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行列環のテンソル積空間上の部分転置写像について

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This is a survey of [1], which is a joint work with Tsuyoshi Ando. Here are some of the results in [1]. For more details, we refer the reader to [1].

1. 部分転置写像とその関連

Theorem 1.1. *Let $A_k \in M_m$, $B_k \in M_n$. Then for every unitarily invariant norm $|||\cdot|||$*

$$\sup_{X \in M_{m,n}} \frac{|||\sum_k A_k X B_k^T|||}{|||X|||} \leqq \sqrt{\min(m,n)}\ \|\sum_k A_k \otimes B_k\|_\infty.$$

Theorem 1.2. *Let $A_k \in M_m$, $B_k \in M_n$. Then for every unitarily invariant norm $|||\cdot|||$*

$$\sup_{X \in M_{m,n}} \frac{|||\sum_k A_k X B_k|||}{|||X|||} \leqq \min(m,n)\ \|\sum_k A_k \otimes B_k\|_\infty.$$

The partial transpose map Θ on $M_m \otimes M_n$ is defined as

$$\Theta(\sum_k A_k \otimes B_k) = \sum_k A_k \otimes B_k^T.$$

Theorem 1.3. *For every unitarily invariant norm $|||\cdot|||$*

$$|||\Theta(\mathbf{X})||| \leqq \min(m,n)\ |||\mathbf{X}|||.$$

For the Hilbert-Schmidt norm $\|\cdot\|_2$

$$\|\Theta(\mathbf{X})\|_2 = \|\mathbf{X}\|_2.$$

Notice that this inequality for the spectral norm was originally established by Tomiyama

2. 部分転置写像の制限

Let

$$K_{m,n} := \{A \otimes I_n + I_m \otimes B : A \in M_m,\ B \in M_n\}.$$

We observe the restriction Θ_0 of the map Θ to the subspace $K_{m,n}$:

$$\Theta_0(\mathbf{X}) = A \otimes I_n + I_m \otimes B^T \quad \text{for } \mathbf{X} = A \otimes I_n + I_m \otimes B.$$

Theorem 2.1. *The restriction Θ_0 is positive:*

$$\mathbf{X} \geqq 0 \implies \Theta_0(\mathbf{X}) \geqq 0.$$

Theorem 2.2. *Let $\mathbf{X} = A \otimes I_n + I_m \otimes B$.*
(i) *If $n = 2$ or B is normal, then $\mathbf{X}$ and $\Theta_0(\mathbf{X})$ are unitarily similar in $M_m \otimes M_n$.*
(ii) *If $m = 2$ or A is normal, then $\mathbf{X}$ and $\Theta_0(\mathbf{X})^T$ are unitarily similar in $M_m \otimes M_n$.*

In any case, $|\mathbf{X}|$ and $|\Theta_0(\mathbf{X})|$ are unitarily similar in $M_m \otimes M_n$; hence, $|||\Theta_0(\mathbf{X})||| = |||\mathbf{X}|||$ for every unitarily invariant norm $|||\cdot|||$.

Theorem 2.3. *If $\mathbf{X} \in K_{m,n}$ is normal, then $\mathbf{X}$ and $\Theta_0(\mathbf{X})$ are unitarily similar and $|||\Theta_0(\mathbf{X})||| = |||\mathbf{X}|||$ for every unitarily invariant norm $|||\cdot|||$.*

Theorem 2.4. *For every unitarily invariant norm $|||\cdot|||$*

$$|||\Theta_0(\mathbf{X})||| \leqq 2\,|||\mathbf{X}||| \quad (\mathbf{X} \in K_{m,n}).$$

We recall the definition of the *numerical range* $W(X)$ of a square matrix $X \in M_n$:

$$W(X) := \{\langle Xa, a\rangle : a \in \mathbb{C}^n, \|a\| = 1\}$$

The *numerical radius* $w(X)$ is defined as

$$w(X) = \sup\{|\xi| : \ \xi \in W(X)\} = \sup\{|\langle Xa, a\rangle| : a \in \mathbb{C}^n, \|a\| = 1\}$$

Theorem 2.5. *For $\mathbf{X} \in K_{m,n}$*

$$w(\Theta_0(\mathbf{X})) = w(\mathbf{X}) \quad \text{and more precisely,} \quad W(\Theta_0(\mathbf{X})) = W(\mathbf{X}).$$

Theorem 2.6. *Let $A \in M_m, B \in M_n$. Then for every unitarily invariant norm $|||\cdot|||$*

$$\sup_{X \in M_{m,n}} \frac{|||AX + XB^T|||}{|||X|||} \leqq \sqrt{2}\, \|A \otimes I_n + I_m \otimes B\|_\infty,$$

and

$$\sup_{X \in M_{m,n}} \frac{|||AX + XB|||}{|||X|||} \leqq \sqrt{2}\, \|A \otimes I_n + I_m \otimes B\|_\infty.$$

Theorem 2.7. *For $A \in M_m$ and $B \in M_n$*

$$\|A \otimes I_m + I_n \otimes B^T\|_\infty \leqq \sqrt{2}\, \|A \otimes I_n + I_m \otimes B\|_\infty.$$

Recall the definition of the *Schatten p-norm* $\|\cdot\|_p$ for $1 \leqq p < \infty$:

$$\|X\|_p := \{\sum_i s_i(X)^p\}^{1/p}.$$

Theorem 2.8. *For $p = 2k$ $(k = 1, 2, 3, 4, 5)$*

$$\|\Theta_0(\mathbf{X})\|_p = \|\mathbf{X}\|_p \quad (\mathbf{X} \in K_{m,n}).$$

References

[1] T. Ando and T. Sano, *Norm estimates of the partial transpose map on the tensor products of matrices,* Positivity, 12 (2008), 9-24.