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A VARIATIONAL APPROACH TO A CHOREOGRAPHY IN THE PARALLELOGRAM FOUR-BODY PROBLEM

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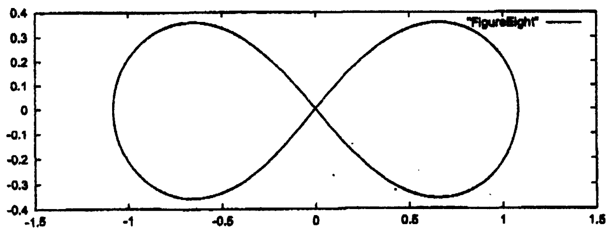
Abstract

This paper indicates one possible direction of studying choreography. Here, we focus on one in the parallelogram four-body problem, the so-called *Super Eight Solution*. It is remarkable that another choreography appears after introducing relative position vectors, which forms Super Eight again. In addition, a projection of the "relative Super Eight" becomes more similar to an ellipse than its original Super Eight. Unfortunately, no proofs of the existence of the Super Eight are included in this article.

1 Background

The Figure Eight Solution in the three-body problem with equal masses was numerically discovered by C. Moore in 1993[6] (see Fig. 1 (left)). In 2000, A. Chenciner and R. Montgomery proved its existence by using the direct method in the variational problem[1]. After the existence proof of the Figure Eight, *choreography* or *choreographic solution* was introduced as a new class of periodic solutions of the N-body problem in which all particles chase each other with tracing the same path. A simple choreography is a periodic solution in which all particles chase each other along a single orbit. A multiple choreography is a periodic solution in which all particles chase each other along plural orbits, each of which is occupied with plural particles.



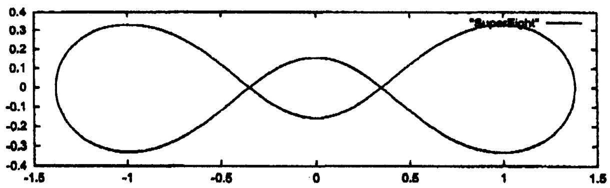


Figure 1: The Figure Eight in the three-body problem (left) and the Super-Eight in the four-body problem (right)

There are known a lot of solutions in the N-body problem [8]. For $N \geq 3$, only Euler's collinear solution ($N = 3$), Lagrange's equilateral solution ($N = 3$), and Moulton's collinear solution ($N \geq 3$) had been rigorously guaranteed to exist before the work by Chenciner and Montgomery. At the present, many choreographies in the N-body problem were discovered by numerical computations while few choreographies were given their existence proofs. It is remarkable that computer-assisted proofs of non-symmetric choreographies were given by Tomasz Kapela and Carles Simó in 2007[5].

The Super Eight solution (see Fig. 1 (right), [7]) is the first choreography in the four-body problem. It was numerically discovered by J. Gerver in 1999, just after the first announcement of the success in the existence proof of the Figure Eight. The existence proof of the Super Eight solution was exhibited by T. Kapela and P. Zgliczyński in 2003[4], whose method is a computer-assisted proof. However, there is no rigorous proof of the existence.

In section 2, we give a formulation of the parallelogram four-body problem and show the way to prove the existence of a solution in the N-body problem via a variational method. In section 3, we give some numerical data about the Super-Eight. In section 4, a new set of variables is introduced, which describes the parallelogram four-body problem much simpler. Finally in section 5, a perspective of a rigorous proof of the existence of the Super-Eight is given.

2 The Parallelogram Four-Body Problem

2.1 Formulation

The Lagrangian of the N-Body Problem: $\mathcal{L} : \mathbb{R}^{dN}\backslash\Delta \times \mathbb{R}^{dN} \to \mathbb{R}$ $(d = 1, 2, 3)$ is defined by $\mathcal{L}(q, \dot{q}) = \mathcal{T}(\dot{q}) + \mathcal{U}(q)$, where $\Delta = \{q \in \mathbb{R}^{dN} \mid q_i = q_j, \text{ for some } i \neq j\}$ is called *collision set*. $\mathcal{T} : \mathbb{R}^{dN} \to \mathbb{R}$ and $\mathcal{U} : \mathbb{R}^{dN}\backslash\Delta \to \mathbb{R}$ are called kinetic energy and force function, respectively. They are defined as follows.

$$\mathcal{T}(\dot{q}) = \frac{1}{2}\sum_{i=1}^{N} m_i\|\dot{q}_i\|^2, \quad \mathcal{U}(q) = \sum_{i<j} \frac{m_i m_j}{\|q_i - q_j\|},$$

where we assume the gravitational constant $G = 1$. The Newtonian equation of motion is

$$M\frac{d^2q}{dt^2} = \frac{\partial \mathcal{U}}{\partial q}, \tag{1}$$

where $M = \mathrm{diag}(m_1, \cdots, m_N)$ is a diagonal matrix, and $m_i \in \mathbb{R}^+$ is mass of ith particle. The equation of motion (1) is not defined at a collision $(q \in \Delta)$. However, the solutions can be smoothly extended beyond a collision if the collision is binary one.

On the other hand, the Lagrangian equation of motion is

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}, \tag{2}$$

which is equivalent to the equation (1). This is derived from the principle of the least action that the solutions of an equation of motion (2) minimizes the following action functional.

$$\mathcal{A}(q, \dot{q}) = \int_0^T \mathcal{L}(q, \dot{q})\, dt, \tag{3}$$

where T is certain positive number. Oppositely, if a minimizer q_0 of $\mathcal{A}(q, \dot{q})$ is collisionless, *i.e.*, $q_0 : [0, T] \to \mathbb{R}^{dN} \backslash \Delta$, then the pass q_0 is a solution of the equation (2). The direct method in the variational problem seeks a minimizer of the action functional $\mathcal{A}(q, \dot{q})$ without solving the differential equation (2). Here is a restriction in applying a variational method to find a solution of the differential equation (2).

The parallelogram four-body problem is a planar subproblem of four-body problem ($d = 2$, *i.e.*, $q_i \in \mathbb{R}^2$), in which particles with equal masses (here to be unity) always occupy vertices of a parallelogram. If a set of initial conditions is given as

$$q_3(0) = -q_1(0), \quad q_4(0) = -q_2(0), \quad \dot{q}_3(0) = -\dot{q}_1(0), \quad \dot{q}_4(0) = -\dot{q}_1(0),$$

then the motion keeps always a parallelogram configuration (Fig. 2):

$$q_3(t) \equiv -q_1(t), \; q_4(t) \equiv -q_2(t). \tag{4}$$

Therefore, the Lagrangian $\mathcal{L}(q, \dot{q})$ can be written down as follows.

$$\mathcal{L}(q, \dot{q}) = \|\dot{q}_1\|^2 + \|\dot{q}_2\|^2 + \frac{1}{2\|q_1\|} + \frac{1}{2\|q_2\|} + \frac{2}{\|q_1 + q_2\|} + \frac{2}{\|q_1 - q_2\|}. \tag{5}$$

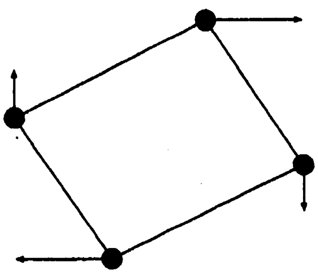


Figure 2: Configuration of the parallelogram four-body problem

2.2 A proof strategy via the variational method

Since $\mathcal{T} \geq 0$ and $\mathcal{U} > 0$, there are extremely many local minimizers of $\mathcal{A}$. Therefore, first, it is necessary to restrict the path-space $H^1([0, T], \mathbb{R}^{dN})$ by using some symmetries. The Super Eight has eight symmetries as will be seen later. Second, it is necessary to estimate the action functional $\mathcal{A}$ on suitable paths in order to show

$$\inf(\mathcal{A}_{\textit{all collision paths}}) > \mathcal{A}_{\textit{non-collision test path}},$$

where any path belongs to the path-space $H^1([0, T], \mathbb{R}^{dN})$. Main difficulty is to obtain good estimates.

3 Description of the Super Eight

One simple choreography and many double choreographies are known in the parallelogram four-body problem. The Super-Eight is the known simple choreography. Since the system is scalable, one can take initial conditions such that the period T becomes 2π. Then, we have the initial conditions $(q_1, q_2, \dot{q}_1, \dot{q}_2)$ as follows.

$$\begin{pmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 1.382857 & 0 \\ 0 & 0.157030 \\ 0 & 0.584873 \\ 1.87935 & 0 \end{pmatrix} \tag{6}$$

The values of the Hamiltonian $\mathcal{H} \equiv \mathcal{T} - \mathcal{U}$ and the angular momentum $L \equiv \sum q_i \times \dot{q}_i$ are constant along any solution. In the case of the Super Eight, the values take

$$\mathcal{H} \approx -2.57354955, \qquad L \approx 1.029691538.$$

The value of the action functional for the Super Eight is

$$\mathcal{A} \approx 48.5475515. \tag{7}$$

On the other hand, the values of the potential $\mathcal{U}$ and the moment of inertia $\mathcal{I} \equiv \sum q_i^2$ are not constant along the Super Eight. Time variations in $\mathcal{U}$ and $\mathcal{I}$ are both periodic with their period $\pi/2$ (see Fig. 3). The mean values and maximum ranges of variations

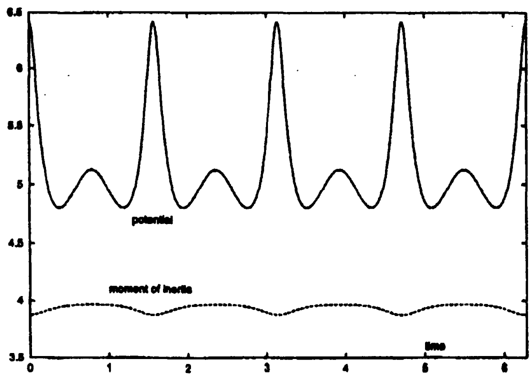


Figure 3: Variation in $\mathcal{U}$ (upper) and in $\mathcal{I}$ (lower) along the Super-Eight

in $\mathcal{U}$ and $\mathcal{I}$ are as follows.

$$\overline{\mathcal{U}} \approx 5.14677, \quad \Delta\mathcal{U} \approx 1.61674, \quad \overline{\mathcal{I}} \approx 3.91874, \quad \Delta\mathcal{I} \approx 8.99215 \times 10^{-2}.$$

These values were calculated by the trapezoidal formula.

Particles in the Super Eight repeat three different type of configurations along the solution. At the initial condition 6, the particles form a rhomboid (Fig. 4 (left)). The next configuration is collinear before the particles form a rectangle (Fig. 4 (right)). The

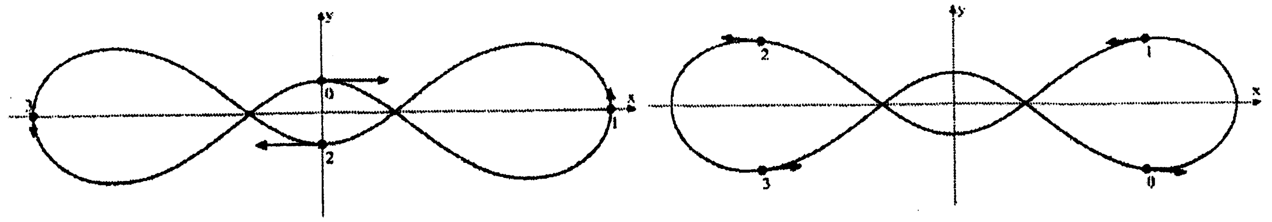


Figure 4: Motion of the Super-Eight
$\cdots \to$ Rhomboidal$\to$Collinear$\to$Rectangular$\to$Collinear $\to \cdots$

next configuration is collinear again before another rhomboid which are equivalent to the initial configuration except for the permutation of particles. After four times of these changes of the form, they return to the initial positions. Therefore, the position vectors satisfy the following condition.

$$q_{i+i}(t+\frac{T}{4}) = q_i(t). \tag{8}$$

Since two configurations: rhomboid and rectangle are symmetry with respect to collinear configuration, the Super Eight has eight symmetries.

4 Change of Variables

4.1 Polar transformation

We introduce a transformation $(q_1, q_2) \to (r, \varphi, \theta_1, \theta_2)$ defined by

$$q_1 = (r\cos\varphi\cos\theta_1, r\cos\varphi\sin\theta_1), \quad q_2 = (r\sin\varphi\cos\theta_2, r\sin\varphi\sin\theta_2), \tag{9}$$

with $r > 0$, $\varphi \in (0, \pi/2)$ and $\theta_i \in (0, 2\pi)$. This leads a point-transformation of the Lagrangian $\mathcal{L}$. The Lagrangian involves θ_i in the form of $\theta_1 - \theta_2$ only. If we introduce additional transformation: $2\theta = \theta_1 - \theta_2$, $2\psi = \theta_1 + \theta_2$, then we can eliminate the variable ψ from the equation of motion. The resulting Lagrangian $\mathcal{L}(r, \varphi, \theta, \dot{r}, \dot{\varphi}, \dot{\theta})$ is

$$\begin{aligned} \mathcal{L} &= \dot{r}^2 + r^2(\dot{\varphi}^2 + \dot{\theta}^2\sin^2 2\varphi) + \frac{c^2}{r^2} + \frac{1}{2r}\left(\frac{1}{\cos\varphi} + \frac{1}{\sin\varphi}\right) \\ &\quad + \frac{2}{r}\left(\frac{1}{\sqrt{1+\sin 2\varphi\cos 2\theta}} + \frac{1}{\sqrt{1-\sin 2\varphi\cos 2\theta}}\right), \end{aligned} \tag{10}$$

where c is a value of the angular momentum L which satisfies

$$c = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = 2r^2(\dot{\psi} + \dot{\theta}\cos 2\varphi) = L. \tag{11}$$

Here, we give a meaning of the new variables r, θ and φ (see Table 1). The size of the system r satisfies $\mathcal{I} = 2r^2$ while θ and φ indicate the shape formed by the particles.

Table 1: Configurations corresponding to the values of (θ, φ)

(θ, φ)	q_1, q_2	Configuration
$\theta = 0,\ \pi/2$	$q_1 \parallel q_2$	Collinear
$\theta = \pi/4$	$q_1 \perp q_2$	Rhomboidal
$\varphi = \pi/4$	$\|q_1\| = \|q_2\|$	Rectangular
$\varphi = 0$	$q_2 = 0$	Binary collision of m_2 and m_4
$\varphi = \pi/2$	$q_1 = 0$	Binary collision of m_1 and m_3
$(\theta, \varphi) = (0, \pi/4)$	$q_1 = q_2$	Simultaneous binary collisions
$(\theta, \varphi) = (\pi/2, \pi/4)$	$q_1 = -q_2$	Simultaneous binary collisions

Variations of r, θ and φ are plotted in Fig. 5 (left). A phase point on the (θ, φ) plane draws an oval (Fig. 5 (right)). As well as $\mathcal{I}$ r changes periodically with its period $\pi/2$. These variables θ and φ oscillate with their period π. Therefore, a phase point on the (θ, φ) plane revolves the oval twice during the period 2π. This oval has two symmetries with respect to the horizontal and the vertical axes because the Super Eight has eight symmetries along its motion as mentioned above.

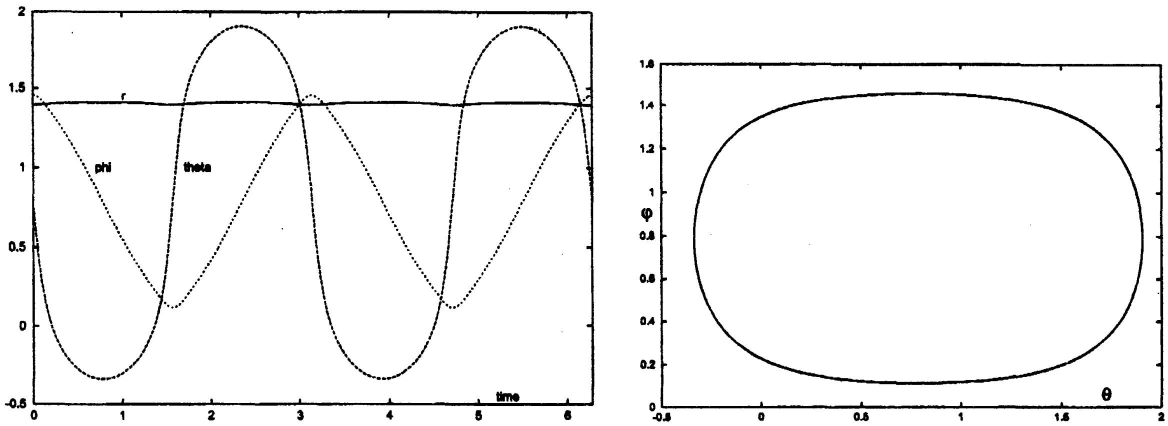


Figure 5: Additional descriptions of the Super Eight
Variations in $r(t)$, $\theta(t)$ and $\varphi(t)$ (left) and the orbit on the (θ, φ) plane (right)

4.2 Approximation of $r(t)$, $\theta(t)$ and $\varphi(t)$

From the facts mentioned in the previous subsection, we can assume the following Fourier expansions of $r(t)$, $\theta(t)$ and $\varphi(t)$.

$$r(t) \sim \sum_{k=0} a_k \cos 4kt, \tag{12}$$

$$\theta(t) \sim \frac{\pi}{4} + \sum_{k=1} b_k \sin 2kt, \quad \varphi(t) \sim \frac{\pi}{4} + \sum_{k=1} c_k \cos 2kt. \tag{13}$$

The corresponding Fourier spectra are shown in Table 2. Collecting three terms with larger amplitude, the maximum differences between the raw data and the approximations are obtained as follows.

$$|a_0 + a_1 \cos 4t + a_4 \cos 16t + a_5 \cos 20t - r(t)| \leq 0.008622584 \tag{14}$$

$$|\frac{\pi}{4} + b_1 \sin 2t + b_3 \sin 6t + b_5 \sin 10t - \theta(t)| \leq 0.068110115 \tag{15}$$

$$|\frac{\pi}{4} + c_1 \cos 2t + c_3 \cos 6t + c_5 \cos 10t - \varphi(t)| \leq 0.009116417 \tag{16}$$

Table 2: Fourier Spectra for the Super Eight

k	a_k	b_k	c_k
0	1.403205921	$\pi/4$	$\pi/4$
1	−0.005838473	−1.283479123	0.613771791
2	−0.000826698	−0.000013778	0.001680927
3	−0.001086241	−0.220851381	0.037820524
4	−0.001428413	−0.000016983	0.001678808
5	0.001536807	−0.083451852	0.015908543

Suppose that, instead of the series (12),

$$r \sim r_0 - \Delta r \cos 4t, \tag{17}$$

where $r_0 = (r_{max} + r_{min})/2 = 1.399775$ and $\Delta r = r_{max} - r_{min} = 0.00803$. Then, we have the following estimate instead of the inequality (14).

$$|r_0 - \Delta r \cos 4t - r(t)| \leq 0.005124335, \tag{18}$$

whose convergence is faster than that of the cosine series of $r(t)$ (12). Figure 6 shows variation in $r(t)$ and $r_0 - \Delta \cos 4t$ and the schematic view of the path in the (θ,φ) plane. Therefore, the θ-φ path should have two symmetries with respect to $\theta = \pi/4$ and $\varphi = \pi/4$, and two restriction to avoid to reach the collision set Δ which is expressed as two bold lines and two dots in Figure 6 (right).

The advantage in the transformation (9) is that shape of the θ-φ path is simple while the disadvantage are that expression of the potential is complicated, therefore, that the estimate of the action functional becomes hard.

4.3 Another Transformation and Relative Super Eight

Let $Q_1 = (q_1 + q_2)/2$ and $Q_2 = (q_1 - q_2)/2$, then the potential and the kinetic energy become simpler as follows.

$$\mathcal{U}(q,Q) = \sum_{i=1}^{2} \left[\frac{1}{2\|q_i\|} + \frac{1}{\|Q_i\|}\right], \quad \mathcal{T}(\dot{q},\dot{Q}) = \sum_{i=1}^{2} \left[\frac{1}{2}\dot{q}_i^2 + \dot{Q}_i^2\right].$$

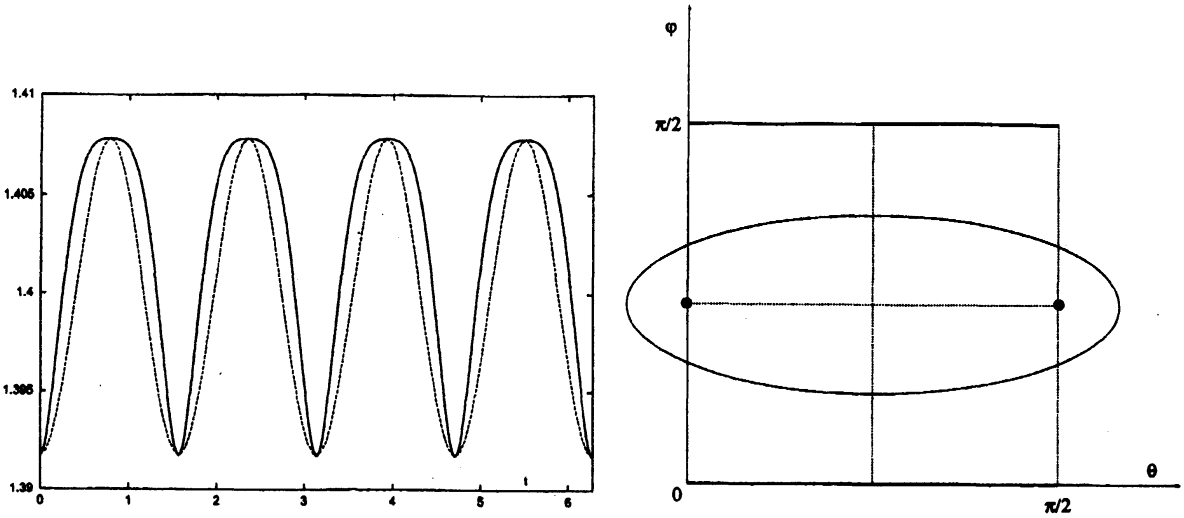


Figure 6: Variation in $r(t)$ and its approximation (right) and the approximation of the path in the (θ, φ) plane (left). Two bold lines and two dots show collisions (see Table 1).

Therefore, the Lagrangian becomes simpler.

$$\mathcal{L}(q, Q, \dot{q}, \dot{Q}) = \sum_{i=1}^{2} \left[\frac{1}{2} \left(\dot{q}_i^2 + \frac{1}{\|q_i\|} \right) + \left(\dot{Q}_i^2 + \frac{1}{\|Q_i\|} \right) \right] \tag{19}$$

Using this new Lagrangian, the problem becomes minimization of the action functional

$$\mathcal{A}(q, Q, \dot{q}, \dot{Q}) = \int_0^T \mathcal{L}(q, Q, \dot{q}, \dot{Q}) dt \tag{20}$$

under the restriction of $Q_1 = (q_1 + q_2)/2$ and $Q_2 = (q_1 - q_2)/2$, or $q_1 = Q_1 + Q_2$ and $q_2 = Q_1 - Q_2$.

Let us introduce $Q_3 = (q_3 + q_4)/2$ and $Q_4 = (q_3 - q_4)/2$. Since q keeps parallelogram ($q_1 + q_3 = q_2 + q_4 = 0$), we have $Q_1 + Q_3 = Q_2 + Q_4 = 0$, *i.e.*, Q keeps another parallelogram again. The correspondence of configurations of q and Q is summarized in Table 3.

In addition, it is remarkable that relative positions Q_l and Q_2 draw Super Eight again (see Fig. 7 (left)). The new "particles" chase each other on the same path, which form a choreography. I would like to call it *relative choreography* only in this article. Note that this terminology is meant differently from the one used in [2], etc. We can execute the polar transformation from Q to (R, Θ, Φ) by the same procedure as expression (9). From the relation between q and Q, we have $R = \|Q\| = \|q\|/\sqrt{2} = r/\sqrt{2}$. The transformed path in the (Θ, Φ) plane forms oval again, which is closer to an ellipse. At the first glance, I guess this oval is similar to an ellipse with its ratio of two axes being 2. After scaling it by a factor 0.9, we have a superposition of the oval and the ellipse (see Fig. 7 (right)). The difference between the oval and the ellipse is very small. It is good to use these variables Q, Θ and Φ.

If we extend the system described with q to the other one described with (q, Q), we obtain the solution of a double choreography (see Fig. 8 (left)).

Table 3: Correspondence of configurations of q and Q

q	Configuration of q	Q	Configuration of Q
$q_1 \parallel q_2$	Collinear	$Q_1 \parallel Q_2$	Collinear
$q_1 \perp q_2$	Rhomboidal	$\|Q_1\| = \|Q_2\|$	Rectangular
$\|q_1\| = \|q_2\|$	Rectangular	$Q_1 \perp Q_2$	Rhomboidal
$q_1 = q_3 = 0$	Binary collision	$Q_1 = Q_4,\ Q_2 = Q_3$	Simultaneous B.C.
$q_2 = q_4 = 0$	Binary collision	$Q_1 = Q_2,\ Q_3 = Q_4$	Simultaneous B.C.
$q_1 = q_2,\ q_3 = q_4$	Simultaneous B.C.	$Q_2 = Q_4 = 0$	Binary collision
$q_1 = q_4,\ q_2 = q_3$	Simultaneous B.C.	$Q_1 = Q_3 = 0$	Binary collision

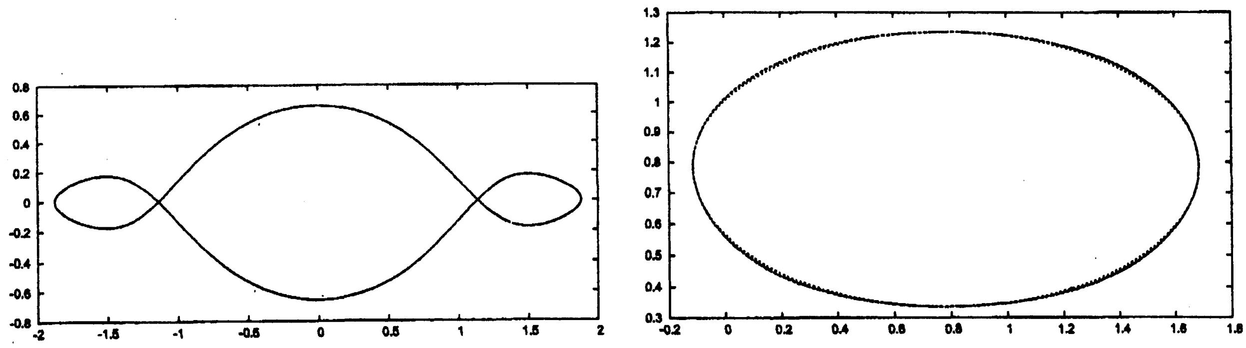


Figure 7: Another choreography (left) and the superposition of the new oval with an ellipse: $(\Theta - \pi/4)^2 + (2\Phi - \pi/2)^2 = 0.9^2$ (right)

4.4 Approximation of $\Theta(t)$ and $\Phi(t)$

We can assume the following Fourier expansions of $\Theta(t)$ and $\Phi(t)$ with their coefficients summarized in Table 4. The expression of $R(t)$ is omitted because $R(t) = r(t)/\sqrt{2}$.

$$\Theta(t) \sim \frac{\pi}{4} + \sum_{k=1} d_k \cos 2kt, \quad \Phi(t) \sim \frac{\pi}{4} + \sum_{k=1} e_k \sin 2kt. \tag{21}$$

Table 4: Fourier Spectra for the relative choreography

k	d_k	e_k
1	−0.866803634	0.470667924
2	−0.000128659	−0.000009556
3	−0.010538565	0.029637011
4	−0.000130458	−0.000015164
5	−0.013824220	0.011852837

Collecting three terms with larger amplitude, the maximum differences between the

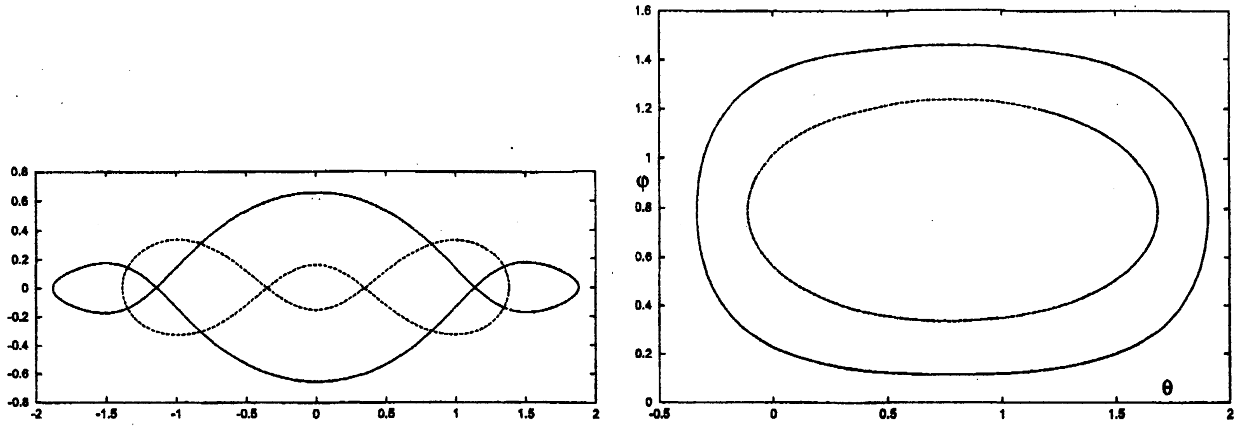


Figure 8: Double choreography induced from the Super-Eight

raw data and the approximations are obtained as follows.

$$|\frac{\pi}{4} + d_1 \cos 2t + d_3 \cos 6t + d_5 \cos 10t - \Theta(t)| \leq 0.007301567 \tag{22}$$

$$|\frac{\pi}{4} + e_1 \sin 2t + e_3 \sin 6t + e_5 \sin 10t - \Phi(t)| \leq 0.007313565 \tag{23}$$

The approximation (21) becomes better than (13), especially in θ and Θ. These approximated paths of the Super-Eight will be appropriate candidates to give better estimates of the action functional $\mathcal{A}$.

5 Towards a rigorous proof for the Super Eight

As we mentioned in the subsection 2.2, we need to have the lowest estimate of $\mathcal{A}_{collision}$ whose paths are taken in the same path space as that of the test path. One choice of the path space is a set of smooth curves connecting two linear segments σ_1 and σ_2 in the (Θ, Φ) plane with $\Theta \geq \pi/4$ and $\Phi \geq \pi/4$:

$$\sigma_1 \equiv \{(\Theta, \Phi) \mid \Theta = \frac{\pi}{4}, \frac{\pi}{4} < \Phi < \frac{\pi}{2}\},\ \sigma_2 \equiv \{(\Theta, \Phi) \mid \Phi = \frac{\pi}{4}, \frac{\pi}{2} < \Theta < \frac{3\pi}{4}\}.$$

At the both ends, the curves must be perpendicular to σ_1 and σ_2. A path in this space is one eighth of the whole path of the Super Eight. So, the action functional is an integral on the interval $0 \leq t \leq T/8 = \pi/4$. $\mathcal{A}_{collision}$ will be estimated in this path space whose path has at least one collision at an arbitrary instance between two ends. Since the longer path gives a larger value of the action functional, the collisional path should be shorter, therefore, should have only one collision, which is one from two choices: single or simultaneous binary collision. This means that estimate of $\mathcal{A}_{single\ collision}$ and $\mathcal{A}_{simultaneous\ collision}$ are required.

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