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On the Brauer categories of p-blocks of finite groups related by the Glauberman-Dade correspondence

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For a prime p, let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system where $\mathcal{O}$ is a complete discrete valuation ring having the residue field k of characteristic p which is algebraically closed and having the quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be large enough for any finite group we consider in this article. Let $\mathcal{R} \in \{\mathcal{O}, k\}$. Below, by characters, we mean $\mathcal{K}$-characters.

Glauberman showed in [4] that, when a finite group G is acted by S where S is a finite solvable group such that $(|\mathsf{G}|, |\mathsf{S}|) = 1$, there is a one-to-one correspondence between the set $\mathrm{Irr}(\mathsf{G})^{\mathsf{S}}$ of S-invariant irreducible characters of G and the set $\mathrm{Irr}(C_{\mathsf{G}}(\mathsf{S}))$ of irreducible characters of $C_{\mathsf{G}}(\mathsf{S})$.

Watanabe showed in [10] that when an S-invariant block b of G has a defect group centralized by S (often called *Watanabe's situation*), then all irreducible characters in b are S-invariant, all of them are mapped by the Glauberman correspondence bijectively to the irreducible characters belonging to a single block $w(\mathsf{b})$ of $C_{\mathsf{G}}(\mathsf{S})$, and b and $w(\mathsf{b})$ have equivalent Brauer categories

In p-block theory, some "good relation" between blocks having equivalent Brauer categories is expected. See, for example, articles by Uno and Narasaki for a formulation in terms of characters.

Recently, Dade gave in [3] a new approach to the Glauberman correspondence and partly generalized it.

In this article, we note that, under some assumptions (see Conditions 2.1, 3.2 and 4.1), blocks related by the correspondence of Dade have equivalent Brauer categories, emphasizing the relation $\mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}\mathrm{Pr}^{E}_{C_E(P)} = \mathrm{Pr}^{E'}_{C_{E'}(P)}\mathrm{Pr}^{E}_{E'}$, for groups E, E' and P below. In particular, with [10, Proposition 1], this gives an alternative proof of above mentioned Watanabe's result.

In fact, under our assumptions, the correspondence of Dade induces a perfect isometry (isotypy) between groups of generalized characters of related blocks, as in the case of the Glauberman correspondence under Watanabe's situation ([10]). A perfect isometry (isotypy) is a phenomenon in the character level which is said to be a shadow of a (splendid) derived equivalence, see [1] and [7], and we may expect a (splendid) derived equivalence between related blocks.

For details, see [8], and for standard facts, see [9] and [5]. We also referred to [11] in writing this article.

Notations: For a ring R, we denote by $R^\times$ the multiplicative group consisting of all units of R and by $Z(R)$ the center of R. Let E be a finite group. Denote by $\alpha^*\in k\mathsf{E}$ the canonical image of $\alpha\in\mathcal{O}\mathsf{E}$. For a subset E_0 of E, $\mathcal{R}\mathsf{E}_0$ is an $\mathcal{R}$-subspace of the group algebra $\mathcal{R}\mathsf{E}$ spanned by elements of E_0. When E_0 is invariant by conjugation action of E, denote by $(\mathcal{R}\mathsf{E}_0)^{\mathsf{E}}$ the $\mathcal{R}$-subspace of $\mathcal{R}\mathsf{E}_0$ consisting of E-invariant elements. Let E' be a subgroup of E. Denote by $\mathrm{Pr}^{\mathsf{E}}_{\mathsf{E}'}$ an $\mathcal{R}$-linear map from $\mathcal{R}\mathsf{E}$ to $\mathcal{R}\mathsf{E}'$ defined by $\mathrm{Pr}^{\mathsf{E}}_{\mathsf{E}'}(\mathsf{x}) = \mathsf{x}$ if $\mathsf{x}\in\mathsf{E}'$ and $\mathrm{Pr}^{\mathsf{E}}_{\mathsf{E}'}(\mathsf{x}) = 0$ if $\mathsf{x}\in\mathsf{E}-\mathsf{E}'$, which induces an $\mathcal{R}$-linear map from $Z(\mathcal{R}\mathsf{E})$ to $Z(\mathcal{R}\mathsf{E}')$. Denote by $\mathrm{Tr}^{\mathsf{E}}_{\mathsf{E}'}$ an $\mathcal{R}$-linear map from $(\mathcal{R}\mathsf{E})^{\mathsf{E}'}(\supset Z(\mathcal{R}\mathsf{E}'))$ to $Z(\mathcal{R}\mathsf{E})$ defined by $\mathrm{Tr}^{\mathsf{E}}_{\mathsf{E}'}(\tau) = \sum_{\mathsf{y}\in[\mathsf{E}'\backslash\mathsf{E}]}\tau^{\mathsf{y}}$ for $\tau\in(\mathcal{R}\mathsf{E})^{\mathsf{E}'}$. For a subset C of E, we denote $\hat{\mathsf{C}} = \sum_{\mathsf{x}\in\mathsf{C}}\mathsf{x} \in \mathcal{R}\mathsf{E}$. For $\psi \in \mathrm{Irr}(\mathsf{E})$, there is an algebra homomorphism ω_ψ from $Z(\mathcal{O}\mathsf{E})$ to $\mathcal{O}$ determined by $\omega_\psi(\widehat{\mathsf{C}(\mathsf{x})}) = |\mathsf{E}|\psi(\mathsf{x})/|C_{\mathsf{E}}(\mathsf{x})|\psi(1)$ where $\mathsf{C}(\mathsf{x})$ is a conjugacy class of E containing x, see [5, III, 2.5]. Let F be a cyclic group and $\check{\mathsf{F}}=\mathrm{Hom}(\mathsf{F},\mathcal{K}^\times)$ the dual group of F. If there is an epimorphism $\pi : \mathsf{E} \to \mathsf{F}$, F acts on $\mathrm{Irr}(\mathsf{E})$, denoted by left multiplication, by $(\lambda\psi)(\mathsf{x})=\lambda(\pi(\mathsf{x}))\psi(\mathsf{x})$ for $\lambda\in\check{\mathsf{F}}$, $\psi\in\mathrm{Irr}(\mathsf{E})$ and $\mathsf{x}\in\mathsf{E}$, see [3, Proposition 1.15, and (1.16)]. Let $\mathsf{G}=\mathrm{Ker}(\pi)$. If $\phi\in\mathrm{Irr}(\mathsf{G})$ is E-invariant, that is, $\phi\in\mathrm{Irr}(\mathsf{G})^{\mathsf{E}}=\{\phi\in\mathrm{Irr}(\mathsf{G})|\phi(\mathsf{g}^{\mathsf{x}})=\phi(\mathsf{g})$ for any $\mathsf{g}\in\mathsf{G}$ and $\mathsf{x}\in\mathsf{E}\}$, then ϕ has $|\mathsf{F}|$ distinct extensions to characters of E, in fact, which form a set $\mathrm{Irr}(\mathsf{E}|\phi)=\{\psi\in\mathrm{Irr}(\mathsf{E}) \mid [\phi,\psi\downarrow^{\mathsf{E}}_{\mathsf{G}}]_G \neq 0\}$ where $[\cdot,\cdot]$ is the usual inner product, and $\mathrm{Irr}(\mathsf{E}|\phi)=\{\lambda\psi|\lambda\in\check{\mathsf{F}}\}$ where ψ is any element of $\mathrm{Irr}(\mathsf{E}|\phi)$, see [3, Proposition 1.19]. Denote by e_ϕ the primitive idempotent of $Z(\mathcal{K}\mathsf{G})$ corresponding to $\phi\in\mathrm{Irr}(\mathsf{G})$, see [5, III, 2.4].

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We say that $(*)_{\mathsf{E},\mathsf{G},\mathsf{E}',\mathsf{G}',r}$ holds if the following holds:

> E is a finite group with a normal subgroup G such that the quotient group $\mathsf{F} = \mathsf{E}/\mathsf{G}$ is cyclic of order r. Let $\pi : \mathsf{E} \to \mathsf{F}$ be the canonical epimorphism. E' is a subgroup of E such that $\mathsf{E}=\mathsf{G}\mathsf{E}'$. G' is a normal subgroup of E' defined by $\mathsf{G}' = \mathsf{G}\cap\mathsf{E}'$. Let F_0 be the set of generators of F, $\mathsf{E}_0 = \pi^{-1}(\mathsf{F}_0)$ and $\mathsf{E}'_0 = \pi^{-1}(\mathsf{F}_0)\cap\mathsf{E}'$. E'_0 is a trivial intersection subset of E with E' as its normalizer, that is, $\mathsf{E}'_0 \cap {\mathsf{E}'_0}^{\tau} = \emptyset$, the empty set, for any $\tau \in \mathsf{E}-\mathsf{E}'$.

Under the above condition, Dade gives a one-to-one correspondence between $\mathrm{Irr}(\mathsf{G})^{\mathsf{E}}$ and $\mathrm{Irr}(\mathsf{G}')^{\mathsf{E}'}$. [3, Theorems 6.8 and 6.9] should be referred for precise statements. Below, we always assume the following:

Condition 2.1. $r = q^n$ for a prime q and $n \in \mathbb{Z}_{>0}$.

Throughtout this article, let E, G, E', G' be such that $(*)_{E,G,E',G',r}$ holds. Let $F=E/G$ and let $\pi : E \to F$ be the canonical epimorphism. For a subgroup H of E such that $\pi(H) = F$ we will consider the action of the dual group $\check{F}$ of F on $\mathrm{Irr}(H)$ defined by the restriction of π to H.

Under Condition 2.1, the correspondence of Dade can be described as:

Theorem 2.2. (Dade) *There is a bijection*

$$\mathrm{Irr}(G)^E \longrightarrow \mathrm{Irr}(G')^{E'}, \qquad \phi \mapsto \phi_{(G')} \tag{$*$}$$

which satisfies the following:

- *When q is odd, there are a unique sign $\epsilon_\phi \in \{\pm 1\}$ and a unique bijection*
$$\mathrm{Irr}(E \mid \phi) \longrightarrow \mathrm{Irr}(E' \mid \phi_{(G')}), \qquad \psi_i \mapsto \psi_{i(E')} \tag{$**$}$$
such that
$$(\psi - \lambda_q \psi) \downarrow^E_{E'} \;=\; \epsilon_\phi(\psi_{(E')} - \lambda_q \psi_{(E')}) \tag{$***$}$$
holds as generalized characters for any element $\lambda_q \in \check{F}$ of order q.
- *When q is 2, if we choose a sign ϵ_ϕ arbitrary, there is a unique bijection $(**)$ such that $(***)$ holds.*

We call both the correspondences $(*)$ and $(**)$ in Theorem 2.2 the *Glauberman-Dade correspondence* of characters. For a relation to the Glauberman correspondence, see [3, Section 7].

Below, we denote by $C(x)$ the conjugacy class of E containing $x \in E$ and by $C(x')'$ the conjugacy class of E' containing $x' \in E'$.

Remark 2.3. Since $E_0 = \sqcup_{t \in [E' \backslash E]} (E'_0)^t$ (disjoint union), see [3, Lemma 6.5], we see that there is a one-to-one correspondence between the set of conjugacy classes of E contained in E_0 and the set of conjugacy classes of E' contained in E'_0, and $\mathrm{Pr}^E_{E'}(\widehat{C(x')}) = \widehat{C(x')'}$ for $x' \in E'_0$. Hence, $\mathrm{Pr}^E_{E'}$ induces an isomorphism between $\mathcal{R}$-spaces $(\mathcal{R}E_0)^E$ and $(\mathcal{R}E'_0)^{E'}$.

The correspondence in Theorem 2.2 can be described in terms of central primitive idempotents corresponding to characters, see also [6]:

Proposition 2.4. $\mathrm{Pr}^E_{E'}(\widehat{C(x')e_\phi}) = \widehat{C(x')'}e_{\phi_{(G')}}$ *for $x' \in E'_0$ and $\phi \in \mathrm{Irr}(G)^E$.*

For Proposition 2.5, see also the proof of [10, Proposition 2(ii)]:

Proposition 2.5. *Let $\phi \in \mathrm{Irr}(G)^E$ and $\psi \in \mathrm{Irr}(E|\phi)$. Then, for $\sigma \in (\mathcal{O}E_0)^E$, it holds that*

$$\omega_\psi(\sigma) \;=\; \epsilon_\phi \frac{|G|}{\phi(1)} \frac{\phi_{(G')}(1)}{|G'|} \omega_{\psi_{(E')}}(\mathrm{Pr}^E_{E'}(\sigma)).$$

In the remainder of this section, let P be an arbitrary subgroup of G' such that there is some element $s \in E'_0$ centralizing P.

Lemma 2.6. $(*)_{C_E(P), C_G(P), C_{E'}(P), C_{G'}(P), r}$ *and* $(*)_{C_E(P)/Z(P), C_G(P)/Z(P), C_{E'}(P)/Z(P), C_{G'}(P)/Z(P), r}$ *hold.*

Remark 2.7.

$$\begin{tikzcd}
(\mathcal{R}E_0)^E \arrow[r, "\mathrm{Pr}^E_{E'}"] \arrow[d, "\mathrm{Pr}^E_{C_E(P)}"'] \arrow[dr, phantom, "\circlearrowleft"] & (\mathcal{R}E'_0)^{E'} \arrow[d, "\mathrm{Pr}^{E'}_{C_{E'}(P)}"] \\
(\mathcal{R}C_E(P)_0)^{C_E(P)} \arrow[r, "\mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}"] & (\mathcal{R}C_{E'}(P)_0)^{C_{E'}(P)}
\end{tikzcd}$$

Lemma 2.8. *$N_E(P)=N_{E'}(P)C_E(P)$ and $N_G(P)=N_{G'}(P)C_G(P)$.*

Remark 2.9. Note that $\mathrm{Tr}_{E'}^{E}$ is the inverse of $\mathrm{Pr}_{E'}^{E} : (\mathcal{R}E_0)^E \to (\mathcal{R}E_0')^{E'}$. For $\phi' \in \mathrm{Irr}(G')^{E'}$ and $\psi' \in \mathrm{Irr}(E'|\phi')$, denote by $\phi'_{(G)}$ and $\psi'_{(E)}$ the Glauberman-Dade corresponding characters respectively. Similar statements as in Propositions 2.4 and 2.5 and Remark 2.7 hold, replacing E, G, E', G', ϕ, $\phi_{(G')}$, ψ, $\psi_{(E')}$, σ, $\mathrm{Pr}_{E'}^{E}$ and $\mathrm{Pr}_{C_{E'}(P)}^{C_E(P)}$ by E', G', E, G, $\phi'(\in \mathrm{Irr}(G')^{E'})$, $\phi'_{(G)}$, $\psi'(\in \mathrm{Irr}(E'|\phi'))$, $\psi'_{(E)}$, $\sigma'(\in (\mathcal{O}E_0')^{E'})$, $\mathrm{Tr}_{E'}^{E}$ and $\mathrm{Tr}_{C_{E'}(P)}^{C_E(P)}$, respectively.

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For a finite group G, a primitive idempotent of $Z(\mathcal{R}\mathsf{G})$ of $\mathcal{R}\mathsf{G}$ is called a (p-) *block* (*idempotent*). $\mathsf{b} \mapsto \mathsf{b}^*$ determines a bijection between blocks of G over $\mathcal{O}$ and k, and so blocks over k are denoted with superscript $*$. Denote $\mathrm{Irr}(\mathsf{b}) = \{\phi \in \mathrm{Irr}(\mathsf{G}) | \mathsf{b}e_\phi \neq 0\}$. If $\phi \in \mathrm{Irr}(b)$, then ϕ is said to be *in* b and b *contains* ϕ. For blocks b of G and $\hat{\mathsf{b}}$ of E where E is a finite group having G as a normal subgroup, we say that $\hat{\mathsf{b}}$ *covers* b if $\hat{\mathsf{b}}\mathsf{b} \neq 0$.

For a finite group E and a block $\hat{\mathsf{b}}$ of E, we denote by $\omega^*_{\hat{\mathsf{b}}}$ the algebra homomorphism from $Z(\mathcal{O}\mathsf{E})$ to k determined by $(\omega_\psi(\sigma))^*$ for $\sigma \in Z(\mathcal{O}\mathsf{E})$, where ψ is any element in $\mathrm{Irr}(\hat{\mathsf{b}})$, see [5, III. 6.4].

Lemma 3.1 follows from results in [2].

Lemma 3.1. *Let E be a finite group with a normal subgroup G such that E/G is cyclic of prime power order r. Let $\pi : \mathsf{E} \to \mathsf{E}/\mathsf{G}$ be the canonical epimorphism and E_0 the inverse image by π of the set of generators of E/G. Let b be a block of G and $\hat{\mathsf{b}}$ any block of E covering b. Then:*

(1) The following conditions are equivalent: (In (ii) and (iii), $C(\mathsf{s})$ is the conjugacy class of E containing s.)

  (i) b is covered by r distinct blocks of E.

  (ii) There is an element $\mathsf{s} \in \mathsf{E}_0$ such that $\omega^*_{\hat{\mathsf{b}}}(\widehat{C(\mathsf{s})}) \neq 0$.

  (iii) There is an element $\mathsf{s} \in \mathsf{E}_0$ such that $\widehat{C(\mathsf{s})}\mathsf{b} \in Z(\mathcal{O}\mathsf{E}\mathsf{b})^\times$.

(2) If the conditions in (1) hold, then $p \neq q$, b is E-invariant and $\mathrm{Irr}(\mathsf{b})^{\mathsf{E}} = \mathrm{Irr}(\mathsf{b})$.

For groups E and G and a block b under the situation in Lemma 3.1, we say that $(\star)_{\mathsf{E},\mathsf{G},\mathsf{b}}$ holds, if the equivalent conditions in Lemma 3.1(1) hold.

Below, we always assume the following:

Condition 3.2. b is a block of G such that $(\star)_{E,G,b}$ holds.

A subgroup D of a finite group G is called a *defect group* of a block b of G if D is a maximal p-subgroup of G such that $\mathrm{Pr}^{\mathsf{G}}_{C_{\mathsf{G}}(\mathsf{D})}(\mathsf{b}^*) \neq 0$, which is uniquely determined up to G-conjugation. If $|\mathsf{D}| = p^{\mathsf{d}}$, d is called a (p)-*defect* of b. A block b has defect 0 if and only if $\mathrm{Irr}(\mathsf{b})$ consists of only one character, called a characrer of (p)-*defect* 0. For a p-subgroup P of G, $\mathrm{Pr}^{\mathsf{G}}_{C_{\mathsf{G}}(\mathsf{P})} : (k\mathsf{G})^{\mathsf{P}} \to kC_{\mathsf{G}}(\mathsf{P})$ becomes a ring epimorphism, and $\mathrm{Pr}^{\mathsf{G}}_{C_{\mathsf{G}}(\mathsf{P})}(\mathsf{b}^*)$ is a central idempotent of $kC_{\mathsf{G}}(\mathsf{P})$.

A block e of $C_{\mathsf{G}}(\mathsf{P})$ such that $\mathrm{Pr}^{\mathsf{G}}_{C_{\mathsf{G}}(\mathsf{P})}(\mathsf{b}^*)\mathsf{e}^*=\mathsf{e}^*$ is said to be *associated with* b. $\mathrm{Pr}^{\mathsf{G}}_{C_{\mathsf{G}}(\mathsf{D})}(\mathsf{b}^*)$ becomes a sum of blocks of $C_{\mathsf{G}}(\mathsf{D})$ which are $N_{\mathsf{G}}(\mathsf{D})$-conjugate. Every block of $C_{\mathsf{G}}(\mathsf{D})$ appearing in the decomposition of $\mathrm{Pr}^{\mathsf{G}}_{C_{\mathsf{G}}(\mathsf{D})}(\mathsf{b}^*)$ contains the unique irreducible character such that $Z(\mathsf{D})$ is contained in its kernel, which is called a *canonical character* of b. Canonical characters can be viewed as irreducible characters of $C_{\mathsf{G}}(\mathsf{D})/Z(\mathsf{D})$, which have defect 0. Canonical characters of b are determined up to G-conjugation.

Proposition 3.3. *$\{\phi_{(G')}|\phi\in\mathrm{Irr}(b)\}$ are contained in some uniquely determined block $b_{(G')}$ of G'.*

Remark 3.4. $\mathrm{Pr}^E_{E'}(\widehat{C(x)}b)=\mathrm{Pr}^E_{E'}(\widehat{C(x)}b)b_{(G')}$. Hence $\mathrm{Pr}^E_{E'}((\mathcal{O}E_0)^E b)\subseteq(\mathcal{O}E'_0)^{E'}b_{(G')}$.

Lemma 3.5. *There are some $s\in E'_0$ and a defect group D of b such that $D\leq G'$, s centralizes D and $\widehat{C(s)}b\in Z(\mathcal{O}Eb)^{\times}$.*

Below, let D and s be as in Lemma 3.5.

Lemma 3.6. *If $(\star)_{\mathsf{E},\mathsf{G},\mathsf{b}}$ holds and, for a p-subgroup* P *of* G, e *is a block of* $C_{\mathsf{G}}(\mathsf{P})$ *associated with* b, *then* $(\star)_{C_{\mathsf{E}}(\mathsf{P}),C_{\mathsf{G}}(\mathsf{P}),\mathsf{e}}$ *holds.*

By Remarks 2.7 and 3.4 and Lemmas 2.6 and 3.6, we have:

Proposition 3.7. *Let P be a subgroup of D and e a block of $C_G(P)$ associated with b. Then $e_{(C_{G'}(P))}$ is a block of $C_{G'}(P)$ associated with $b_{(G')}$. In particular, $b_{(G')}$ have a defect group containing D.*

If a block $\hat{\mathsf{e}}$ of $C_{\mathsf{E}}(\mathsf{P})$ is associated with a block $\hat{\mathsf{b}}$ of a finite group E, then

$$\omega^*_{\hat{\mathsf{b}}}(\sigma)=\omega^*_{\hat{\mathsf{e}}}(\mathrm{Pr}^{\mathsf{E}}_{C_{\mathsf{E}}(\mathsf{P})}(\sigma))\quad\text{for}\quad\sigma\in Z(\mathcal{O}\mathsf{E}),\tag{†}$$

see [5, V, Theorem 3.5].

Proposition 3.8. *Let $\overline{\zeta}\in\mathrm{Irr}(C_G(D)/Z(D))$ be such that its inflation to $C_G(D)$ is a canonical character $\zeta\in\mathrm{Irr}(C_G(D))$ of b. Then the following are equivalent:*

(i) $(\star)_{E',G',b_{(G')}}$ *holds.*
(ii) $b_{(G')}$ *has the same defect as* b.
(iii) $\overline{\zeta}_{(C_{G'}(D)/Z(D))}$ *is a character of defect 0.*

Proof. (i)⇒(ii) follows from Proposition 3.7 and Remark 3.10 below.

(ii)⇒(iii) follows from the commutativity of the Glauberman-Dade correspondence and the inflation.

We show (iii)⇒(i). Let $\hat{\zeta}\in\mathrm{Irr}(C_E(D)|\zeta)$ and $\hat{\phi}\in\mathrm{Irr}(E|\phi)$ for $\phi\in\mathrm{Irr}(b)$ be such that the block containing $\hat{\zeta}$ is associated with the block containing $\hat{\phi}$. Let $\psi'\in\mathrm{Irr}(E')$ be such that the block containing $\hat{\zeta}_{(C_{E'}(D))}$ is associated with the block $\hat{b}'$ containing ψ'. Note that $\hat{b}'$ covers $b_{(G')}$. Note also that $\overline{\zeta}_{(C_{G'}(D)/D)}$

being defect 0 implies $\epsilon_\zeta \frac{|C_G(D)|}{\zeta(1)} \frac{\zeta_{(C_{G'}(D))}(1)}{|C_{G'}(D)|} \in \mathcal{O}^\times$. Then, by (†) and Proposition 2.5,

$$
\begin{aligned}
0 &\neq \left(\omega_{\hat{\phi}}(\widehat{C(s)})\right)^* \\
&= \left(\omega_{\hat{\zeta}}\left(\mathrm{Pr}^E_{C_E(D)}(\widehat{C(s)})\right)\right)^* \\
&= \left(\epsilon_\zeta \frac{|C_G(D)|}{\zeta(1)} \frac{\zeta_{(C_{G'}(D))}(1)}{|C_{G'}(D)|} \omega_{\hat{\zeta}_{(C_{E'}(D))}}\left(\mathrm{Pr}^{C_E(D)}_{C_{E'}(D)}\left(\mathrm{Pr}^E_{C_E(D)}(\widehat{C(s)})\right)\right)\right)^* \\
&= \left(\epsilon_\zeta \frac{|C_G(D)|}{\zeta(1)} \frac{\zeta_{(C_{G'}(D))}(1)}{|C_{G'}(D)|}\right)^* \left(\omega_{\hat{\zeta}_{(C_{E'}(D))}}\left(\mathrm{Pr}^{E'}_{C_{E'}(D)}\left(\mathrm{Pr}^E_{E'}(\widehat{C(s)})\right)\right)\right)^* \\
&= \left(\epsilon_\zeta \frac{|C_G(D)|}{\zeta(1)} \frac{\zeta_{(C_{G'}(D))}(1)}{|C_{G'}(D)|}\right)^* \left(\omega_{\hat{\zeta}_{(C_{E'}(D))}}\left(\mathrm{Pr}^{E'}_{C_{E'}(D)}(\widehat{C(s)'})\right)\right)^* \\
&= \left(\epsilon_\zeta \frac{|C_G(D)|}{\zeta(1)} \frac{\zeta_{(C_{G'}(D))}(1)}{|C_{G'}(D)|}\right)^* \left(\omega_{\psi'}(\widehat{C(s)'})\right)^*
\end{aligned}
$$

Hence, $\omega^*_{\hat{b}'}(\widehat{C(s)'}) \neq 0$, and so (i) holds for $b_{(G')}$, see Lemma 3.1. □

Remark 3.9. Assume that the equivalent conditions (i)–(iii) in Proposition 3.8 hold and that q is odd. We use above notations. We can show that there is some block $\hat{b}_{(E')}$ of E' such that $\mathrm{Irr}(\hat{b}_{(E')})=\{\psi_{(E')} | \psi \in \mathrm{Irr}(\hat{b})\}$, see [8, Proposition 3.5(3)]. We can also show that $\hat{b}_{(E')}=\hat{b}'$, see [8, Lemma 5.4], and we may take $\hat{\phi}_{(E')}$ for ψ'. Then we have, for any $\sigma \in (\mathcal{O}E_0)^E$,

$$\omega^*_{\hat{b}}(\sigma) = \left(\epsilon_\zeta \frac{|C_G(D)|}{\zeta(1)} \frac{\zeta_{(C_{G'}(D))}(1)}{|C_{G'}(D)|}\right)^* \omega^*_{\hat{b}_{(E')}}(\mathrm{Pr}^E_{E'}(\sigma)).$$

On the other hand, by Proposition 2.5, we have, for any $\phi \in \mathrm{Irr}(b)$,

$$\omega^*_{\hat{b}}(\widehat{C(s)}) = \left(\epsilon_\phi \frac{|G|}{\phi(1)} \frac{\phi_{(G')}(1)}{|G'|}\right)^* \omega^*_{\hat{b}_{(E')}}(\widehat{C(s)'})$$

and we see that

$$\left(\epsilon_\phi \frac{|G|}{\phi(1)} \frac{\phi_{(G')}(1)}{|G'|}\right)^* = \left(\epsilon_\zeta \frac{|C_G(D)|}{\zeta(1)} \frac{\zeta_{(C_{G'}(D))}(1)}{|C_{G'}(D)|}\right)^*.$$

Remark 3.10. Starting by the condition $(\star)_{E',G',b'}$ for a block b' of G', statements as in this section hold, replacing E, G, E', G', b, ϕ, $\phi_{(G')}$, $\mathrm{Pr}^E_{E'}$, $\cdots$ by E', G', E, G, b', $\phi'(\in \mathrm{Irr}(G')^{E'})$, $\phi'_{(G)}$, $\mathrm{Tr}^E_{E'}$, $\cdots$, respectively. We see immediately that even when ϕ' has defect 0, $\phi'_{(G)}$ is not necessarilly has defect 0. On the other hand, we do not solve the problem to find explicit example such that the equivalent conditions (i)–(iii) in Proposition 3.8 does not hold (or to prove that the conditions always hold) under $(\star)_{E,G,b}$.

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Below, we assume the following:

Condition 4.1. $(\star)_{E',G',b_{(G')}}$ holds.

Then, in particular, b and $b_{(G')}$ have a defect group D. Below, for simplicity we denote $b'=b_{(G')}$, and denote $(b^*)'=(b')^*$.

Proposition 4.2. $\mathrm{Irr}(b')=\{\phi_{(G')}|\phi\in\mathrm{Irr}(b)\}$, *and so* $\mathrm{Pr}^E_{E'}(\widehat{C(x')}b)=\widehat{C(x')}'b'$ *for* $x'\in E'_0$.

Remark 4.3. $b^{(*)}=b$ means $b^{(*)}=b$ if $\mathcal{R}=\mathcal{O}$ and $b^{(*)}=b^*$ if $\mathcal{R}=k$:

$$\begin{tikzcd}
(\mathcal{R}E_0)^E \arrow[r, "\mathrm{Pr}^E_{E'}"] \arrow[d, "(\cdot)b^{(*)}"'] \arrow[dr, phantom, "\circlearrowleft"] & (\mathcal{R}E'_0)^{E'} \arrow[d, "(\cdot)(b^{(*)})'"] \\
(\mathcal{R}E_0)^E b^{(*)} \arrow[r, "\mathrm{Pr}^E_{E'}"] & (\mathcal{R}E'_0)^{E'}(b^{(*)})'
\end{tikzcd}$$

A pair $(\mathsf{P},\mathsf{e}^*)$ of a p-subgroup P of a finite group G and a block e^* of $C_{\mathsf{G}}(\mathsf{P})$ is called a *Brauer pair*. G acts on Brauer pairs by $(\mathsf{P},\mathsf{e}^*)^{\mathsf{g}}=(\mathsf{P}^{\mathsf{g}},(\mathsf{e}^*)^{\mathsf{g}})$ where $\mathsf{g}\in\mathsf{G}$. $(\mathsf{P},\mathsf{e}^*)$ is called a b-*Brauer pair* if e is associated with a block b of G. For a Brauer pair $(\mathsf{P},\mathsf{e}^*)$ and a normal subgroup Q of P, there exists a unique P-invariant block f of $C_{\mathsf{G}}(\mathsf{Q})$ such that $\mathrm{Pr}^{C_{\mathsf{G}}(\mathsf{Q})}_{C_{\mathsf{G}}(\mathsf{P})}(\mathsf{f}^*)\mathsf{e}^*=\mathsf{e}^*$, in which case denoted by $(\mathsf{P},\mathsf{e}^*)\trianglerighteq(\mathsf{Q},\mathsf{f}^*)$. See [9, Section 40] for the definition of the relation $(\mathsf{P},\mathsf{e}^*)\geq(\mathsf{R},\mathsf{l}^*)$ for Brauer pairs $(\mathsf{P},\mathsf{e}^*)$ and $(\mathsf{R},\mathsf{l}^*)$, which makes the set of Brauer pairs of G a partially ordered set. It is known that $(\mathsf{P},\mathsf{e}^*)\geq(\mathsf{R},\mathsf{l}^*)$ if and only if $(\mathsf{P},\mathsf{e}^*)\trianglerighteq(\mathsf{P}_1,\mathsf{e}_1^*)\trianglerighteq\cdots\trianglerighteq(\mathsf{P}_n,\mathsf{e}_n^*)=(\mathsf{R},\mathsf{l}^*)$ for a sequence of subgroups P_i of P such that $\mathsf{P}\trianglerighteq\mathsf{P}_1\trianglerighteq\cdots\trianglerighteq\mathsf{P}_n=\mathsf{R}$. In fact, for a subgroup R of P and a Brauer pair $(\mathsf{P},\mathsf{e}^*)$, there exists a unique block l^* of $C_{\mathsf{G}}(\mathsf{R})$ such that $(\mathsf{P},\mathsf{e}^*)\geq(\mathsf{R},\mathsf{l}^*)$. $(\mathsf{P},\mathsf{e}^*)$ is a b-Brauer pair if and only if $(\mathsf{P},\mathsf{e}^*)\geq(1,\mathsf{b}^*)$.

The *Brauer category* $\mathcal{B}_{\mathsf{G}}(\mathsf{b})$ of a block b of G is a category such that

$$\mathrm{Ob}(\mathcal{B}_{\mathsf{G}}(\mathsf{b}))=\{(\mathsf{P},\mathsf{e}^*)\mid(\mathsf{P},\mathsf{e}^*)\text{ is a }\mathsf{b}\text{-Brauer pair}\}$$

and, for $(\mathsf{P},\mathsf{e}^*),(\mathsf{Q},\mathsf{f}^*)\in\mathrm{Ob}(\mathcal{B}_{\mathsf{G}}(\mathsf{b}))$,

$$\mathrm{Mor}((\mathsf{Q},\mathsf{f}^*),(\mathsf{P},\mathsf{e}^*))$$
$$=\{\varphi:\mathsf{Q}\to\mathsf{P}\mid\text{there exists }\mathsf{g}\in\mathsf{G}\text{ such that }(\mathsf{Q},\mathsf{f}^*)^{\mathsf{g}}\leq(\mathsf{P},\mathsf{e}^*)\text{ and }\varphi(\mathsf{u})=\mathsf{u}^{\mathsf{g}}\text{ for all }\mathsf{u}\in\mathsf{Q}\}.$$

For a b-Brauer pair $(\mathsf{D},\mathsf{b}_{\mathsf{D}})$ where D is a defect group of b, $\mathcal{B}_{\mathsf{G}}(\mathsf{b})_{\leq(\mathsf{D},\mathsf{b}^*_{\mathsf{D}})}$ is a full subcategory of $\mathcal{B}_{\mathsf{G}}(\mathsf{b})$ such that $\mathrm{Ob}(\mathcal{B}_{\mathsf{G}}(\mathsf{b})_{\leq(\mathsf{D},\mathsf{b}^*_{\mathsf{D}})})=\{(\mathsf{P},\mathsf{e}^*)\mid(\mathsf{P},\mathsf{e}^*)\leq(\mathsf{D},\mathsf{b}^*_{\mathsf{D}})\}$, which is equivalent to $\mathcal{B}_{\mathsf{G}}(\mathsf{b})$, see [9, Lemma 47.1 and p.428].

We fix a b-Brauer pair (D,b^*_D) of b and, for a subgroup P of D, denote (P,b^*_P) the uniquely determined b-Brauer pair such that $(D,b^*_D)\geq(P,b^*_P)$. Note that $(\star)_{C_E(P),C_G(P),b_P}$ holds. For simplicity of notations, we denote $(b_P)'=(b_P)_{(C_{G'}(P))}$, which is associated with b' by Proposition 3.7 and hence $(\star)_{C_{E'}(P),C_{G'}(P),(b_P)'}$ holds by Lemma 3.6. Denote $(b^*_P)'=((b_P)')^*$.

We denote by $C(x)_{(P)}$ the conjugacy class of $C_E(P)$ containing $x\in C_E(P)$ and by $C(x')'_{(P)}$ the conjugacy class of $C_{E'}(P)$ containing $x'\in C_{E'}(P)$.

Theorem 4.4. *The Brauer categories $\mathcal{B}_G(b)$ and $\mathcal{B}_{G'}(b')$ are equivalent.*

Proof. It suffices to show that categories $\mathcal{B}_G(b)_{\leq(D,b_D^*)}$ and $\mathcal{B}_{G'}(b')_{\leq(D,(b_D^*)')}$ are isomorphic.

Firstly note that, for $x' \in G'$,

$$C(\widehat{s^{x'})'_{(P^{x'})}}((b_P^*)')^{x'} = \left(\widehat{C(s)'_{(P)}}(b_P^*)'\right)^{x'} = \left(\mathrm{Pr}_{C_{E'}(P)}^{C_E(P)}(\widehat{C(s)_{(P)}}b_P^*)\right)^{x'}$$
$$= \mathrm{Pr}_{C_{E'}(P)^{x'}}^{C_E(P)^{x'}}\left((\widehat{C(s)_{(P)}}b_P^*)^{x'}\right) = \mathrm{Pr}_{C_{E'}(P^{x'})}^{C_E(P^{x'})}\left(C(\widehat{s^{x'})_{(P^{x'})}}(b_P^*)^{x'}\right) = C(\widehat{s^{x'})'_{(P^{x'})}}((b_P^*)^{x'})'.$$

Hence, we have

$$((b_P^*)')^{x'} = ((b_P^*)^{x'})' \text{ for } x' \in G'. \tag{$\sharp$}$$

We show that for any objects (P, b_P^*), (Q, b_Q^*) of $\mathcal{B}_G(b)_{\leq(D,b_D^*)}$ such that $(P, b_P^*) \geq (Q, b_Q^*)$, it holds that $(P, (b_P^*)') \geq (Q, (b_Q^*)')$. It suffices to show the case $(P, b_P^*) \trianglerighteq (Q, b_Q^*)$. Note that P normalizes $C_E(Q)$, $(b_Q^*)'$ is P-invariant by ($\sharp$) and $\mathrm{Pr}_{C_E(P)}^{C_E(Q)} : (kC_E(Q))^P \to kC_E(P)$ is a ring homomorphism. We have

$$\begin{aligned}
&\mathrm{Pr}_{C_{E'}(P)}^{E'}(\widehat{C(s)'}(b^*)')(b_P^*)' \\
&= \mathrm{Pr}_{C_{E'}(P)}^{C_E(P)}\left(\mathrm{Pr}_{C_E(P)}^{C_E(Q)}\left(\mathrm{Pr}_{C_E(Q)}^{E}(\widehat{C(s)}b^*)\right)\right)(b_P^*)' \\
&= \mathrm{Pr}_{C_{E'}(P)}^{C_E(P)}\left(\mathrm{Pr}_{C_E(P)}^{C_E(Q)}\left(\mathrm{Pr}_{C_E(Q)}^{E}(\widehat{C(s)}b^*)\right)b_P^*\right) \\
&= \mathrm{Pr}_{C_{E'}(P)}^{C_E(P)}\left(\mathrm{Pr}_{C_E(P)}^{C_E(Q)}\left(\mathrm{Pr}_{C_E(Q)}^{E}(\widehat{C(s)}b^*)\right)\mathrm{Pr}_{C_E(P)}^{C_E(Q)}\left(b_Q^*\right)b_P^*\right) \\
&= \mathrm{Pr}_{C_{E'}(P)}^{C_E(P)}\left(\mathrm{Pr}_{C_E(P)}^{C_E(Q)}\left(\mathrm{Pr}_{C_E(Q)}^{E}(\widehat{C(s)}b^*)b_Q^*\right)\right)(b_P^*)' \\
&= \mathrm{Pr}_{C_{E'}(P)}^{C_{E'}(Q)}\left(\mathrm{Pr}_{C_{E'}(Q)}^{C_E(Q)}\left(\mathrm{Pr}_{C_E(Q)}^{E}(\widehat{C(s)}b^*)b_Q^*\right)\right)(b_P^*)' \\
&= \mathrm{Pr}_{C_{E'}(P)}^{C_{E'}(Q)}\left(\mathrm{Pr}_{C_{E'}(Q)}^{C_E(Q)}\left(\mathrm{Pr}_{C_E(Q)}^{E}(\widehat{C(s)}b^*)\right)(b_Q^*)'\right)(b_P^*)' \\
&= \mathrm{Pr}_{C_{E'}(P)}^{C_{E'}(Q)}\left(\mathrm{Pr}_{C_{E'}(Q)}^{C_E(Q)}\left(\mathrm{Pr}_{C_E(Q)}^{E}(\widehat{C(s)}b^*)\right)\right)\mathrm{Pr}_{C_{E'}(P)}^{C_{E'}(Q)}\left((b_Q^*)'\right)(b_P^*)' \\
&= \mathrm{Pr}_{C_{E'}(P)}^{E'}(\widehat{C(s)'}(b^*)')\mathrm{Pr}_{C_{E'}(P)}^{C_{E'}(Q)}((b_Q^*)')(b_P^*)'.
\end{aligned}$$

Then since $\mathrm{Pr}_{C_{E'}(P)}^{E'}(\widehat{C(s)'}(b^*)') \in Z\big(kC_{E'}(P)\mathrm{Pr}_{C_{E'}(P)}^{E'}((b^*)')\big)^\times$, multiplying the inverse, we have $(b_P^*)' = \mathrm{Pr}_{C_{E'}(P)}^{C_{E'}(Q)}((b_Q^*)')(b_P^*)'$, and so $(P, (b_P^*)') \trianglerighteq (Q, (b_Q^*)')$.

On the other hand, G' controls fusin in $\mathcal{B}_G(b)_{\leq(D,b_D^*)}$, that is, we may assume that morphisms in $\mathcal{B}_G(b)_{\leq(D,b_D^*)}$ are induced by conjugations of elements of G', see Lemma 2.8 and [9, Section 49].

Hence, the assertion follows from above arguments. □

Remark 4.5. With the notations in above proof,

$$
\begin{tikzcd}
(kC_E(P)_0)^{C_E(P)}b_P^* \arrow[r, "\mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}"] \arrow[d, "(\cdot)^{x'}"'] \arrow[dr, phantom, "\circlearrowleft"] & (kC_{E'}(P)_0)^{C_{E'}(P)}(b_P^*)' \arrow[d, "(\cdot)^{x'}"] \\
(kC_E(P^{x'})_0)^{C_E(P^{x'})}b^*_{P^{x'}} \arrow[r, "\mathrm{Pr}^{C_E(P^{x'})}_{C_{E'}(P^{x'})}"] & (kC_{E'}(P^{x'})_0)^{C_{E'}(P^{x'})}(b^*_{P^{x'}})'
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(kC_E(Q)_0)^{C_E(Q)}b_Q^* \arrow[r, "\mathrm{Pr}^{C_E(Q)}_{C_{E'}(Q)}"] \arrow[d, "\mathrm{Pr}^{C_E(Q)}_{C_E(P)}"'] \arrow[dr, phantom, "\circlearrowleft"] & (kC_{E'}(Q)_0)^{C_{E'}(Q)}(b_Q^*)' \arrow[d, "\mathrm{Pr}^{C_{E'}(Q)}_{C_{E'}(P)}"] \\
(kC_E(P)_0)^{C_E(P)}\mathrm{Pr}^{C_E(Q)}_{C_E(P)}(b_Q^*) \arrow[r, "\mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}"] \arrow[d, "(\cdot)b_P^*"'] \arrow[dr, phantom, "\circlearrowleft"] & (kC_{E'}(P)_0)^{C_{E'}(P)}\mathrm{Pr}^{C_{E'}(Q)}_{C_{E'}(P)}((b_Q^*)') \arrow[d, "(\cdot)(b_P^*)'"] \\
(kC_E(P)_0)^{C_E(P)}b_P^* \arrow[r, "\mathrm{Pr}^{C_E(P)}_{C_{E'}(P)}"] & (kC_{E'}(P)_0)^{C_{E'}(P)}(b_P^*)'
\end{tikzcd}
$$

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