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Club guessing on the least uncountable cardinal and CH

宮元　忠敏 (Tadatoshi, MIYAMOTO)

南山大学, 数理情報学部

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Abstract

We consider a principle which not only negates weak club guessing but also codes every subset of the least uncountable cardinal. In particular, the continuum hypothesis must fail under this principle.

Introduction

By [Sh2] and [Sa], we know the following are all consistent.

(1) $2^{\omega} = \omega_1$ and club guessing fails.

(2) 2^{ω} is large and club guessing fails.

The argument in (1) is a combination of many ideas over a period of at least two decades and introduces new kinds of, say, appropriately proper notions of forcing. Please see [Sh2]. The construction in (2) is by Cohen forcing with a nice treatment of clubs and ladder systems. Please see [Sa]. Hence if we do nothing intentionally, then the continuum stays.

Now we may intentionally code the subsets of ω_1 to blow up the continuum. For example, we know a family of almost disjoint subsets of ω can be used for the purpose by c.c.c. forcing. However, we know that c.c.c. p. o. sets are ω-proper and that club guessing remains under ω-proper forcing ([I]).

We consider a principle, denoted by *Code(even-odd)*, which intentionally codes the subsets of ω_1 using a ladder system by proper + σ-Baire forcing. This coding introduces a club in ω_1 so that the given ladder system fails to be weak club guessing.

This principle needs no large cardinals. We just iterate proper + σ-Baire forcing ω_2-times. New reals are only created at limit stages. Code(even-odd) implies $2^{\omega} = 2^{\omega_1}$. However we do not know whether Code(even-odd) implies $2^{\omega} = \omega_2$.

We know there are coding principles, say, ψ_{AC} and v_{AC} which not only imply $2^{\omega} = 2^{\omega_1}$ but also $2^{\omega} = \omega_2$. These principles are related to large cardinals ([A], [W], [L-S], [D-D], [Mo] and [Mi]).

Preliminary

Definition 0.1. Let us denote $\Omega = \{\delta < \omega_1 \mid \delta \text{ is limit}\}$ in order to use a shorter notation. For $\delta \in \Omega$, A *ladder* A at δ means that A is a cofinal subset of δ and is of order-type ω. We write $\langle A(n) \mid n < \omega\rangle$ when we list the elements of A in the strict increasing order. A *ladder system* $\langle A_\delta \mid \delta \in \Omega\rangle$ means that for all $\delta \in \Omega$, A_δ is a ladder at δ.

For a club D in ω_1 and a ladder A at $\delta \in \Omega$, we write $A \subseteq^* D$, if there exists $n_0 < \omega$ such that for all $n \geq n_0$, we have $A(n) \in D$. Hence we may say that A is *almost included* in D.

For a club D in ω_1, we denote the set of countable limit ordinals which are accumulation points of D by $\overline{D}$. Hence if $\delta \in \overline{D}$, then $\delta \in \Omega \cap D$ and $D \cap \delta$ is cofinal below δ.

Definition 0.2. *Club guessing (CG)* holds, if there exists a ladder system $\langle A_\delta \mid \delta \in \Omega\rangle$ such that for any club D in ω_1, there exists $\delta \in \Omega$ with $A_\delta \subseteq^* D$.

Notice that there actually are stationary many δ's as above.

Definition 0.3. *Weak club guessing (WCG)* holds, if there exists a ladder system $\langle A_\delta \mid \delta \in \Omega\rangle$ such that for any club D in ω_1, there exists $\delta \in \Omega$ with $\mid A_\delta \cap D \mid = \omega$.

The following is trivial.

Proposition 0.4. CG implies WCG.

□

The converse does not hold due to [Sa]. Hence WCG is indeed weaker.

Theorem 0.5. ([Sa]) WCG does not entail CG.

Proof. (Out-line) First get $\neg$ CG. Then add a Cohen real. We have WCG and $\neg$ CG remains in the extension. For more on this, please consult [Sa].

□

§1. **Good Parameters**

We formulate an equivalent condition to the negation of the weak diamond of Shelah and Devlin.

Definition 1.1. Let θ be a regular cardinal with $\theta \geq \omega_2$. For any countable elementary substructure N of H_θ, we define

$$N^* = \{f(N \cap \omega_1) \mid f \in N\}.$$

Lemma 1.2. Let θ and N be as above. Then N^* is the $\subseteq$-least countable elementary substructure M of H_θ with $N \cup \{N \cap \omega_1\} \subseteq M$.

Proof. Via Tarski's criterion.

□

Lemma 1.3. Let θ be a regular cardinal with $\theta \geq \omega_2$. For any countable elementary substructrure N, we may associate a sequence $\langle N_i \mid i < \omega_1\rangle$ of countable elementary substructures of H_θ such that

- $N_0 = N$,
- $N_{i+1} = N_i^* = \{f(N_i \cap \omega_1) \mid f \in N_i\}$,
- For limit i, $N_i = \bigcup\{N_j \mid j < i\}$.

□

We may call $\langle N_i \mid i < \omega_1\rangle$ *the canonical sequence of extensions* of N in H_θ.

Definition 1.4. Let θ be a regular cardinal with $\theta \geq \omega_2$. We say $p \in H_\theta$ is a *good parameter* in H_θ, if for any two countable elementary substructures N_1, N_2 of H_θ with $p \in N_1 \cap N_2$, if $\pi : N_1 \longrightarrow N_2$ is an $\in$-isomorphism with $\pi(p) = p$, then there exists an $\in$-isomorphism $\pi^* : N_1^* \longrightarrow N_2^*$ extending π.

The following is implicit in [W].

Lemma 1.5. (Good Parameter Lemma) The following are equivalent.

(1) There exists a good parameter p in some H_θ, where θ is a regular cardinal with $\theta \geq \omega_2$.
(2) $2^\omega = 2^{\omega_1}$.

Proof. (1) implies (2): Fix p and θ. Then let F consist of all $((\overline{N}, \overline{p}), \overline{N_{\omega_1}})$, where

- N is a countable elementary substructure of H_θ with $p \in N$,
- $\overline{N}$ denotes the transitive collapse of N and $\overline{p}$ denotes the image of p under the collapse,
- $\langle N_i \mid i < \omega_1\rangle$ is the canonical sequence of extensions of N in H_θ and let $N_{\omega_1} = \bigcup\{N_i \mid i < \omega_1\}$,
- $\overline{N_{\omega_1}}$ denotes the transitive collapse of N_{ω_1}.

By (1), F is a well-defined function from Dom $= \{(\overline{N}, \overline{p}) \mid$ N is a countable elementary substructure of H_θ with $p \in N\}$ onto Ran $= \{\overline{N_{\omega_1}} \mid$ there exists a canonical sequence $\langle N_i \mid i < \omega_1\rangle$ of extensions of some countable elementary substructure N of H_θ with $p \in N$ and $N_{\omega_1} = \bigcup\{N_i \mid i < \omega_1\}\}$. Notice that Dom $\subseteq H_{\omega_1}$ and so Dom is of size 2^ω. On the other hand, $\mathcal{P}(\omega_1) = \{B \mid B \subseteq \omega_1\} \subseteq \bigcup$ Ran. Hence $2^{\omega_1} \leq 2^\omega \cdot \omega_1 = 2^\omega$.

(2) implies (1): Let $f : \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega_1)$ be a bijection. Let θ be regular cardinal with $f \in H_\theta$. Then f is a good parameter in H_θ.

□

§2. The principle Code(even-odd)

We introduce our coding principle which requires no large cardinals.

Definition 2.1. Let $\langle A_\delta \mid \delta \in \Omega\rangle$ be a ladder system, we denote *Code($\langle A_\delta \mid \delta \in \Omega\rangle$, even-odd)*, if for any $B \subseteq \omega_1$, there exists two clubs C and D in ω_1 such that for any $\delta \in \overline{C}$,

- If $\delta \in B$, then $\mid A_\delta \cap D \mid < \omega$ is odd,
- If $\delta \notin B$, then $\mid A_\delta \cap D \mid < \omega$ is even.

We denote *Code(even-odd)*, if for all ladder systems $\langle A_\delta \mid \delta \in \Omega\rangle$, Code($\langle A_\delta \mid \delta \in \Omega\rangle$, even-odd) hold.

Proposition 2.2. If Code($\langle A_\delta \mid \delta \in \Omega\rangle$, even-odd) holds, then $\langle A_\delta \mid \delta \in \Omega\rangle$ is a good parameter in H_{ω_2} and so $2^\omega = 2^{\omega_1}$ holds.

Proof. Let $\pi : N_1 \longrightarrow N_2$ with $\pi(p) = p$, where we set $p = \langle A_i \mid i \in \Omega\rangle$.
If $\pi^* : N_1^* \longrightarrow N_2^*$ were to extend π, we would have

$$\pi^*(f(\delta)) = \pi^*(f)(\pi^*(\delta)) = \pi(f)(\delta),$$

where we denote $\delta = N_1 \cap \omega_1 = N_2 \cap \omega_1$.

Suppose $f, g \in N_1$ with $f(\delta) = g(\delta)$. We want to show $\pi(f)(\delta) = \pi(g)(\delta)$. Let $B = B(f,g) = \{\alpha < \omega_1 \mid f(\alpha) = g(\alpha)\}$. Then $\delta \in B \in N_1$. By Code(p, even-odd), we have two clubs C and D. We may assume $C, D \in N_1$. Then via π, for all $i \in \overline{\pi(C)}$,

- If $i \in \pi(B)$, then $\mid \pi(p)(i) \cap \pi(D) \mid < \omega$ is odd,
- If $i \notin \pi(B)$, then $\mid \pi(p)(i) \cap \pi(D) \mid < \omega$ is even.

Since $\delta \in \overline{C}$, $\pi(C)$ is a club in ω_1 and $C \cap \delta = \pi(C) \cap \delta$, we have $\delta \in \overline{\pi(C)}$. Since $\delta \in B$, we have $\mid A_\delta \cap D \mid < \omega$ is odd. Since $\pi(p) = p$, we have $A_\delta = p(\delta) = \pi(p)(\delta)$. Hence $A_\delta \cap D = \pi(p)(\delta) \cap D = \pi(p)(\delta) \cap \pi(D)$ and so $\mid \pi(p)(\delta) \cap \pi(D) \mid < \omega$ is odd. Hence $\delta \in \pi(B) = B(\pi(f), \pi(g))$ and so $\pi(f)(\delta) = \pi(g)(\delta)$.

This establishes that $\pi^*(f(\delta)) = \pi(f)(\delta)$ is well-defined from N_1^* into N_2^*. We may show this π^* is an $\in$-isomophism in a similar manner.

□

Proposition 2.3. If Code(even-odd) holds, then weak club guessing gets negated.

Proof. For any ladder system $\langle A_\delta \mid \delta \in \Omega\rangle$, we have two clubs C and D such that for all $\delta \in \overline{C}$, $A_\delta \cap D$ is finite. Hence weak club guessing fails.

□

§3. Forcing Code(even-odd)

We first design a notion of forcing which is proper and σ-Baire.

Definition 3.1. Let $\langle A_\delta \mid \delta \in \Omega\rangle$ be a ladder system and $B \subseteq \omega_1$. We define a notion of forcing $P = P(\langle A_\delta \mid \delta \in \Omega\rangle, B)$ as follows;

$p = (\alpha^p, C^p, D^p) \in P$, if

(1) $\alpha^p < \omega_1$,

(2) C^p and D^p are closed subsets of $\alpha^p + 1$ with $\alpha^p \in C^p \cap D^p$,

(3) For each $\delta \in \overline{C^p}$ $(= \{\alpha \leq \alpha^p \mid \alpha \in \Omega,\ C^p \cap \alpha$ is cofinal below $\alpha\})$,

- If $\delta \in B$, then $\mid A_\delta \cap D^p \mid < \omega$ is odd,
- If $\delta \notin B$, then $\mid A_\delta \cap D^p \mid < \omega$ is even.

For $p, q \in P$, let $q \leq p$, if

- $\alpha^p \leq \alpha^q$,
- $C^p = C^q \cap (\alpha^p + 1)$ and $D^p = D^q \cap (\alpha^p + 1)$.

The following is from [Sh2]. Due to this, there is no need to deal with $\in$-chains $\langle N_n \mid n < \omega\rangle$ of countable elementary substructures of H_θ and a countable elementary substructure N of H_χ with $P,\ H_\theta \in N$ such that $\bigcup\{N_n \mid n < \omega\} = H_\theta \cap N$, where θ and χ are regular cardinals with $P \in H_\theta \in H_\chi$.

Lemma 3.2. Let $p \in P$ and D be a dense subset of P. Then consider $f = f_{pD} : (\alpha^p, \omega_1) \longrightarrow \omega_1$ such that $\xi < f(\xi) = \alpha^q$ for some $q \in D$ with $q \leq p' = (\xi, C^p \cup \{\xi\}, D^p \cup \{\xi\}) \leq p$. Let $D(f) = \{\beta < \omega_1 \mid \forall \xi \in (\alpha^p, \beta)\ f(\xi) < \beta\}$. Then $D(f)$ is a club in ω_1.

□

Lemma 3.3. P is proper and σ-Baire.

Proof. Let θ be a sufficiently large regular cardinal and let M be a countable elementary substructure of H_θ with $P \in M$. Given $p \in M \cap P$, we may construct a (P, M)-generic sequence $\langle p_n \mid n < \omega\rangle$ such that $p_0 = p$ and

- If $M \cap \omega_1 \in B$, then $\mid D^{p_1} \cap A_{M\cap\omega_1} \mid < \omega$ is odd and for all $n \geq 1$, $D^{p_n} \cap A_{M\cap\omega_1} = D^{p_1} \cap A_{M\cap\omega_1}$.
- If $M \cap \omega_1 \notin B$, then $\mid D^{p_1} \cap A_{M\cap\omega_1} \mid < \omega$ is even and for all $n \geq 1$, $D^{p_n} \cap A_{M\cap\omega_1} = D^{p_1} \cap A_{M\cap\omega_1}$.

This is possible by lemma 3.2. Now let,

- $\alpha^q = M \cap \omega_1$,
- $C^q = \bigcup\{C^{p_n} \mid n < \omega\} \cup \{M \cap \omega_1\}$,
- $D^q = \bigcup\{D^{p_n} \mid n < \omega\} \cup \{M \cap \omega_1\}$.

Then $q \in P$, $q \leq p$ and q is (P, M)-generic.

□

Note 3.4. (1) P can not be ω-proper, since ω-proper is iterable under countable support and ω-proper preserves club guessing ([Sh1]).

(2) P is not only σ-Baire but forces $\diamondsuit$. In particular, we have CH in V^P. But when we iterate this type of p.o. sets, we must add new reals at some limit stages. This is because we have $2^\omega = 2^{\omega_1} > \omega_1$ in the end. The reals added are far from being, say, Cohen reals, since ω^ω-bounding + proper is iterable under countable support. And we are certainly iterating with ω^ω-bounding + proper notions of forcing.

Theorem 3.5. There exists a countable support iteration $\langle P_\alpha \mid \alpha \leq \omega_2\rangle$ of length ω_2 such that Code(even-odd) holds in the generic extensions of V via P_{ω_2}.

Proof. Since we iterate notions of forcing of size ω_1 under CH, we have the ω_2-c.c. as long as iteration is of length at most ω_2 ([Sh1]). Hence by suitable book-keeping, we may take care of every pair of a ladder system and a subset of ω_1 in ω_2 steps.

□

Question 3.6. (1) Code(even-odd) implies $\neg$ weak club guessing and $2^\omega = 2^{\omega_1}$. Does Code(even-odd) imply $2^\omega = \omega_2$? We have a coding principle which implies $2^\omega = 2^{\omega_1} = \omega_2$ and whose consistency strength is exactly that of a strongly inaccessible cardinal ([Mi]).

(2) It is shown Con(2^ω is large + $\neg$ club guessing) via Cohen forcing in [Sa]. How about, as pointed out in [Sa], Con(2^ω is large + $\neg$ weak club guessing) ?

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miyamoto@nanzan-u.ac.jp
Division of Mathematics
Nanzan University
27 Seirei-cho, Seto, Aichi
489-0863 JAPAN