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On the displacement decomposition of a Q-polynomial distance-regular graph

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1 Leonard systems

We begin by recalling the notion of a Leonard system, following [20]. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let d denote a positive integer and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space of all $d+1$ by 1 matrices that have entries in $\mathbb{K}$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$. We observe this module is irreducible. Let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$ and let V denote an irreducible left $\mathcal{A}$-module. We remark that V is unique up to isomorphism of $\mathcal{A}$-modules, and that V has dimension $d+1$. Let A denote an element of $\mathcal{A}$. We say A is *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Let A denote a multiplicity-free element of $\mathcal{A}$. Let $\theta_0, \theta_1, \dots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \leqslant i \leqslant d$ put

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j},$$

where I denotes the identity of $\mathcal{A}$. We observe (i) $AE_i = \theta_i E_i$ $(0 \leqslant i \leqslant d)$; (ii) $E_i E_j = \delta_{ij} E_i$ $(0 \leqslant i, j \leqslant d)$; (iii) $\sum_{i=0}^{d} E_i = I$; (iv) $A = \sum_{i=0}^{d} \theta_i E_i$. We call E_i the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0 V + E_1 V + \cdots + E_d V \quad \text{(direct sum)}.$$

For $0 \leqslant i \leqslant d$, $E_i V$ is the (one dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace.

Definition 1.1 ([20, Definition 1.4]). By a *Leonard system* in $\mathcal{A}$ we mean a sequence

$$\Phi = \left(A; A^*; \{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d}\right)$$

that satisfies (i)-(v) below:

(i) Each of A, A^* is a multiplicity-free element in $\mathcal{A}$.

(ii) $\{E_i\}_{i=0}^{d}$ is an ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^{d}$ is an ordering of the primitive idempotents of A^*.

(iv) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$

(v) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \quad (0 \leqslant i, j \leqslant d).$

We refer to d as the *diameter* of Φ, and say Φ is *over* $\mathbb{K}$. We call $\mathcal{A}$ the *ambient algebra* of Φ. For notational convenience, we set $E_i = E_i^* = 0$ if $i < 0$ or $i > d$.

Note 1.2. Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote the Leonard system from Definition 1.1. Then the sequence $\Phi^* = (A^*; A; \{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d)$ is a Leonard system in $\mathcal{A}$.

2 Balanced bilinear forms

Let Φ denote the Leonard system from Definition 1.1. Let $\Phi' = (A'; A^{*\prime}; \{E_i'\}_{i=0}^{d'}; \{E_i^{*\prime}\}_{i=0}^{d'})$ denote a Leonard system over $\mathbb{K}$ with diameter d', where $d \geqslant d'$. For any object f that we associate with Φ, we let f' denote the corresponding object for the Leonard system Φ'; an example is $V' = V(\Phi')$.

Definition 2.1. A nonzero bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle : V \times V' \longrightarrow \mathbb{K}$ is said to be *balanced* with respect to Φ, Φ' if (i), (ii) hold below:

(i) There exists an integer ρ $(0 \leqslant \rho \leqslant d - d')$ such that $\langle\!\langle E_i^* V, E_j^{*\prime} V' \rangle\!\rangle = 0$ if $i - \rho \neq j$ $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant d')$.

(ii) $\langle\!\langle E_i V, E_j' V' \rangle\!\rangle = 0$ if $i < j$ or $i > j + d - d'$ $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant d')$.

We refer to ρ as the *endpoint* of $\langle\!\langle\ ,\ \rangle\!\rangle$.

The *parameter array* of the Leonard system Φ is a sequence of the form $p(\Phi) = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$, where θ_i (resp. θ_i^*) denotes the eigenvalue for A (resp. A^*) associated with E_i (resp. E_i^*) for $0 \leqslant i \leqslant d$, and the φ_i, ϕ_i $(1 \leqslant i \leqslant d)$ are certain nonzero scalars in $\mathbb{K}$. See [22]. The central result of this paper is the following characterization of the existence of a balanced bilinear form in terms of the parameter arrays of Φ and Φ':

Theorem 2.2. *There exists a bilinear form* $\langle\!\langle\ ,\ \rangle\!\rangle : V \times V' \longrightarrow \mathbb{K}$ *that is balanced with respect to* Φ, Φ' *and with endpoint* ρ $(0 \leqslant \rho \leqslant d - d')$ *if and only if the parameter arrays of* Φ, Φ' *satisfy* (i), (ii) *below:*

(i) *There exist scalars* $\xi^*, \zeta^* \in \mathbb{K}$ $(\xi^* \neq 0)$ *such that* $\theta_i^{*\prime} = \xi^* \theta_{\rho+i}^* + \zeta^*$ $(0 \leqslant i \leqslant d')$.

(ii) $\dfrac{\phi_i'}{\varphi_i'} = \dfrac{\phi_{\rho+i}}{\varphi_{\rho+i}}$ $(1 \leqslant i \leqslant d')$.

Moreover, if (i), (ii) *hold above, then* $\langle\!\langle\ ,\ \rangle\!\rangle$ *is unique up to multiplication by a nonzero scalar in* $\mathbb{K}$.

In [21] the parameter array of a Leonard system is given in parametric form. See Appendix A. The following result is a restatement of Theorem 2.2 in terms of this form.

Theorem 2.3. *Let the parameter array of* Φ *be given as in Theorem A.1. Then there exists a bilinear form* $\langle\!\langle\ ,\ \rangle\!\rangle : V \times V' \longrightarrow \mathbb{K}$ *that is balanced with respect to* Φ, Φ' *and with endpoint* ρ *if and only if the following* (i)-(iii) *hold:*

(i) *For Case* III, ρ *is even if* d *is odd; and* $d-d'$ *is even if* $d' \geqslant 2$.

(ii) *For Case* IV, $(d',\rho) \in \{(1,0),(1,2),(3,0)\}$.

(iii) *The parameter array of* Φ' *is of the following form:*

$$
p(\Phi') = \begin{cases}
p(\mathrm{IIC}; r', s', s^{*\prime}, \theta'_0, \theta^{*\prime}_0, 1) \text{ where } s's^{*\prime}/r' = 1-\phi_{\rho+1}/\varphi_{\rho+1} & \text{for } d'=1,\\
p(\mathrm{I}; q, h', h^{*\prime}, r_1 q^\rho, r_2 q^\rho, s q^{d-d'}, s^* q^{2\rho}, \theta'_0, \theta^{*\prime}_0, d') & \text{for Case I},\\
p(\mathrm{IA}; q, h^{*\prime}, r', s', \theta'_0, \theta^{*\prime}_0, d') \text{ where } s'/r' = q^{d-d'-\rho} s/r & \text{for Case IA},\\
p(\mathrm{II}; h', h^{*\prime}, r_1+\rho, r_2+\rho, s+d-d', s^*+2\rho, \theta'_0, \theta^{*\prime}_0, d') & \text{for Case II},\\
p(\mathrm{IIA}; h', r+\rho, s+d-d', s^{*\prime}, \theta'_0, \theta^{*\prime}_0, d') & \text{for Case IIA},\\
p(\mathrm{IIB}; h^{*\prime}, r+\rho, s', s^*+2\rho, \theta'_0, \theta^{*\prime}_0, d') & \text{for Case IIB},\\
p(\mathrm{IIC}; r', s', s^{*\prime}, \theta'_0, \theta^{*\prime}_0, d') \text{ where } s's^{*\prime}/r' = ss^*/r & \text{for Case IIC},\\
p(\mathrm{III}; h', h^{*\prime}, r_1+\rho, r_2+\rho, s-d+d', s^*-2\rho, \theta'_0, \theta^{*\prime}_0, d') & \text{for Case III, } d \text{ even,}\\
& d' \text{ even, } \rho \text{ even;}\\
& \text{or Case III, } d \text{ odd,}\\
& d' \text{ odd, } \rho \text{ even,}\\
p(\mathrm{III}; h', h^{*\prime}, r_2+\rho, r_1+\rho, s-d+d', s^*-2\rho, \theta'_0, \theta^{*\prime}_0, d') & \text{for Case III, } d \text{ even,}\\
& d' \text{ even, } \rho \text{ odd,}\\
p(\mathrm{IV}; h', h^{*\prime}, r, s, s^*, \theta_0, \theta^*_0) & \text{for Case IV, } d'=3.
\end{cases}
$$

Our third result characterizes the Leonard system Φ' in terms of the balanced bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$. To state the result we recall a concept from linear algebra. Let V' denote a vector space over $\mathbb{K}$ with finite positive dimension $d'+1$. By a *decomposition* of V' we mean a sequence $\{U_i\}_{i=0}^{d'}$ consisting of one-dimensional subspaces of V' such that $V' = U_0 + U_1 + \cdots + U_{d'}$ (direct sum).

Theorem 2.4. *Let* d' *denote a positive integer such that* $d \geqslant d'$. *Let* $\mathcal{A}'$ *denote a* $\mathbb{K}$-*algebra isomorphic to* $\mathrm{Mat}_{d'+1}(\mathbb{K})$ *and let* V' *denote an irreducible left* $\mathcal{A}'$-*module. Let* $\{U_i\}_{i=0}^{d'}$, $\{U_i^*\}_{i=0}^{d'}$ *denote decompositions of* V'. *Assume there exists a bilinear form* $\langle\!\langle\ ,\ \rangle\!\rangle : V \times V' \longrightarrow \mathbb{K}$ *that satisfies* (i)-(iii) *below:*

(i) *There exists an integer* ρ $(0 \leqslant \rho \leqslant d-d')$ *such that* $\langle\!\langle E_i^*V, U_j^*\rangle\!\rangle = 0$ *if* $i-\rho \neq j$ $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant d')$.

(ii) $\langle\!\langle E_iV, U_j\rangle\!\rangle = 0$ *if* $i<j$ *or* $i>j+d-d'$ $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant d')$.

(iii) $\langle\!\langle\ ,\ \rangle\!\rangle$ *has full-rank.*

With reference to Theorem A.1, we further assume (iv), (v) *below:*

(iv) *For Case* III, ρ *is even if* d *is odd; and* $d-d'$ *is even if* $d' \geqslant 2$.

(v) *For Case* IV, $(d',\rho) \in \{(1,0),(1,2),(3,0)\}$.

Then there exists a Leonard system $\Phi' = \left(A'; A^{*\prime}; \{E'_i\}_{i=0}^{d'}; \{E^{*\prime}_i\}_{i=0}^{d'}\right)$ *in* $\mathcal{A}'$ *such that* $E'_iV' = U_i$, $E^{*\prime}_iV' = U_i^*$ *for* $0 \leqslant i \leqslant d'$. *In particular,* $\langle\!\langle\ ,\ \rangle\!\rangle$ *is balanced with respect to* Φ, Φ' *with endpoint* ρ. *Moreover, this Leonard system is unique up to affine transformations of* A', $A^{*\prime}$.

3 Motivations: Q-polynomial distance-regular graphs

In this section we discuss how balanced bilinear forms arise in the theory of Q-polynomial distance-regular graphs. We refer the reader to [2, 3, 17] for terminology and background materials on this topic. Throughout we let $\Gamma = (\mathsf{X}, \mathsf{R})$ denote a Q-polynomial distance-regular graph with diameter D. Let $\mathbb{C}$ denote the complex number field. Let $\mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $\mathsf{V} = \mathbb{C}^{\mathsf{X}}$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ acts on V by left multiplication. We endow V with the Hermitian inner product $\langle\ ,\ \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in \mathsf{V}$, where ^{t} denotes transpose and $^-$ denotes complex conjugation. For all $y \in \mathsf{X}$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. Let $\mathsf{A}_0 = \mathsf{I}, \mathsf{A}_1, \ldots, \mathsf{A}_D \in \mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ denote the distance matrices of Γ and let $\mathsf{E}_0 = |\mathsf{X}|^{-1}\mathsf{J}, \mathsf{E}_1, \ldots, \mathsf{E}_D$ denote the primitive idempotents (in the Q-polynomial ordering) for the Bose-Mesner algebra $\mathsf{M} = \langle \mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_D\rangle$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$. We set $\mathsf{A} = \mathsf{A}_1$ and recall A generates M.

We briefly recall the Terwilliger algebra of Γ. Fix a vertex $x \in \mathsf{X}$. We call x the *base vertex*. For $0 \leqslant i \leqslant D$ let $\mathsf{E}_i^* = \mathsf{E}_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ with yy entry $(\mathsf{E}_i^*)_{yy} = (\mathsf{A}_i)_{xy}$ for all $y \in \mathsf{X}$. These *dual idempotents* form a basis for the *dual Bose-Mesner algebra* $\mathsf{M}^* = \mathsf{M}^*(x)$ of Γ with respect to x. Let $\mathsf{A}^* = \mathsf{A}^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ with yy entry $(\mathsf{A}^*)_{yy} = |\mathsf{X}|(\mathsf{E}_1)_{xy}$ for all $y \in \mathsf{X}$. We recall A^* generates M^*. The *Terwilliger algebra* (or *subconstituent algebra*) $\mathsf{T} = \mathsf{T}(x)$ of Γ with respect to x is the subalgebra of $\mathrm{Mat}_{\mathsf{X}}(\mathbb{C})$ generated by M and M^* [17, 18, 19]. Let $\mathsf{W} \subseteq \mathsf{V}$ denote an irreducible T-module. By the *endpoint* (resp. *dual endpoint*) of W we mean $\nu = \min\{i \mid 0 \leqslant i \leqslant D,\ \mathsf{E}_i^*\mathsf{W} \neq 0\}$ (resp. $\nu^* = \min\{i \mid 0 \leqslant i \leqslant D,\ \mathsf{E}_i\mathsf{W} \neq 0\}$). By the *diameter* (resp. *dual diameter*) of W we mean $d = |\{i \mid 0 \leqslant i \leqslant D,\ \mathsf{E}_i^*\mathsf{W} \neq 0\}| - 1$ (resp. $d^* = |\{i \mid 0 \leqslant i \leqslant D,\ \mathsf{E}_i\mathsf{W} \neq 0\}| - 1$). In fact, by [11, Corollary 3.3] we find $d = d^*$. We say W is *thin* whenever $\dim \mathsf{E}_i^*\mathsf{W} \leqslant 1$ for $0 \leqslant i \leqslant D$. Suppose W is thin. Then $\Phi = (\mathsf{A}|_{\mathsf{W}}; \mathsf{A}^*|_{\mathsf{W}}; \{\mathsf{E}_{\nu+i}|_{\mathsf{W}}\}_{i=0}^d; \{\mathsf{E}^*_{\nu^*+i}|_{\mathsf{W}}\}_{i=0}^d)$ defines a Leonard system in the $\mathbb{C}$-algebra of all linear transformations on W, where for all $\mathsf{B} \in \mathsf{T}$ we let $\mathsf{B}|_{\mathsf{W}}$ denote the action of B on W (cf. [18, Theorem 4.1]). We say Φ is *associated with* W. We recall the *primary* T-module $\mathsf{M}\hat{x}$ is a unique irreducible T-module in V with diameter D, and moreover it is thin [17, Lemma 3.6].

Subsets with minimal width plus dual width

Let C denote a proper subset of X. We let $\chi = \chi_C$ denote the characteristic vector of C; i.e., $\chi = \sum_{y\in C}\hat{y}$. Brouwer, Godsil, Koolen and Martin [4] introduced two parameters, width and dual width, for C. By the *width* of C we mean $w = \max\{i \mid 0 \leqslant i \leqslant D,\ \chi^t\mathsf{A}_i\chi \neq 0\}$. By the *dual width* of C we mean $w^* = \max\{i \mid 0 \leqslant i \leqslant D,\ \chi^t\mathsf{E}_i\chi \neq 0\}$. They showed $w + w^* \geqslant D$, and if $w + w^* = D$ then C is completely-regular and induces a Q-polynomial w-class association scheme [4, Section 5]. Subsets with $w + w^* = D$ arise quite naturally when Γ is associated with a regular semilattice [4, Theorem 5], and we expect that such subsets will play a potential role in the theory of Q-polynomial distance-regular graphs. We remark that subsets with $w + w^* = D$ have been applied to *Erdős-Ko-Rado theorem* in extremal set theory [13, Theorem 3] and (implicitly) to *Assmus-Mattson theorem* in coding theory [14, Example 5.4]. Now suppose C is connected and satisfies $w + w^* = D$. Then by [4, Theorem 3] the induced subgraph Γ_C on C is a Q-polynomial distance-regular graph with diameter w. Suppose the base vertex x is taken from C. Let M' denote the Bose-Mesner algebra of Γ_C and let T' denote the Terwilliger algebra of Γ_C with respect to x. Let Φ, Φ' denote the Leonard systems associated with the primary T-module $\mathsf{M}\hat{x}$ and the primary T'-module $\mathsf{M}'\hat{x}$, respectively. By carefully analyzing some of the arguments in [4, 10], we will show in a subsequent paper [16] that the bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle : \mathsf{M}\hat{x} \times \mathsf{M}'\hat{x} \longrightarrow \mathbb{C}$ defined by $\langle\!\langle u, v\rangle\!\rangle = \langle u, \overline{v}\rangle$ $(u \in \mathsf{M}\hat{x},\ v \in \mathsf{M}'\hat{x})$ is balanced with respect to Φ, Φ' and with endpoint 0. By this fact, for instance, our results will enable us to explicitly determine when C is connected (so that Γ_C is a Q-polynomial distance-regular graph). Moreover, we will also show that if $0 < w < D$ then C is convex (i.e., geodetically closed) precisely when Γ has *classical parameters* [3, p. 193]. Known families of Q-polynomial distance-regular graphs with unbounded diameter that have classical parameters form natural hierarchical structures. Embedding as a subset with $w + w^* = D$ is a special (but very important) case of these structures. The classification of subsets satisfying $w + w^* = D$ is complete for Hamming, Johnson, Grassmann, bilinear forms and dual polar graphs [4, 13]. Our results will then lead to the classification of such subsets for Doob, alternating forms, Hermitian forms, quadratic forms and also *twisted Grassmann graphs* [6, 8, 1].

Short irreducible modules for the Terwilliger algebra

Recall $\mathsf{T} = \mathsf{T}(x)$ denotes the Terwilliger algebra of Γ with respect to x. Let W denote an irreducible T-module in V with endpoint ν, dual endpoint ν^* and diameter d. Caughman [5, Lemma 5.1] showed $2\nu + d \geqslant D$, and if $2\nu + d = D$ then by the results of [12] we find W is thin. See also [9] for discussions concerning the case $\nu = 1$. Now suppose W satisfies $2\nu + d = D$ and pick any $y \in \mathsf{X}$ such that $\langle \hat{y}, \mathsf{E}^*_\nu(x)\mathsf{W}\rangle \neq 0$. Let $\mathsf{T}' = \mathsf{T}(y)$ denote the Terwilliger algebra of Γ with respect to y. Let Φ, Φ' denote the Leonard systems associated with the primary T'-module $\mathsf{M}\hat{y}$ and W, respectively. Then it is easy to show that the bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle : \mathsf{M}\hat{y} \times \mathsf{W} \longrightarrow \mathbb{C}$ defined by $\langle\!\langle u, v\rangle\!\rangle = \langle u, \overline{v}\rangle$ $(u \in \mathsf{M}\hat{y},\ v \in \mathsf{W})$ is balanced with respect to Φ^*, Φ'^* and with endpoint ν^*. Around 1990 Terwilliger [17, 18, 19] began the systematic study of thin irreducible T-modules and found how the Leonard systems associated with these modules are described. We remark Theorem 2.3, when applied to the above pair of Leonard systems, recovers his results for those modules satisfying $2\nu + d = D$. See [18, Theorem 4.6]. The current approach to Leonard pairs and Leonard systems was established in Terwilliger's 2001 paper [20], and since then it has been an active area of research; so it should be a natural and important project to reconstruct and extend his theory on thin irreducible modules based on this new treatment. We may also view this paper as providing the starting point of this project.

A The list of parameter arrays

In this appendix we display all the parameter arrays of Leonard systems. The data in Theorem A.1 below is from [21], with a change of presentation to be consistent with the notation in [2, 17, 18, 19] which we will follow for describing various Q-polynomial distance-regular graphs in a subsequent paper [16]. In Theorem A.1 the following implicit assumptions apply: the scalars θ_i, θ^*_i $(0 \leqslant i \leqslant d)$, φ_i, ϕ_i $(1 \leqslant i \leqslant d)$ are contained in $\mathbb{K}$, and the scalars $q, h, h^*, \dots$ are contained in the algebraic closure of $\mathbb{K}$.

Theorem A.1 ([21, Theorem 5.16]). *Let Φ denote the Leonard system from Definition 1.1 and let $p(\Phi) = \left(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d\right)$ denote the parameter array of Φ. Then at least one of the following cases* I, IA, II, IIA, IIB, IIC, III, IV *hold (the expressions $p(.;\dots)$ below are labels):*

(I) $p(\Phi) = p(\mathrm{I}; q, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta^*_0, d)$ *where* $r_1 r_2 = s s^* q^{d+1}$,

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1q^i)(1-r_2q^i),$$
$$\phi_i = \begin{cases} hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1-s^*q^i)(r_2-s^*q^i)/s^* & \text{if } s^* \neq 0, \\ hh^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-r_1q^{i-d-1}-r_2q^{i-d-1}) & \text{if } s^* = 0 \end{cases}$$

for $1 \leqslant i \leqslant d$.

(IA) $p(\Phi) = p(\mathrm{IA}; q, h^*, r, s, \theta_0, \theta^*_0, d)$ *where*

$$\theta_i = \theta_0 - sq(1-q^i),$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i}$$

for $0 \leqslant i \leqslant d$ *and*

$$\varphi_i = -rh^*q^{1-i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = h^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1})$$

for $1 \leqslant i \leqslant d$.

(II) $p(\Phi) = p(\mathrm{II}; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta^*_0, d)$ *where* $r_1 + r_2 = s + s^* + d + 1$,

$$\theta_i = \theta_0 + hi(i+1+s),$$
$$\theta^*_i = \theta^*_0 + h^*i(i+1+s^*)$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = hh^*i(i-d-1)(i+r_1)(i+r_2),$$
$$\phi_i = hh^*i(i-d-1)(i+s^*-r_1)(i+s^*-r_2)$$

for $1 \leqslant i \leqslant d$.

(IIA) $p(\Phi) = p(\mathrm{IIA}; h, r, s, s^*, \theta_0, \theta^*_0, d)$ *where*

$$\theta_i = \theta_0 + hi(i+1+s),$$
$$\theta^*_i = \theta^*_0 + s^*i$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = hs^*i(i-d-1)(i+r),$$
$$\phi_i = hs^*i(i-d-1)(i+r-s-d-1)$$

for $1 \leqslant i \leqslant d$.

(IIB) $p(\Phi) = p(\mathrm{IIB}; h^*, r, s, s^*, \theta_0, \theta_0^*, d)$ *where*

$$\theta_i = \theta_0 + si,$$
$$\theta_i^* = \theta_0^* + h^* i(i + 1 + s^*)$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = h^* si(i - d - 1)(i + r),$$
$$\phi_i = -h^* si(i - d - 1)(i + s^* - r)$$

for $1 \leqslant i \leqslant d$.

(IIC) $p(\Phi) = p(\mathrm{IIC}; r, s, s^*, \theta_0, \theta_0^*, d)$ *where*

$$\theta_i = \theta_0 + si,$$
$$\theta_i^* = \theta_0^* + s^* i$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = ri(i - d - 1),$$
$$\phi_i = (r - ss^*)i(i - d - 1)$$

for $1 \leqslant i \leqslant d$.

(III) $p(\Phi) = p(\mathrm{III}; h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$ *where* $r_1 + r_2 = -s - s^* + d + 1$,

$$\theta_i = \theta_0 + h(s - 1 + (1 - s + 2i)(-1)^i),$$
$$\theta_i^* = \theta_0^* + h^*(s^* - 1 + (1 - s^* + 2i)(-1)^i)$$

for $0 \leqslant i \leqslant d$, *and*

$$\varphi_i = \begin{cases} -4hh^* i(i + r_1) & \text{if } i \text{ even, } d \text{ even,} \\ -4hh^*(i - d - 1)(i + r_2) & \text{if } i \text{ odd, } d \text{ even,} \\ -4hh^* i(i - d - 1) & \text{if } i \text{ even, } d \text{ odd,} \\ -4hh^*(i + r_1)(i + r_2) & \text{if } i \text{ odd, } d \text{ odd,} \end{cases}$$

$$\phi_i = \begin{cases} 4hh^* i(i - s^* - r_1) & \text{if } i \text{ even, } d \text{ even,} \\ 4hh^*(i - d - 1)(i - s^* - r_2) & \text{if } i \text{ odd, } d \text{ even,} \\ -4hh^* i(i - d - 1) & \text{if } i \text{ even, } d \text{ odd,} \\ -4hh^*(i - s^* - r_1)(i - s^* - r_2) & \text{if } i \text{ odd, } d \text{ odd} \end{cases}$$

for $1 \leqslant i \leqslant d$.

(IV) $p(\Phi) = p(\mathrm{IV}; h, h^*, r, s, s^*, \theta_0, \theta_0^*)$ *where* $\mathrm{char}(\mathbb{K}) = 2$, $d = 3$, *and*

$$\begin{aligned} &\theta_1 = \theta_0 + h(s + 1), && \theta_2 = \theta_0 + h, && \theta_3 = \theta_0 + hs, \\ &\theta_1^* = \theta_0^* + h^*(s^* + 1), && \theta_2^* = \theta_0^* + h^*, && \theta_3^* = \theta_0^* + h^* s^*, \\ &\varphi_1 = hh^* r, && \varphi_2 = hh^*, && \varphi_3 = hh^*(r + s + s^*), \\ &\phi_1 = hh^*(r + s(1 + s^*)), && \phi_2 = hh^*, && \phi_3 = hh^*(r + s^*(1 + s)). \end{aligned}$$

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