any k, m with $0 \leq k < m$, bounded by a positive number h_M, the two curves C_k and C_m intersect only at $h = k + m$ in $h \in [0, h_M]$. Let N be the maximal integer less than equal to h_M. Then at $h = 2n - 1$ $(1 \leq 2n - 1 < N)$*

$$\sigma_{n-1} = \sigma_n < \sigma_{n-2} = \sigma_{n+1} < \cdots < \sigma_1 = \sigma_{2n-2} < \sigma_{2n} < \cdots < \sigma_N,$$

while at $h = 2n$ $(2 \leq 2n < N)$

$$\sigma_n < \sigma_{n-1} = \sigma_{n+1} < \sigma_{n-2} = \sigma_{n+2} < \cdots < \sigma_0 = \sigma_{2n} < \sigma_{2n+1} < \cdots < \sigma_N$$

holds.

This lemma tells that under the assumption of the lemma only the critical points of $C_{n-1} \cap C_n$ $(n = 1, \ldots, (N+1)/2)$ allow stable bifurcating solutions if we apply the local bifurcation theory near the critical points. In other words, in a neighborhood of the other critical points we are not able to obtain a stable solution by the local bifurcation theory.

Next we specify the perturbation so that $a(x, y)$ is given by

$$a(x, y) = (1 + \nu_1 p(x))(1 + \nu_2 q(y)), \quad |\nu_j| = O(\epsilon^2) \ (j = 1, 2), \quad q(-y) = q(y). \tag{2.1}$$

where ϵ is as in (1.4). Then

$$\frac{\nabla a}{a} = (\nu_1 p_x, \nu_2 q_y) + O(|(\nu_1, \nu_2)|^2).$$

Set

$$\lambda_{k+m} := \sigma_k(k + m) \quad (= \sigma_m(k + m)), \tag{2.2}$$

and for $h = k + m$ define

$$a_1 := 2\pi \int_0^1 (w^{(k)})^2 x dx, \qquad a_2 := 2\pi \int_0^1 (w^{(m)})^2 x dx = 1 - a_1,$$

$$b_1 := 2\pi \int_0^1 (w^{(k)})^2 x^2 dx, \qquad b_2 := 2\pi \int_0^1 (w^{(m)})^2 x^2 dx = b_1 - 2a_1 + 1,$$

$$A_1 := \{-2ka_1 + 2(k+m)b_1\}d^2, \quad A_2 := \{-2ma_2 + 2(k+m)b_2\}d^2,$$

$$C := 2\pi\lambda_{k+m} \int_0^1 (w^{(k)})^4 dx = 2\pi\lambda_{k+m} \int_0^1 (w^{(m)})^4 dx, \tag{2.3}$$

$$D := 2\pi\lambda_{k+m} \int_0^1 (w^{(k)} w^{(m)})^2 dx,$$

$$\gamma_1 := 2\pi \int_0^1 p_x w_x^{(k)} w^{(k)} dx, \quad \tilde{\gamma}_1 := 2\pi \int_0^1 p_x w_x^{(m)} w^{(m)} dx,$$

$$\gamma_2 := \frac{m-k}{2} \int_0^1 w^{(k)} w^{(m)} dx \int_0^{2\pi} q_y \sin(m-k)y \, dy$$

Note that at $h = k + m$, $w^{(m)}(x) = w^{(k)}(1-x)$ holds ([3]).

Denote

$$\langle u, v \rangle := \int_0^1 dx \int_0^{2\pi} u(x,y)\overline{v(x,y)} dy,$$

and

$$\varphi_k := w^{(k)}(x) \exp(iky) \quad (k = 0, 1, 2, \ldots), \quad \|\varphi_k\|_{L^2} = 1.$$

We decompose a function of X_1 as

$$\psi = z_0 \varphi_k + z_1 \varphi_m + \hat{\psi},$$

where $\hat{\psi}$ is orthogonal to φ_k and φ_m, that is,

$$\langle \hat{\psi}, \varphi_k \rangle = \langle \hat{\psi}, \varphi_m \rangle = 0.$$

Then we can naturally define the projection $Q : \psi \mapsto \hat{\psi} = Q\psi$.

Applying the center manifold theorem at $h = k + m$ to the equation (1.3), we obtain the following proposition:

Proposition 2.2 *Let $0 \le k < m$. Assume that only the two curves C_k and C_m intersect at $(h_c, \lambda_c) := (k+m, \sigma_k(k+m))$. Let $\xi := h - h_c$ and $\eta := \lambda - \lambda_c$. Then there exist a neighborhood*

$$\mathcal{U} := \{(z_0, z_1, \xi, \eta, \nu_1, \nu_2, \hat{\psi}) : |(z_0, z_1, \xi, \eta, \nu_1, \nu_2)| < \delta_1, \|\hat{\psi}\|_{X_1} < \delta_2\}$$

and a smooth map $K(z_0, z_1, \xi, \eta, \nu_1, \nu_1) : \mathbb{C}^2 \times \mathbb{R}^4 \to QX_1$ *such that the graph of* K *is contained in* $\mathcal{U}$ *and the semiflow generated by solutions* $\psi(\cdot, \cdot, t) = z_0(t)\varphi_k + z_1(t)\varphi_m + \hat{\psi}(\cdot, \cdot, t)$ *to* (1.3) *can be reduced on the manifold, that is, any bounded trajectory contained in* $\mathcal{U}$ *lies in the manifold. Moreover, by scaling*

$$z_0, z_1 = O(\epsilon), \qquad \xi, \eta, \nu_1, \nu_2 = O(\epsilon^2),$$

the flow on the manifold is given by the system of ordinary differential equations:

$$\begin{cases} \dot{z}_0 = \{-A_1\xi + \eta - Cz_0^2 - 2Dz_1^2\}z_0 + \nu_1\gamma_1 z_0 - \nu_2\gamma_2 z_1 + R_0(z_0, z_1, \xi, \eta), \\ \dot{z}_1 = \{-A_2\xi + \eta - 2Dz_0^2 - Cz_1^2\}z_1 + \nu_1\tilde{\gamma}_1 z_1 - \nu_2\gamma_2 z_0 + R_1(z_0, z_1, \xi, \eta), \end{cases} \tag{2.4}$$

where $R_j = O(\epsilon^4), j = 0, 1$.

The next corollary is proved by the attractivity of the manifold and the invariant foliation along the manifold (see [6], [5] and [4]).

Corollary 2.3 *For* $k = 0, m = 1$, *a non-degenerate stable (resp. an unstable) equilibrium of* (2.4) *for sufficiently small* ϵ *gives a stable (resp. an unstable) equilibrium solution to* (1.3) *by* $\psi = z_0\varphi_k + z_1\varphi_m + K(z_0, z_1, \xi, \eta, \nu_1, \nu_2)$. *If the assumption of Lemma 2.1 holds, then for any* $(k, m) = (n-1, n)$ $(n \leq (N+1)/2)$ *the same assertion holds.*

We remark that a non-degenerate equilibrium is meant by a solution at which the linearized operator has a simple zero eigenvalue corresponding to the invariance for the transformation $\psi \mapsto e^{ic}\psi$ and the other eigenvalues being away from zero.

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