数理解析研究所講究録
第 1592 巻 2008 年 69-74

UNIFORM L^2-STABILITY FOR THE BOLTZMANN EQUATION

SEUNG-YEAL HA AND SEOK-BAE YUN

ABSTRACT. We discuss a recent progress on the uniform L^2-stability for the Boltzmann equation in a close-to-Maxwellian regime.

1. INTRODUCTION

The purpose of this article is to present a recent formulation [6] on the uniform L^2-stability for the Boltzmann equation near a global Maxwellian. Consider the Boltzmann equation describing the phase space evolution of a distribution function $F = F(x, \xi, t)$ of moderately dilute gas particles with the physical position $x \in \Omega$ and the velocity $\xi \in \mathbb{R}^3$ at time $t \in \mathbb{R}_+$:

$$\begin{aligned} &\partial_t F + \xi \cdot \nabla_x F = Q(F, F), \quad x \in \Omega,\ \xi \in \mathbb{R}^3,\ t > 0, \\ &F(x, \xi, 0) = F^{in}(x, \xi), \end{aligned} \tag{1.1}$$

where $Q(F, F)$ is a quadratic collision operator whose explicit form will defined below.

Let (ξ', ξ'_*) be the post-collisional velocities defined in terms of pre-collisional velocities (ξ, ξ_*) and $\omega \in \mathbb{S}^2_+$:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega \quad \text{and} \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega. \tag{1.2}$$

In this case, the collision operator is given by the following form:

$$Q(F, F)(\xi) \equiv \frac{1}{\kappa} \iint_{\mathbb{R}^3 \times \mathbb{S}^2_+} q(\xi - \xi_*, \omega)(F' F'_* - F F_*) d\omega d\xi_*. \tag{1.3}$$

Here κ is the Knudsen number which is the ratio between the mean free path and the characteristic length of the flow, $\mathbb{S}^2_+ \equiv \{\omega \in \mathbb{S}^2 : (\xi - \xi_*) \cdot \omega > 0\}$, and we used standard abbreviated notations:

$$F' \equiv F(x, \xi', t), \quad F'_* \equiv F(x, \xi'_*, t), \quad F \equiv F(x, \xi, t) \quad \text{and} \quad F_* \equiv F(x, \xi_*, t).$$

We assume that the collision kernel $q(\cdot, \cdot)$ satisfies the inverse power law and the angular cut-off assumption:

$$q(\xi - \xi_*, \omega) = |\xi - \xi_*|^\gamma b_\gamma(\theta), \quad -\frac{3}{2} < \gamma \le 1 \quad \text{and} \quad \frac{b_\gamma(\theta)}{\cos\theta} \le b_* < \infty,$$

where θ is the angle between $\xi - \xi_*$ and ω:

$$\theta \equiv \cos^{-1}\Big(\frac{(\xi - \xi_*) \cdot \omega}{|\xi - \xi_*|}\Big).$$

The spatial domain Ω is assumed to be either whole space $\mathbb{R}^3$ or a torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{L}^3$ ($\mathbb{L}$: any 3-dimensional lattice in $\mathbb{R}^3$) to focus on the initial value problem. Throughout the paper, we shall restrict ourselves to the Boltzmann equation in a maxwellian regime, and denote by C the generic constant independent of time t.

In a global maxwellian regime, there are many literatures available for the existence theory of solutions and convergence toward a global maxwellian (see [2, 3] for a detailed survey). We next briefly review only the global existence theory of solutions to (1.1). In [10], Ukai first established the global existence of mild solutions to the Boltzmann equation for hard potential and hard sphere models combining a spectral analysis and a bootstrapping argument. Later Caflisch[1] and Ukai-Asano [11] further extended Ukai's seminal work to the moderately soft potentials $\gamma \in (-1, 0]$ on a periodic domain and whole space respectively. For the general case of $\gamma \in (-3, 0]$, the global existence of classical solutions was finally settled by Guo [5] employing an energy method. A global existence theory in an energy space $H^s_x(L^2_\xi)$ $(s \geq 8)$ became available only in recent years due to Liu-Yang-Yu [8] and Guo [4]. In particular, Liu, Yang and Yu in [7] introduced a macro-microscopic decomposition of the solution so that the Boltzmann equation can be rewritten as a new fluid type system and an equation for a non-fluid component. Hence the existence theory for (1.1) in a global Maxwellian regime is now in a good shape for small perturbations.

The rest of this paper is organized as follows. In Section 2, we review the basic properties of the linearized collision operator and micro-macro decomposition of a solution and the Boltzmann equation, and key trilinear estimates for the stability analysis. In Section 3, we discuss a priori uniform L^2-stability estimates [6] for the Boltzmann equation with moderately soft potentials $-\frac{3}{2} < \gamma \leq 1$.

Notations: Throughout the paper, we use various local and global norms on Ω, $\mathbb{R}^3_\xi$ and $\Omega \times \mathbb{R}^3_\xi$. Let $h = g(x, t, \xi)$ be a measurable function on $\Omega \times \mathbb{R}_t \times \mathbb{R}^3_\xi$. Below, $p, q \in [1, \infty]$:

$$\|h(x,t)\|_{L^q_\xi} \equiv \begin{cases} \left(\int_{\mathbb{R}^3} |f(x,\xi,t)|^q d\xi\right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \mathrm{esssup}_{\xi\in\mathbb{R}^3} |f(x,\xi,t)|, & q = \infty, \end{cases}$$

$$\|h(t)\|_{L^p_x(L^q_\xi)} \equiv \begin{cases} \left(\int_{\mathbb{R}^3} \|h(x,t)\|^p_{L^q_\xi} dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \mathrm{esssup}_{x\in\mathbb{R}^3} \|h(x,t)\|_{L^q_\xi}, & p = \infty, \end{cases} \qquad \|h(t)\|_{L^p} \equiv \|h(t)\|_{L^p_x(L^p_\xi)}.$$

2. PRELIMINARIES

In this section, we review the basic properties of collision operators around a global Maxwellian, and micro-macro decomposition introduced in [7, 8]. Consider the Boltzmann equation

$$\partial_t F + \xi \cdot \nabla_x F = Q(F, F), \qquad x \in \Omega,\ \xi \in \mathbb{R}^3,\ t \in \mathbb{R}_+,$$
$$F(0, x, \xi) = F_0(x, \xi).$$

We now introduce a symmetric bilinear operator $Q[F, G]$ associated with $Q(F, F)$:

$$Q[F,G](\xi) \equiv \frac{1}{2\kappa} \iint_{\mathbb{R}^3\times\mathbb{S}^2_+} q(\xi - \xi_*, \omega)\Big(F'G'_* + F'_*G' - FG_* - F_*G\Big) d\omega d\xi_*.$$

Then it is easy to see that

$$Q[F, F] \equiv Q(F, F).$$

2.1. **The Boltzmann equation near M.** In this part, we study the linearization of the Boltzmann equation around a global Maxwellian. We first introduce the perturbation f as

$$F = M + M^{\frac{1}{2}} f, \qquad M \equiv \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{|\xi|^2}{2}}. \tag{2.1}$$

Then the perturbation f satisfies the linearized Boltzmann equation:

$$\partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f, f), \tag{2.2}$$

where $L(\cdot)$ and $\Gamma(\cdot,\cdot)$ are linear and nonlinear collision operators

$$L(f) \equiv 2M^{-\frac{1}{2}} Q[M, M^{\frac{1}{2}} f] \qquad \text{and} \qquad \Gamma(f,f) \equiv M^{-\frac{1}{2}} Q[M^{\frac{1}{2}} f, M^{\frac{1}{2}} f].$$

We formally define a quadratic form $\Gamma[\cdot,\cdot]$ associated with $\Gamma(\cdot,\cdot)$:

$$\Gamma[g,h] \equiv M^{-\frac{1}{2}} Q[M^{\frac{1}{2}} g, M^{\frac{1}{2}} h].$$

Proposition 2.1. [2] *For the Boltzmann equation (2.2), there exist positive constants $\nu_1 = \nu_1(\gamma), \nu_2 = \nu_2(\gamma), \sigma, k_1, k_2, k_3, k_4$ such that*

(1) *L has the decomposition*

$$L = -\nu(\xi) I + K,$$

where Id is an identity operator and $\nu(\xi)$ is a collision frequency satisfying

$$\nu_1 \langle \xi \rangle^{\gamma} \le \nu(\xi) \le \nu_2 \langle \xi \rangle^{\gamma}, \qquad \langle \xi \rangle = 1 + |\xi|, \quad \xi \in \mathbb{R}^3,$$

and K is a compact operator.

(2) *L is a non-positive self-adjoint operator on L^2_ξ with the estimate*

$$\langle Lh, h \rangle \le -\sigma \langle \nu^{\frac{1}{2}} \mathbf{P}_1 h, \mathbf{P}_1 h \rangle.$$

where $\langle \cdot,\cdot \rangle$ is a usual L^2-inner product.

2.2. **Micro-macro decomposition.** In this part, we briefly present the micro-macro decomposition which enable us to see the multi-scale nature of the Boltzmann equation. This beautiful idea of decompose the solution and the Boltzmann equation to see its corresponding fluid part and non-fluid part directly at a time was introduced by Liu and Yu in [7] to the study of the positivity of Boltzmann shock. This micro-macro decomposition will play a key role in our L^2-stability analysis for hard potential case in Section 3.2.

The linear collision operator L defines an unbounded symmetric operator on L^2_ξ:

$$L^2_\xi \equiv (L^2_\xi(\mathbb{R}^3), \langle \cdot,\cdot \rangle) \;\text{ and }\; \langle f, g \rangle \equiv \int_{\mathbb{R}^3} f(\xi) g(\xi) d\xi \qquad \text{for } f, g \in L^2_\xi.$$

The null space $\mathcal{N}$ of L is a five-dimensional vector space spanned by an orthonormal basis $\{\chi_i\}_{i=0}^{i=4}$:

$$\mathcal{N} \equiv \text{span}\{\chi_0, \chi_1, \chi_2, \chi_3, \chi_4\},$$

$$\chi_0 = M^{\frac{1}{2}}, \quad \chi_i = \xi_i M^{\frac{1}{2}}, \quad \chi_4 = \frac{1}{\sqrt{6}}(|\xi|^2 - 3) M^{\frac{1}{2}}, \quad \langle \chi_i, \chi_j \rangle = \delta_{ij}, \qquad i = 1,2,3.$$

We decompose Hilbert space L^2_ξ as a direct sum of $\mathcal{N}$ and its orthogonal component $\mathcal{N}^\perp$, and we denote by $\mathbf{P}_0$ the projection on this null space and $\mathbf{P}_1$ the complementary projection:

$$\begin{cases} f = \mathbf{P}_0 f + \mathbf{P}_1 f = f_0 + f_1, \\ f_0 = \mathbf{P}_0 f \equiv \rho(x,t)\chi_0 + \sum_{i=1}^3 m_i(x,t)\chi_i + e(x,t)\chi_4, \\ \rho(x,t) = \langle f, \chi_0 \rangle, \ m_i(x,t) = \langle f, \chi_i \rangle \ (i = 1,2,3), \ e(x,t) = \langle f, \chi_4 \rangle, \\ f_1 = \mathbf{P}_1 f = f - f_0, \end{cases}$$

We next present trilinear estimates for nonlinear term $\Gamma[f+g, f-g](f-g)$. The property of $\Gamma[f+g, f-g] \in \mathcal{N}^\perp$ and Cauchy-Schawarz yield the following estimates.

Lemma 2.1. [6] *Let $-\frac{3}{2} \le \gamma \le 1$, and f, g be measurable functions in $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$ satisfying*

$$||\nu^{\frac{1}{2}}(f+g)||_{L^\infty_x(L^2_\xi)} < \infty, \quad ||f-g||_{L^2} + ||\nu^{\frac{1}{2}}\mathbf{P}_1(f-g)||_{L^2} < \infty.$$

Then there exists a positive constant C independent of t such that

(*i*) $-\dfrac{3}{2} \le \gamma \le 0$;

$$\Big|\int_{\mathbb{R}^3} \langle \Gamma[f+g, f-g], f-g\rangle(x)dx\Big| \le C\Big(||\nu^{\frac{1}{2}}f(t)||^2_{L^\infty_x(L^2_\xi)} + ||\nu^{\frac{1}{2}}g(t)||^2_{L^\infty_x(L^2_\xi)}\Big)||f(t)-g(t)||^2_{L^2} + \frac{\sigma}{2}||\nu^{\frac{1}{2}}\mathbf{P}_1(f(t)-g(t))||^2_{L^2}.$$

(*ii*) $0 < \gamma \le 1$;

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \langle \Gamma[f+g, f-g], f-g\rangle(x)dx\Big| &\le C\Big(||\nu^{\frac{1}{2}}f(t)||^2_{L^\infty_x(L^2_\xi)} + ||\nu^{\frac{1}{2}}g(t)||^2_{L^\infty_x(L^2_\xi)}\Big)||f(t)-g(t)||^2_{L^2} \\ &+ \Big[C(||f(t)||_{L^\infty_x(L^2_\xi)} + ||g(t)||_{L^\infty_x(L^2_\xi)}) + \frac{\sigma}{2}\Big]||\nu^{\frac{1}{2}}\mathbf{P}_1(f(t)-g(t))||^2_{L^2}.\end{aligned}$$

3. A PRIORI UNIFORM L^2-STABILITY

In this section, we briefly present a priori uniform L^2-stability estimates. For details, we refer to [6]. Let f and g be two classical solutions to the Boltzmann equation (2.2) and $f, g \in L^\infty(\mathbb{R}_+; L^2_{x,\xi} \cap L^\infty_x(L^2_\xi))$. Then f and g satisfy

$$\partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f,f), \tag{3.1}$$

$$\partial_t g + \xi \cdot \nabla_x g = L(g) + \Gamma(g,g). \tag{3.2}$$

We subtract (3.2) from (3.1) and multiply $(f-g)$ to both sides to find

$$\partial_t|f-g|^2 + \xi\cdot\nabla_x|f-g|^2 = L(f-g)(f-g) + \Gamma[f+g, f-g](f-g). \tag{3.3}$$

We now integrate (3.3) with respect to (x,ξ) using the boundary condition and Proposition 2.1 to see

$$\begin{aligned}\frac{d}{dt}||f(t)-g(t)||^2_{L^2} &= \int_\Omega \langle L(f-g), f-g\rangle dx + \int_\Omega \langle \Gamma[f+g, f-g], f-g\rangle dx \\ &\le -\sigma||\nu^{\frac{1}{2}}\mathbf{P}_1(f(t)-g(t))||_{L^2} + \Big|\int_\Omega \langle \Gamma[f+g, f-g], f-g\rangle dx\Big|.\end{aligned} \tag{3.4}$$

We set the uniform L^2-stability criterion as follows.

$$\int_0^\infty \Big(||\nu^{\frac{1}{2}}f(s)||^2_{L^\infty_x(L^2_\xi)} + ||\nu^{\frac{1}{2}}g(t)||^2_{L^\infty_x(L^2_\xi)}\Big)dt < \infty. \tag{3.5}$$

3.1. **Soft potential and Maxwellian molecule:** $-\frac{3}{2} < \gamma \le 0$. Suppose two smooth perturbations f and g satisfy the stability condition (3.5). In (3.4), we use Lemma 2.1 to derive a Gronwall type inequality:

$$\frac{d}{dt}\|f(t)-g(t)\|_{L^2}^2 \le -\frac{\sigma}{2}\|\nu^{\frac12}\mathbf{P}_1(f(t)-g(t))\|_{L^2}^2 + C\Big(\|\nu^{\frac12}f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac12}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\|f(t)-g(t)\|^2_{L^2}.$$

Then Gronwall's lemma yields

$$\begin{aligned}&\|f(t)-g(t)\|^2_{L^2} + \frac{\sigma}{2}\int_0^t \|\nu^{\frac12}\mathbf{P}_1(f(s)-g(s))\|^2_{L^2}ds\\ &\le \exp\Big[C\int_0^t\Big(\|\nu^{\frac12}f(s)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac12}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)dt\Big]\|f^{in}-g^{in}\|^2_{L^2}\\ &\le C\|f^{in}-g^{in}\|^2_{L^2}.\end{aligned}$$

This yields the uniform L^2-stability estimate.

Theorem 3.1. [6] *For $\gamma \in (-\frac{3}{2}, 0]$ and let F and G be two classical solutions to* (1.1) *in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac12}d\xi dx)\cap L^\infty_x(L^2(M^{-\frac12}d\xi)))$ corresponding to initial data F^{in}, G^{in} respectively. Suppose the smooth perturbations f and g satisfy the condition* (3.5). *Then we have*

$$\sup_{0\le t<\infty}\|F(t)-G(t)\|_{L^2(M^{-1/2}d\xi dx)} \le C\|F^{in}-G^{in}\|_{L^2(M^{-1/2}d\xi dx)},$$

where C is a positive constant independent of t.

Remark 3.1. *As a direct application of the above theorem, the classical solutions in* [1, 5, 11] *are uniformly L^2-stable.*

3.2. **Hard potential and hard sphere model:** $0 < \gamma \le 1$. Suppose two smooth perturbations f and g satisfy the stability condition (3.5) and the smallness condition:

$$\|f(t)\|_{L^\infty_x(L^2_\xi)} + \|g(t)\|_{L^\infty_x(L^2_\xi)} \ll \frac{\sigma}{4}. \tag{3.6}$$

In (3.4), we use Lemma 2.1 to get

$$\begin{aligned}\frac{d}{dt}\|f(t)-g(t)\|^2_{L^2} \le\ & C\Big(\|\nu^{\frac12}f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac12}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\|f(t)-g(t)\|^2_{L^2}\\ &+\Big[-\frac{\sigma}{2} + C\Big(\|f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\Big]\|\nu^{\frac12}\mathbf{P}_1(f(t)-g(t))\|^2_{L^2}.\end{aligned} \tag{3.7}$$

We use (3.7) to find

$$\begin{aligned}\frac{d}{dt}\|f(t)-g(t)\|^2_{L^2} \le\ & C\Big(\|\nu^{\frac12}f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac12}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\|f(t)-g(t)\|^2_{L^2}\\ &- \frac{\sigma}{4}\|\nu^{\frac12}\mathbf{P}_1(f(t)-g(t))\|^2_{L^2}.\end{aligned}$$

Then Gronwall's lemma yield the following stability estimate.

Theorem 3.2. [6] *For $\gamma \in (0,1]$ and let F and G be two small classical solutions to* (1.1) *in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac12}d\xi dx)\cap L^\infty_x(L^2(M^{-\frac12}d\xi)))$ corresponding to small initial data F^{in}, G^{in}*

respectively. Suppose the smooth perturbations f and g satisfy (3.5) *and* (3.6). *Then we have*

$$\sup_{0\leq t<\infty} \|F(t)-G(t)\|_{L^2(M^{-1/2}d\xi dx)} \leq C\|F^{in}-G^{in}\|_{L^2(M^{-1/2}d\xi dx)},$$

where C is a positive constant independent of t.

Remark 3.2. *As a direct application of this theorem, the classical solutions in* [12] *are uniformly L^2-stable.*

Acknowledgment. The first author would like to thank Professor Shinya Nishibata for inviting him to participate in this Workshop at RIMS. The work of SYHA and SBYUN is supported by the grant of KOSEF R01-2006-000-10002-0.

References

[1] Caflisch, R.: The Boltzmann equation with a soft potential, Parts I and II. Commun. Math. Phys. **74**, 1-95, 97-109 (1980)
[2] Cercignani, C., Illner, R., Pulvirenti, M.: The mathematical theory of dilute gases, Applied Mathematical Sciences **106**, Springer-Verlag, New York 1994
[3] Glassey, R. T.: The Cauchy problem in kinetic theory. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
[4] Guo, Y.: The Boltzmann equation in the whole space. Indiana Univ. Math. J. **53**, 1081-1094 (2004)
[5] Guo, Y.: Classical solutions to the Boltzmann equation for molecules with angular cutoff. Arch. Rational Mech. Anal. **169**, 305-353 (2003)
[6] Ha, S.-Y., Yang, X. and Yun, S.-B.: L^2-Stability Theory of the Boltzmann Equation near a global Maxwellian. Submitted.
[7] Liu, T.-P., Yu, S.-H.: Boltzmann equation: micro-macro decompositions and positivity of shock profiles. Comm. Math. Phys. **246**, 133-179 (2004)
[8] Liu, T.-P., Yang, T., Yu, S.-H.: Energy method for Boltzmann equation. Physica D **188**, 178-192 (2004)
[9] Nishida, T., Imai, K.: Global solutions to the initial value problem for the nonlinear Boltzmann equation. Publ. Res. Inst. Math. Sci. **12**, 229-239 (1976)
[10] Ukai, S.: On the existence of global solutions of amixed problem for the nonlinear Boltzmann equation. Proc. Japan Acad., Ser. A **53**, 179-184 (1974)
[11] Ukai, S., Asano, K.: On the Cauchy problem of Boltzmann equation with a soft potential. Publ. Res. Inst. Math. Sci. **18**, 477-519 (1982)
[12] Yang, T., Zhao, H.-J.: A new energy method for the Boltzmann equation. J. Math. Phys. **47**, 053301 (2006)

Department of Mathematics Sciences, Seoul National University, Seoul 151-747, Korea
E-mail address: **syha@snu.ac.kr**

Department of Mathematics Sciences, Seoul National University, Seoul 151-747, Korea
E-mail address: **sbyun@math.snu.ac.kr**