数理解析研究所講究録
第 1596 巻 2008 年 11-17

BLOCK MATRIX OPERATORS FOR p-HYPONORMALITY

George Exner
Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837, USA
e-mail: exner@bucknell.edu
Il Bong Jung
Department of Mathematics, Kyungpook National University, Daegu, 702-701 Korea
e-mail: ibjung@knu.ac.kr
Mi Ryeong Lee [1]
Faculty of Liberal Education, Kyungpook National University, Daegu, 702-701 Korea
e-mail: leemr@knu.ac.kr

ABSTRACT. We introduce a new model of block matrix operator $M(\alpha,\beta)$ induced by two sequences α and β and characterize its p-hyponormality. The model induces a measurable transformation T on the set of nonnegative integers $\mathbf{N}_0$ with point mass and composition operator C_T on $l^2 := l^2(\mathbf{N}_0)$. The techniques via composition operators will be used to treat p-hyponormality of $M(\alpha,\beta)$ and provide some interesting theorems about p-hyponormality. Finally, we apply our results to obtain examples of p-hyponormal making distinct as usual.

1. Introduction and Preliminaries. This was talked at the 2008 RIMS conference: Inequalities on linear operators and its applications, which was held at Kyoto University on January 30-February 1 in 2008.

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, $p \in (0,\infty)$. If $p=1$, T is *hyponormal* and if $p=\frac{1}{2}$, T is *semi-hyponormal* ([Xi]). In particular, T is said to be *∞-hyponormal* if it is p-hyponormal for all $p>0$ ([MS]). The Löwner-Heinz inequality implies that every p-hyponormal operators are q-hyponormal operators for $q \leq p$ and many operator theorists have studied properties in operators in those classes; for examples, spectral theory, operator inequalities, and invariant subspaces, etc. (cf. [BJ], [Fur], [IY], [JKP], [JLPa]). Also, the study of gaps between subnormality and hyponormality has been studied in several areas by many operator theorists, and whose study is growing up still. The p-hyponormality is contained in those studies, but new models for p-hyponormal operators need to be developed still. And also, Jung-Lee-Park constructed examples induced by some block matrix operators in [JLP] and [JLL], in which the classes of those operators are distinct with respect to any positive real number p. Recently Burnap-Jung-Lambert discussed some models via composition operator C_T on L^2 in [BJL] and [BJ], in which such classes of weak hyponormal operators are distinct for each p. Moreover, they used the notion of conditional expectations for studying of p-hyponormality of C_T, which will be also main tool of this note. Here are some terminologies for conditional expectation. Let $(X,\mathcal{F},\mu)$ be a σ finite measure space and let $T: X \to X$ be a transformation such that $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ T^{-1}/d\mu$ is in L^∞. The composition operator C_T acting on $L^2 := L^2(X,\mathcal{F},\mu)$ is defined by $C_T f = f \circ T$.

[1]2000 Mathematics subject classification: 47B20, 47B38.

Key words and phrases: p-hyponormal operator, composition operator, conditional expectation.

The condition $h \in L^\infty$ assures that C_T is bounded. And we denote $Ef = E(f|T^{-1}\mathcal{F})$ for the conditional expectation of f with respect to $T^{-1}\mathcal{F}$. Some useful results will come from [L], [BJL], and [HWh]. In particular, in the proofs and examples below, we will have need of the following special case: if $\mathcal{A}$ is the purely atomic σ-subalgebra of $\mathcal{F}$ generated by the measurable partition of X into sets of positive measure $\{A_k\}_{k\geq 0}$, then

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)} \left(\int_{A_k} f(x)d\mu(x) \right) \chi_{A_k}.$$

The interested readers can find a more extensive list of properties for conditional expectations in [BJL] and [Ra].

This article consists of five sections. In Section 2, we construct a block matrix operator induced by two sequences α and β, which will make distinct classes of p-hyponormal operators with respect to $p > 0$ later section. A block matrix operator $M(\alpha, \beta)$ induced by two sequences α and β provides a measurable transformation T on $\mathbb{N}_0$ with point mass measure on $\mathbb{N}_0$ and its corresponding composition operator C_T on l^2 is equivalent to $M(\alpha, \beta)$. In Section 3, we characterize block matrix operators $M(\alpha, \beta)$ for p-hyponormality and construct a useful form for distinction examples. In Section 4, we discuss a flatness of p-hyponormality about block matrix operator $M(\alpha, \beta)$: the ∞-hyponormality of $M(\alpha, \beta)$ is equivalent to any[some] p-hyponormality under some conditions. Finally, in Section 5, we give some examples being distinct the classes of p-hyponormal operators.

This article will be appeared in other journal as the full version. And so we skip the detail proofs here.

2. Relationships. Let $\alpha := \{a_i^{(n)}\}_{\substack{1\leq i\leq r\\ 0\leq n<\infty}}$ and $\beta := \{b_j^{(n)}\}_{\substack{1\leq j\leq s\\ 0\leq n<\infty}}$ be bounded sequences of positive real numbers. Let $M = [A_{ij}]_{0\leq i,j<\infty}$ be a *block matrix operator* whose blocks are $(r+s)\times(s+1)$ matrices such that $A_{ij} = 0$, $i \neq j$, and

$$A_n := A_{nn} = \begin{pmatrix} a_1^{(n)} & & & \\ \vdots & & O & \\ a_r^{(n)} & & & \\ & b_1^{(n)} & & \\ & & \ddots & \\ O & & & b_s^{(n)} \end{pmatrix}, \tag{2.1}$$

where other entries are 0 except $a_*^{(n)}$ and $b_*^{(n)}$ indicated in (2.1). Obviously such block matrix operator M is bounded.

Definition 2.1. For two bounded sequences $\alpha := \{a_i^{(n)}\}_{\substack{1\leq i\leq r\\ 0\leq n<\infty}}$ and $\beta := \{b_j^{(n)}\}_{\substack{1\leq j\leq s\\ 0\leq n<\infty}}$, the block matrix operator $M := M(\alpha, \beta)$ satisfying (2.1) is called a *block matrix operator with weight sequence* (α, β).

Let M be a block matrix operator with weight sequence (α, β) and let $W_{\alpha,\beta}$ be its corresponding operator on l^2 relative to some orthonormal bases. Then $W_{\alpha,\beta}$ has a duplicate form; for example, if we take $r = 3$, $s = 2$ and $a_i^{(n)} = b_j^{(n)} = 1$ for all $i, j, n \in \mathbb{N}$, then the block matrix operator with (α,β) is unitarily equivalent to the following operator $W_{\alpha,\beta}$ on l^2 defined by

$$W_{\alpha,\beta}(x_1,x_2,x_3,x_4,x_5,\cdots)=(\underbrace{x_1,x_1,x_1}_{(3)},x_2,x_3,\underbrace{x_4,x_4,x_4}_{(3)},x_5,x_6,\underbrace{x_7,x_7,x_7}_{(3)},\cdots).$$

For arbitrary block matrix operator M with weight sequence (α,β), since M is p-hyponormal if and only if αM is p-hyponormal for any[some] positive real number α, we may assume $a_1^{(0)}=1$, which will be assumed throughout this note.

We now return to our work, in particular, consider $X=\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and the power set $\mathcal{P}(X)$ of X for the σ-algebra $\mathcal{F}$. Define a non-singular measurable transformation T on $\mathbb{N}_0$ such that

$$T^{-1}(k(s+1))=\{k(r+s)+i-1:0\leq i\leq r\},\ k=0,1,2,\cdots, \tag{2.2}$$

$$T^{-1}(k(s+1)+i)=k(r+s)+r-1+i,\ \ 1\leq i\leq s,\quad k=0,1,2,\cdots.$$

We write $m(\{i\}):=m_i$ for a point mass measure on X.

Proposition 2.2. *Under the above notation, the composition operator C_T on l^2 defined by $C_Tf=f\circ T$ is unitarily equivalent to the block matrix operator $M(\alpha,\beta)$, where $\alpha: a_i^{(n)}=\sqrt{\frac{m_{n(r+s)+i-1}}{m_{n(s+1)}}}$ $(1\leq i\leq r)$ and $\beta: b_j^{(n)}=\sqrt{\frac{m_{n(r+s)+r+j-1}}{m_{n(s+1)+j}}}$ $(1\leq j\leq s)$, $n\in\mathbb{N}_0$.*

Proposition 2.3. *Let $M(\alpha,\beta)$ be a block matrix with weight sequence (α,β), where $\alpha:=\{a_i^{(n)}\}_{\substack{1\leq i\leq r\\ 0\leq n<\infty}}$, $\beta:=\{b_j^{(n)}\}_{\substack{1\leq j\leq s\\ 0\leq n<\infty}}$, and $a_1^{(0)}=1$. Then there exists a measurable transformation T on a σ finite measure space $(\mathbb{N}_0,\mathcal{P}(\mathbb{N}_0),m)$ such that $M(\alpha,\beta)$ is unitarily equivalent to a composition operator C_T on l^2.*

3. Some Characterizations. Let T be a non-singular measurable transformation on l^2 as in (2.2) and let $m(\{i\})=m_i$ be the point mass on $\mathbb{N}_0$.

Theorem 3.1. *Let $p\in(0,\infty)$. Then the following assertions are equivalent:*
(i) *C_T is p-hyponormal on l^2;*
(ii) *the block matrix operator $M(\alpha,\beta)$ as in Proposition 2.2 is p-hyponormal;*
(iii) *$E(1/h^p)(n)\leq 1/(h^p\circ T)(n)$*
(iv) *it holds that*

$$\frac{1}{m(T^{-1}(T(n)))}\sum_{j\in T^{-1}(T(n))}\frac{m_j^p m_j}{m(T^{-1}(j))^p}\leq\left(\frac{m_{T(n)}}{m(T^{-1}(T(n)))}\right)^p,\qquad n\in\mathbb{N}_0.$$

Remark 3.2. By some formulas in the proof of Theorem 3.1, we have the following assertions:
(i) $M(\alpha,\beta)$ is ∞-hyponormal if and only if $m(T^{-1}(n))/m_n\geq m(T^{-1}(T(n))/m(T(n))$ for all $n\in\mathbb{N}_0$.
(ii) $M(\alpha,\beta)$ is quasinormal if and only if $m(T^{-1}(n))/m_n=m(T^{-1}(T(n))/m(T(n))$ for all $n\in\mathbb{N}_0$.

To obtain more useful and simpler form for p-hyponormality of $M(\alpha,\beta)$, we consider a block matrix operator M as following:

$$
\begin{aligned}
&M(\alpha,\beta): A \equiv A_1 = A_2 = \cdots \text{ (with notation in (2.1)) with}\\
&\qquad \alpha : a_i^{(n)} = a_i,\ n \in \mathbb{N}_0,\ 1 \le i \le r;\\
&\qquad \beta : b_j^{(n)} = b_j,\ n \in \mathbb{N}_0,\ 1 \le j \le s.
\end{aligned} \tag{3.1}
$$

This type will be used usefully to obtain examples being distinct classes of p-hyponormal operators in Section 5.

Theorem 3.3. *Let $M(\alpha,\beta)$ be as in (3.1). Then the block matrix operator $M(\alpha,\beta)$ is p-hyponormal if and only if the following two cases hold:*
(i) *for $n = k(r+s)+i-1$ $(1 \le i \le r)$,*

$$
\sum_{\substack{j\in T^{-1}(T(n))\\ j\equiv 0 \bmod (s+1)}} \left(\frac{1}{\sum_{1\le i\le r} a_i^2}\right)^p \frac{a_{i_j}^2}{\sum_{1\le i\le r} a_i^2} + \sum_{\substack{j\in T^{-1}(T(n))\\ j\not\equiv 0 \bmod (s+1)}} \frac{1}{b_{l_j}^{2p}}\cdot\frac{a_{i_j}^2}{\sum_{1\le i\le r} a_i^2}
$$

$$
\le \left(\frac{1}{\sum_{1\le i\le r} a_i^2}\right)^p, \qquad 1 \le i_j \le r,\ 1 \le l_j \le s, \tag{3.2}
$$

(ii) *for $n = k(r+s)+r+j-1$ $(1 \le j \le s)$,*

(ii-a) $b_j^2 \le \sum_{1\le i\le r} a_i^2$ *if $n \equiv 0 \mod(s+1)$*

(ii-b) $b_j^2 \le b_{t_n}^2$ *if $n \not\equiv 0 \mod(s+1)$ and for some t_n $(1 \le t_n \le s)$.*

The following is a special case of Theorem 3.3, which provides a simple form.

Corollary 3.4. *Let $M := M(\alpha,\beta)$ be as in (3.1) with $a_i^{(n)} = a$ $(1 \le i \le r)$ and $b_j^{(n)} = b$ $(1 \le j \le s)$. Then M is p-hyponormal if and only if the following two cases hold:*
(i) *for $n = k(r+s)+i-1$ $(1 \le i \le r)$,*

$$
\frac{1}{r}\left[\sum_{\substack{j\in T^{-1}(T(n))\\ j\equiv 0 \bmod (s+1)}} \left(\frac{1}{ra^2}\right)^p + \sum_{\substack{j\in T^{-1}(T(n))\\ j\not\equiv 0 \bmod (s+1)}} \frac{1}{b^{2p}}\right] \le \left(\frac{1}{ra^2}\right)^p,
$$

(ii) *for $n = k(r+s)+r+j-1$ $(1 \le j \le s)$, $b^2 \le ra^2$ holds.*

Note that if we are under type of Theorem 3.3 (which will be called "type I") it will be important to know *which* j in $T^{-1}(T(n))$ have various $j \equiv t_j \mod(s+1)$ which if we are under type of Corollary 3.4 (which will be called "type II") it is only important to know *how many* j are of various $j \equiv t_j$ mod(s + 1). Then we have the following remark.

Remark 3.5 (Special case of Corollary 3.4 with $r = N(s+1)$). In this case for $n = n(r+s)+i-1$, $1 \le i \le r$, the set of l in $T^{-1}(T(n))$ contains exactly N elements of each modulus, mod$(s+1)$. So under type II the test (3.2) for such n becomes

$$
N\left(\frac{1}{ra^2}\right)^p \frac{1}{r} + (r-N)\left(\frac{1}{b^2}\right)^p \frac{1}{r} \le \left(\frac{1}{ra^2}\right)^p.
$$

For $n = k(r+s)+r-1+j$, and under type II we either get a condition trivially satisfied for all p, or $1/(ra^2) \le 1/b^2$, the latter only if there is at least one n so that $n = K(r+s)+r-1+j$ and $n = Q(s+1)$. But since $r = N(s+1)$, this is $(K+1)N(s+1)+Ks+j-1 = Q(s+1)$ for some K, Q, j, and take $K = s+1$ and $j = 1$ to obtain a solution, so $1/(ra^2) \le 1/b^2$.

Remark 3.6. We can apply the idea of Theorem 3.3 to the model of general block matrix operator in the Definition 2.1 by the same method; the result formula will be slight complete than that of Theorem 3.3. We leave the exact formula to interested readers.

4. ∞-hyponormality and Flatness. We begin this section with the following fundamental lemma.

Lemma 4.1. *Suppose $p > 1$ and $q > 1$ are relatively prime. Given any l_p, $0 \le l_p \le p-1$, and any l_q, $0 \le l_q \le q-1$, there exists $n \in \mathbb{N}$ so that $n \equiv l_p \bmod p$ and $n \equiv l_q \bmod q$.*

Lemma 4.2. *Suppose that*

$$A := \begin{pmatrix} \sqrt{y_1} & & & & \\ \vdots & & & \text{O} & \\ \sqrt{y_r} & & & & \\ & \sqrt{x_1} & & & \\ & & \ddots & & \\ \text{O} & & & & \sqrt{x_s} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} A & & \\ & A & \\ & & \ddots \end{pmatrix}. \tag{4.1}$$

Assume that $\mathrm{GCD}(r+s, s+1) = 1$. *If M is p-hyponormal for some $p \in (0, \infty)$, then*

$$x_1 = x_2 = \cdots = x_s \le \sum_{1 \le i \le r} y_i. \tag{4.2}$$

Proposition 4.3. *Let A and M be as in* (4.1). *Suppose there exists $N \in \mathbb{N}$ such that $r = N(s+1)$ and* $\mathrm{GCD}(r+s, s+1) = 1$. *Then the following assertions are equivalent:*
(i) *M is p-hyponormal for some $p \in (0, \infty)$;*
(ii) *M is ∞-hyponormal;*
(iii) $x_1 = x_2 = \cdots = x_s = \sum_{1 \le i \le r} y_i$.

5. Examples. Let A and M be as in (4.1) with $r+s = N(s+1)$ for some $N \in \mathbb{N}$ and we will see this is the "opposite" of $r = N(s+1)$ and $\mathrm{GCD}(r+s, s+1) = 1$.

Proposition 5.1. *Let M be the block matrix operator as in* (4.1). *Then M is p-hyponormal if and only if the following inequality holds:*

$$\sum_{\substack{j \not\equiv 0 \bmod (s+1) \\ j \in T^{-1}(T(n))}} \left(\frac{1}{x_{t_j \bmod (s+1)}} \right)^p y_{j+1} \le \frac{1}{(\sum_{1 \le i \le r} y_i)^p} \sum_{\substack{j \not\equiv 0 \bmod (s+1) \\ j \in T^{-1}(T(n))}} y_{j+1}. \tag{5.1}$$

The following corollaries come immediately from Proposition 5.1.

Corollary 5.2. *Let M be the block matrix operator as in* (4.1) *with $x_1 = x_2 = \cdots = x_s = x$. Then* (5.1) *is trivially satisfied as long as $x \ge \sum_{1 \le i \le r} y_i$ with no conditions on the y_j.*

Corollary 5.3. *Let M be the block matrix operator as in* (4.1) *such that the y_{j+1} for $j \equiv 0 \bmod(s+1)$ occur only in $\sum_{1 \le i \le r} y_i$. Thus if we consider some y'_{j+1} for $j \equiv 0 \bmod(s+1)$, as long as $\sum_{j \equiv 0} y'_{j+1} = \sum_{j \equiv 0} y_{j+1}$, then M' is p-hyponormal if and only if M is p-hyponormal.*

Now we close this paper with the following example.

Example 5.4. Let

$$A := \begin{pmatrix} 1 & & & & & \\ & 1 & & & O & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \sqrt{x_1} & \\ O & & & & & \sqrt{x_2} \end{pmatrix} \quad \text{and } M := \begin{pmatrix} A & & \\ & A & \\ & & \ddots \end{pmatrix}.$$

Write Y for $\sum_{1 \le i \le 4} y_i$. Then the condition of

$$\frac{1}{Y^p}\frac{y_1}{Y} + \frac{1}{x_1^p}\frac{y_2}{Y} + \frac{1}{x_2^p}\frac{y_3}{Y} + \frac{1}{Y^p}\frac{y_4}{Y} \le \frac{1}{Y^p}$$

is equivalent to

$$\frac{y_2}{x_1^p} + \frac{y_3}{x_2^p} \le \frac{y_2 + y_3}{4^p}.$$

Inserting the $y_i \equiv 1$, $1 \le i \le 4$, we get

$$\left(\frac{4}{x_1}\right)^p + \left(\frac{4}{x_2}\right)^p \le 2, \tag{5.2}$$

which is equivalent to M is p-hyponormal. Note that (5.2) keeps distinct the classes of p-hyponormal operators with respect to $0 < p < \infty$. To obtain region for ∞-hyponormality of M we use Remark 3.2 and formulas in proof of Theorem 3.3, and there are three cases, Cases 1a, 1b, and 2b, which imply that $m_{3k_1} \ge m_{3k}$, $x_1 \ge 4$ & $x_2 \ge 4$, and $x_1 \ge x_1$ & $x_2 \ge x_2$, respectively. Thus we obtain that

$$M \text{ is } \infty\text{-hyponormal} \Longleftrightarrow x_1 \ge 4 \text{ and } x_2 \ge 4.$$

Of course, since (5.2) is equivalent to $x_2 \ge 4 \cdot (2 - (4/x_1)^p)^{-1/p}$ for $x_1 > 4 \cdot 2^{-1/p}$, taking $p \to \infty$, we may check easily the obtaining conditions ∞-hyponormality of M are $x_1 \ge 4$ and $x_2 \ge 4$. On the other hand, applying Remark 3.2 and formulas in proof of Theorem 3.3 for quasinormality of M, we also obtain that M is quasinormal if and only if $(x_1, x_2) = (4, 4)$.

References

[BJ] C. Burnap and I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl., to appear.

[BJL] C. Burnap, I. Jung and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53**(2005), 381-397.

[CHo1] J. Campbell and W. Hornor, *Seminormal composition operators,* J. Operator Theory, **29**(1993), 323–343.

[EJL] G. Exner, I. Jung and M. Lee, *Block matrix operators and p-hyponormality*, preprint.

[Fur] T. Furuta, *Invitation to linear operators*, Taylor & Francis Inc., 2001.

[L] A. Lambert, *Hyponormal composition operators*, Bull. London Math. Soc., **18**(1986), 395-400.

[HWh] D. Harrington and R. Whitley, *Seminormal Composition Operators*, J. Operator Theory, **11**(1984), 125-135.

[IY] M. Ito and T. Yamazaki, *Relations between two inequalities* $(B^{r/2}A^p B^{r/2})^{r/(p+r)} \geq B^r$ *and* $A^p \geq (A^{p/2}B^r A^{p/2})^{p/(p+r)}$ *and their applications*, Integral Equations Operator Theory, **44**(2002) 442-450.

[JKP] I. Jung, E. Ko, and C. Pearcy, *Aluthge transforms of operators,* Integral Equations Operator Theory, **37**(2000), 437-448.

[JLL] I. Jung, M. Lee and P. Lim, *Gaps of operators, II*, Glasgow Math. J., **47**(2005), 461-469.

[JLP] I. Jung, P. Lim and S. Park, *Gaps of operators*, J. Math. Anal. Appl., **304**(2005), 87-95.

[JLPa] I. Jung, M. Lee and S. Park, *Separating classes of composition operators via subnormal condition*, Proc. A.M.S., to appear

[MS] S. Miyajima and I. Saito, ∞-*hyponormal operators and their spectral properties*, Acta Sci. Math. (Szeged) **67**(2001), 357-371.

[Ra] M. Rao, *Conditional measures and Applications*, Marcel Decker, New York 1993.

[Xi] D. Xia, *Spectral Theory of hyponormal Operators*, Birkhäuser, Boston, 1983.