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Remark on decay estimates for solutions to the critical dissipative quasi-geostrophic equations

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1 Introduction

Let us consider the critical dissipative quasi-geostrophic equations in $\mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial\theta}{\partial t}+(-\Delta)^{\frac{1}{2}}\theta+u\cdot\nabla\theta=0 \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ u=(-R_2\theta,R_1\theta) \quad \text{in} \quad \mathbb{R}^2\times(0,\infty), \\ \theta|_{t=0}=\theta_0 \quad \text{in} \quad \mathbb{R}^2, \end{cases} \tag{QG}$$

where the scalar function θ and the vector field u denote the potential temperature and the fluid velocity, respectively, and $R_i=\frac{\partial}{\partial x_i}(-\Delta)^{-1/2}$ $(i=1,2)$ represents the Riesz transform. We are concerned with the initial value problem for this equation. It is known that (QG) is an important model in geophysical fluid dynamics. Indeed, it is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency. Since there are a number of applications to the theory of oceanography and meteology, a lot of mathematical researches have been devoted to the equations. For example, there are many works on the well-posedness for this equations. Constantin, Cordoba and Wu [4] proved existence of a strong solution for the initial data in H^1 with small L^∞ norm. In [9] and [10], Ju and the author independently proved the local well-posedness for large initial data in H^1. Here it is worth noticing that the spaces L^∞ and $\dot{H}^1$ are both scaling invariant function spaces for (QG). Very recently, Kiselev, Nazarov and Volberg [8] showed a new maximum principle and proved the global well-posedness of (QG) for any $W^{1,\infty}$ periodic initial data. Dong and Du improved the argument of [8] and obtained the global well-posedness of (QG) for any initial data in H^1. Furthermore, Caffarelli and Vasseur [2] showed the regularity of weak solution for initial data in $L^2(\mathbb{R}^2)$ by applying De Giorgi's method to (QG).

The purpose of this note is to show spatial decay estimates of the solutions for fast decaying initial data. It is known that in general, the solution of (QG) does not decay in space variables as fast as the initial data if the initial data dacays very rapidly. This is due to the fact that the fundamental solution of the linearized QG equation decays only with order -3. Therefore it seems to be interesting to investigate the relation between the spatial decay rate of the solution and the initial data. We first show that if the initial data decays with order $-\alpha$ ($\alpha \geq 3$) as $x \to \infty$, the solution decays (at least) with order -3. Moreover, we prove that for such a initial data the solution decays with order $-\alpha$ if and only if the average of the initial data is equal to 0.

In the proof of the result, we will make use of weighted spaces. Because of the presence of the (non-local) drift term $u \cdot \nabla\theta$, it is difficult to obtain weighted estimates of the solution. Indeed the velocity u is expressed by the Riesz transform of θ, and unboundedness of the transform in L^∞ or certain weighted L^p spaces causes difficulties to deal with the drift term. To overcome this difficulty we need weighted estimates of the derivative of solutions to the linearized equation. Combining these estimates with special structure of the drift term, we obtain the weighted estimates of the solution. The decay estimates of the solution also play important role to estimate the growth of the weighted norm.

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2 Definitions and Statement of Theorem

Let us first recall the definition of the Sobolev space. We denote $\mathcal{Z}'$ as the topological dual space of $\mathcal{Z}$ defined by

$$\mathcal{Z} \equiv \{f \in \mathcal{S}; \int x^\alpha f(x)\, dx = 0 \quad \text{for all } \alpha \in \mathbb{N}^n\}.$$

We define the homogeneous and inhomogeneous Sobolev spaces $\dot{H}^{s,p}$, $H^{s,p}$ by

$$\dot{H}^{s,p} \equiv \left\{ f \in \mathcal{Z}'; \|f\|_{\dot{H}^{s,p}} \equiv \|(-\Delta)^{s/2} f\|_p < \infty \right\} \qquad \text{for} \quad s \in \mathbb{R},$$

and

$$H^{s,p} \equiv \left\{ f \in \mathcal{S}'; \|f\|_{H^{s,p}} \equiv \|f\|_{L^p} + \|f\|_{\dot{H}^{s,p}} < \infty \right\} \qquad \text{for} \quad s > 0,$$

respectively. We abbreviate $\dot{H}^{s,2} = \dot{H}^s$ and $H^{s,2} = H^s$.

We next introduce weighted L^p spaces. For $1 \leq p \leq \infty$ and $0 \leq \beta < \infty$, we define homogeneous and inhomogeneous weighted L^p spaces by

$$\dot{L}^{p,\beta} \equiv \{ f \in L^1_{loc}; \|f\|_{\dot{L}^{p,\beta}} \equiv \| |\cdot|^\beta f \|_{L^p} \}$$

and

$$L^{p,\beta} \equiv \{ f \in L^1_{loc}; \|f\|_{L^{p,\beta}} \equiv \| \langle \cdot \rangle^\beta f \|_{L^p} \},$$

respectively where we denote $\langle \cdot \rangle \equiv (1 + |\cdot|^2)^{1/2}$.

Remarks We state some basic property of weighted spaces.

i) It is easy to see the following embedding between the weighted space and the usual L^p space:

$$\dot{L}^{p,\alpha} \quad \hookrightarrow \quad L^q \qquad \text{for} \quad \alpha > n/q - n/p.$$

ii) It is known that the Riesz transform R_i is bounded in $\dot{L}^{p,\alpha}$ if and only if $1 < p < \infty$ and $\alpha < n - n/p$ [3]. This yields that velocity field u does not have the same decay as θ in general. The fact causes some difficulties when one get decay estimate of the nonlinear term.

Our main theorem is stated as follows

Theorem 2.1 *Let $3 < \alpha < 4$. Suppose that the initial data θ_0 belongs to $H^1 \cap L^{\infty,\alpha}$. For $\varepsilon > 0$ there exists positive constant $C > 0$ and a unique solution of (QG) θ in the class*

$$C([0,\infty); H^1) \cap L^2(0,\infty; \dot{H}^{3/2})$$

satisfying the following estimate:

$$\|\theta(t) - AP_t\|_{L^{\infty,\alpha}} \leq Ct^{\alpha-3} \qquad \text{for} \qquad t \geq \varepsilon, \tag{2.1}$$

where $A = \int \theta_0(x)dx$ and $P_t(x) = ct^{-2}(1+\frac{|x|^2}{t^2})^{-\frac{3}{2}}$ where P_t is the fundamental solution to the linear fractional diffusion equation without the drift term.

Remarks

i) The statements of global existence and regularity in this theorem are due to [6, 9, 10], and our contribution of this theorem is the weighted estimate of the asymptotics (2.1). We see that the solution evolving from well-localized initial data decays as fast as constant power of P_t in spatial variables, where the constant is determined by the average of the initial data. Since P_t decays like $|x|^{-3}$, we see that the solution also decays like $|x|^{-3}$ for $t > 0$. In particular, we cannot expect that the solution does not decay faster than $|x|^{-3}$ for $t > 0$ even if the initial data decays rapidly in general. However the solution decays faster than $|x|^{-3}$ if and only if the average of the initial data is equal to 0. So we are able to classfy spatial decay property of the solution for (QG) in terms of the avarage of the initial data.

ii) The growth rate $t^{\alpha-3}$ is optimal in the sense that this rate is equal to the one for the solution to the linearized equation. By the technical reasons due to the presence of the drift term, we need the assumption $t \geq \varepsilon$ to avoid the singularity near $t = 0$. It seems to be difficult to remove this condition by our approach.

iii) Hayashi-Kaikina-Naumkin [7] considered related fractional diffusion equations in one space dimension. They derived the spatial dacay and the asymptotics of the solutions. Brandolese-Karch [1] also studied some class of the fractional diffusion equations with convection terms. They obtained the asymptotics of the solutions toward the linear evolution $e^{-t(-\Delta)^\alpha}\theta_0$. It is worth noticing that they considered the case $\alpha > 1/2$ while we are interested in the critical case $\alpha = 1/2$.

3 Preliminaries

3.1 Linear Estimate in Weighted L^p Spaces

In this section we prepare some basic tools to prove our main results. The following estimates in this subsection is a variant of the estimates proved by Hayashi-Kaikina-Naumkin [7] in one space dimension.

Lemma 3.1 *i) Let $p, q, r \in [1, \infty]$ with $r \geq p, q$ and define p', q' by $1/p' = 1 + 1/r - 1/p$ and $1/q' = 1 + 1/r - 1/q$. Suppose that $a \in \dot{L}^{q,\beta} \cap L^p$ with $\beta \in [2 - 2/p', 3 - 2/p')$ or $(p, r, \beta) = (1, \infty, 3)$. Then we have*

$$\|e^{-t\Lambda}a\|_{\dot{L}^{r,\beta}} \leq Ct^{\beta-2(1-1/p')}\|a\|_{L^p} + Ct^{-2(1-1/q')}\|a\|_{\dot{L}^{q,\beta}} \quad for \quad t > 0, \qquad (3.1)$$

where $\Lambda = (-\Delta)^{1/2}$.

ii) Let $p,q,r \in [1,\infty]$ with $r \geq p,q$ and define p',q' by $1/p' = 1+1/r-1/p$ and $1/q' = 1+1/r-1/q$. Suppose that $a \in \dot{L}^{q,\beta} \cap L^p$ with $\beta \in [3-2/p', 4-2/p')$ or $(p,r,\beta) = (1,\infty,4)$. Then we have

$$\|\nabla e^{-t\Lambda}a\|_{\dot{L}^{r,\beta}} \leq Ct^{\beta-2(1-1/p')-1}\|a\|_{L^p} + Ct^{-2(1-1/q')-1}\|a\|_{\dot{L}^{q,\beta}} \text{ for } t>0. \tag{3.2}$$

Next we show the asymptotics of the solutions to the linear fractional diffusion equation in weighted spaces. Substracting the fundamental solution P_t, we obtain higher order weighted estimates.

Lemma 3.2 *i) Let $1 \leq p \leq \infty$ and $\beta \leq 4$. Suppose that $a \in \dot{L}^{p,\alpha} \cap \dot{L}^{1,1}$. Then we have*

$$\|e^{-t\Lambda}a - AP_t\|_{\dot{L}^{\infty,\beta}} \leq Ct^{\beta-3}\|a\|_{\dot{L}^{1,1}} + Ct^{-\frac{2}{p}}\|a\|_{\dot{L}^{p,\beta}}. \tag{3.3}$$

where $A = \int a(x)dx$.
ii) Let $1 \leq p \leq \infty$ and $\beta \leq 4$. Suppose that $\nabla a \in \dot{L}^{1,1}$ and $a \in \dot{L}^{p,\beta}$. Then we have

$$\|e^{-t\Lambda}\nabla a - A'P_t\|_{\dot{L}^{\infty,\beta}} \leq Ct^{\beta-3}\|\nabla a\|_{\dot{L}^{1,1}} + Ct^{-\frac{2}{p}-1}\|a\|_{\dot{L}^{p,\beta}} + Ct^{-\frac{2}{p}}\|a\|_{\dot{L}^{p,\beta-1}}. \tag{3.4}$$

where $A = \int \nabla a(x)dx$.

3.2 Decay estimates for Derivative of Solution

We next prepare decay estimates of the solution for (QG). These estimates are used for the proof of Theorem 2.1 to control the growth in time of the solution.

Proposition 3.3 *(Constantin-Wu [5]) Assume that θ_0 belongs to $L^1 \cap L^2$ Then there exists a weak solution θ of (QG) such that*

$$\|\theta(t)\|_{L^2} \leq C(1+t)^{-1},$$

where C is constant depending only on the L^1 and L^2 norms of θ_0.

We also recall the existence of the global solution in the critical Sobolev space.

Proposition 3.4 *(Dong-Du [6], Ju [9], Miura [10]) Suppose that the initial data θ_0 belongs to H^1. Then there exists a unique solution of (QG) θ in the class*

$$C([0,\infty);H^1) \cap L^2(0,\infty;\dot{H}^{3/2}).$$

Combining these results with Fourier splitting method by M. Schonbek [12], we are able to show the following sharp decay estimates of the derivative of the solution.

Proposition 3.5 *Suppose that the initial data θ_0 belongs to $L^1 \cap H^1$ and θ is the (unique) corresponding solution of (QG) in the class $C([0,T); H^1) \cap L^2(0,T; H^{3/2})$. Then there exists constant $C > 0$ such that the following estimate holds*

$$\|\theta(t)\|_{\dot{H}^s} \leq C(1+t)^{-s-1} \qquad \textit{for} \qquad t > 0 \quad \textit{and} \quad 0 < s \leq 1 \tag{3.5}$$

and

$$\|\theta(t)\|_{\dot{H}^s} \leq Ct^{-s+1}(1+t)^{-s-1} \qquad \textit{for} \qquad t > 0 \quad \textit{and} \quad 1 \leq s \leq 2. \tag{3.6}$$

4 Sketch of Proof

4.1 Weighted Estimates of solutions

Firstly we show following auxiliary weighted estimates. These estimates are needed for the estimate of the nonlinear term in the proof of the Theorem 2.1. The point is that weights for these estimates are subcritical in terms of boundedness of the Riesz transform. As mentioned before in section 2, the Riesz transform is bounded in $\dot{L}^{m,\alpha}$ if and only if $1 < m < \infty$ and $\alpha < 2 - 2/m$. Hence we obtain weighted estimates using the usual integral representation of (QG):

$$\theta(t) = e^{-t\Lambda}\theta_0 - \int_0^t e^{-(t-s)\Lambda}(u \cdot \nabla\theta)(s)ds \tag{4.1}$$

and Lemma 3.1. Furthermore applying non-local maximum principle by [8], we obtain such estimates globally in time.

Proposition 4.1 *Let $2 < m < \infty$ and $1 - 2/m < \alpha < 2 - 2/m$. Suppose that the initial data θ_0 belongs to $L^{m,\alpha} \cap L^1 \cap \dot{H}^1$. Then there exists a solution of (QG) in $L^\infty(0,\infty; L^{m,\alpha})$.*

Proposition 4.2 *Under the same assumption of the initial data as Proposition 4.1, for every $\varepsilon > 0$ there exist a constant $C > 0$ and a solution of (QG) satisfying*

$$\|\nabla\theta(t)\|_{L^{m,\alpha}} \leq C \qquad \textit{for} \qquad t \geq \varepsilon,$$

where the constant C depends only on θ_0 and ε.

4.2 Asymtotics of the solutions in weighted spaces

In this position we now outline the proof of Theorem 2.1. We estimate the righthand side of the integral representation (4.1) respectively. The estimates for the linear term in (4.1) is obtained by applying Lemma 3.2 directly. As for the estimate of the nonlinear term, we notice the fact that the average of the drift term $u \cdot \nabla\theta$ is equal to 0, because the velocity u satisfies divergence free condition. This and Lemma 3.2 imply that

$$\begin{aligned}
&\|e^{-(t-s)\Lambda}(u \cdot \nabla\theta)\|_{\dot{L}^{\infty,\alpha}} \\
\le&\|e^{-(t-s)\Lambda}(u \cdot \nabla\theta) - AP_t\|_{\dot{L}^{\infty,\alpha}} \\
\le&Ct^{\beta-3}\|u \cdot \nabla\theta\|_{\dot{L}^{1,1}} + Ct^{-\frac{2}{p}}\|u \cdot \nabla\theta\|_{\dot{L}^{p,\alpha}} \\
\le&C(t^{\beta-3} + t^{-\frac{2}{p}})\|u \cdot \nabla\theta\|_{\dot{L}^{p,\alpha}},
\end{aligned}$$

where $A = \int u \cdot \nabla\theta(x)dx$ and embedding $L^{1,1} \hookrightarrow L^{p,\alpha}$. Applying Proposition 4.1, 4.2, we can show that $\|u \cdot \nabla\theta\|_{p,\alpha}$ for sufficiently large p. So we see that the expression $\|e^{-(t-s)\Lambda}(u \cdot \nabla\theta)\|_{\dot{L}^{\infty,\alpha}}$ is finite for $0 < s < t$. In fact, we have to check that the integral is integrable on $[0, t]$ and it is bounded by $t^{\alpha-3}$ for $t > 0$. To this end, we need to split the interval of integral and apply Proposition 3.5 to control the growth in time. The details of these arguments are work out in [11] and will be published elsewhere.

References

[1] Brandolese, L. and Karch, K., *Far field asymptotics of solutions to convection equation with anomalous diffusion*, preprint.

[2] Caffarelli, L. and Vasseur, A., *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation.* preprint.

[3] Coifman, R, Rochberg, R. *Another characterization of BMO*, Proc. Amer. Math. Soc. 79 (1980), no. 2, 249–254.

[4] Constantin, P. and Cordoba, D. and Wu, J., *On the critical dissipative quasi-geostrophic equation*, Indiana Univ. Math. J., **50**, 97-107 (2001).

[5] Constantin, P. and Wu, J., *Behavior of solutions of 2D quasi-geostrophic equations*, SIAM J. Math. Anal. **30**, 937-948 (1999).

[6] Dong, H. and Du, D., *Global well-posedness and a decay estimate for the critical dissipative quasi-geostrophic equation in the whole space*, preprint.

[7] Hayashi,N., Kaikina E. and Naumkin. P., *Asymptotics for large x and t of solutions to fractional nonlinear equations*, preprint.

[8] Kiselev, A., Nazarov, F. and Volberg, A., *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation.* Invent. Math., **167**, 445-453 (2007).

[9] Ju, N., *Dissipative quasi-geostrophic equation: local well-posedness, global regularity and similarity solutions*, Indiana Univ. Math. J., **56**, 187-206(2007).

[10] Miura, H., *Dissipative quasi-geostrophic equation for large initial data in the critical Sobolev space*, Comm. Math. Phys., **1**, 141-157, (2006).

[11] Miura, H., *Weighted estimates for solutions to the critical dissipative quasi-geostrophic equations*, preprint.

[12] Schonbek, M. and Schonbek, T., *Asymptotic behavior to dissipative quasi-geostrophic flows*, SIAM J. Math. Anal., **35**, 357-375 (2003).