Global asymptotic stability for a class of difference equations

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1 Introduction

Consider the following nonlinear difference equation with variable coefficients:

\[ x_{n+1} = qx_{n} - \sum_{j=0}^{m} a_{j} f_{j}(x_{n-j}), \quad n = 0, 1, 2, \ldots, \]  

where \( 0 < q \leq 1 \), \( a_{j} \geq 0 \), \( 0 \leq j \leq m \) and \( \sum_{j=0}^{m} a_{j} > 0 \). We now assume that

\[
\begin{cases}
    f(x) \in C(-\infty, +\infty) \text{ is a strictly monotone increasing function,} \\
    f(0) = 0, \quad 0 < \frac{f(x)}{f'(x)} \leq 1, \quad x \neq 0, \quad 1 \leq j \leq m, \text{ and} \\
    \text{if } f(x) \neq x, \text{ then } \lim_{x \to -\infty} f(x) \text{ is finite, otherwise } f(x) = x.
\end{cases}
\]

The above difference equation has been studied by many literatures (see for example, [1]-[9] and references therein).

**Definition 1.1** The solution \( y^{*} \) of (1.1) is called uniformly stable, if for any \( \varepsilon > 0 \) and non-negative integer \( n_{0} \), there is a constant \( \delta = \delta(\varepsilon) > 0 \) such that \( \sup\{|y_{n_{0}-i} - y^{*}| | 0 \leq i \leq m\} < \delta \), implies that the solution \( \{y_{n}\}_{n=0}^{\infty} \) of (1.1) satisfies \( |y_{n} - y^{*}| < \varepsilon \), \( n = n_{0}, n_{0} + 1, \ldots \).

**Definition 1.2** The solution \( y^{*} \) of (1.1) is called globally attractive, if every solution of (1.1) tends to \( y^{*} \) as \( n \to \infty \).

**Definition 1.3** The solution \( y^{*} \) of (1.1) is called globally asymptotically stable, if it is uniformly stable and globally attractive.

In this paper, we study "semi-contractive" functions and global asymptotic stability of difference equations. In Section 2, we first define semi-contractivity of functions and show the related results on the global asymptotic stability of difference equations.

2 Semi-contractive function

Assume that

\[ g(z_{0}, z_{1}, \ldots, z_{m}) \in C(R^{m+1}) \quad \text{and} \quad g(y, y, \ldots, y) = y \text{ has a unique solution } y = y^{*}. \]  

\[ (2.1) \]
Definition 2.1 The function \( g(z_0, z_1, \cdots, z_m) \) is said to be semi-contractive at \( y^* \), if
(i) for any constants \( \bar{z} < y^* \) and \( z_i \geq \bar{z} \), \( 0 \leq i \leq m \), there exists a constant \( y^* < \bar{z} < +\infty \) such that \( g(z_0, z_1, \cdots, z_m) < \bar{z} \), and for any \( \bar{z} \leq z_i \leq \bar{z} \), \( 0 \leq i \leq m \), there exists a constant \( \bar{z} > z \) such that \( \bar{z} \leq g(z_0, z_1, \cdots, z_m) \), or
(ii) for any constants \( \bar{z} > y^* \) and \( z_i \leq \bar{z} \), \( 0 \leq i \leq m \), there exists a constant \( y^* > \bar{z} > -\infty \) such that \( g(z_0, z_1, \cdots, z_m) \geq \bar{z} \), and for any \( \bar{z} \leq z_i \leq \bar{z} \), \( 0 \leq i \leq m \), there exists a constant \( \bar{z} < \bar{z} \) such that \( \bar{z} \geq g(z_0, z_1, \cdots, z_m) \).

Lemma 2.1 If \( g(y) \in C(R) \) is a strictly monotone decreasing function such that \( g(g(y)) > y \) for any \( y < y^* \), then \( g(z) \) is semi-contractive for \( y^* \).

Lemma 2.2 Assume (2.1) and that each \( g_i(z_0, z_1, \cdots, z_m) \), \( 0 \leq i \leq m \) is semi-contractive for \( y^* \). Then for any \( b_{n,i} \geq 0 \), \( n \geq 0 \), \( 0 \leq i \leq m \) such that \( \sum_{i=0}^{m} b_{n,i} = 1 \) and \( \lim_{n \to \infty} b_{n,i} = b_i \), \( 0 \leq i \leq m \), it holds that \( \sum_{i=0}^{m} b_{n,i} g_i(z_0, z_1, \cdots, z_m) \) is semi-contractive for \( y^* \).

Corollary 2.1 Assume (2.1) and that \( g(z_0, z_1, \cdots, z_m) \) is semi-contractive for \( y^* \). Then for any \( 0 \leq q_n < 1 \), \( g_n(z_0, z_1, \cdots, z_m) \) and \( k \) such that
\[
\lim_{n \to \infty} q_n = q < 1, \quad \text{and} \quad 0 \leq k \leq m,
\]
\[
\lim_{n \to \infty} g_n(z_0, z_1, \cdots, z_m) = g(z_0, z_1, \cdots, z_m) \quad \text{for any} \quad z_0, z_1, \cdots, z_m \in (-\infty, +\infty),
\]
(2.2)
it holds that \( q_n z_k + (1 - q_n) g_n(z_0, z_1, \cdots, z_m) \) is semi-contractive for \( y^* \).

Corollary 2.2 Assume each \( g_i(z) \in C(R) \) and \( g_i(y) = y \) has a unique solution \( y = y^* \), \( 0 \leq i \leq m \), and each \( g_i(z_i) \), \( 0 \leq i \leq m \) is semi-contractive for \( y^* \), then for any \( b_{n,i} \geq 0 \), \( n \geq 0 \), \( 0 \leq i \leq m \) such that \( \sum_{i=0}^{m} b_{n,i} = 1 \) and \( \lim_{n \to \infty} b_{n,i} = b_i \), \( 0 \leq i \leq m \), it holds that \( \sum_{i=0}^{m} b_{n,i} g_i(z_i) \) is semi-contractive for \( y^* \). In particular, for any \( 0 \leq q_n < 1 \) and \( k \) such that \( \lim_{n \to \infty} q_n = q < 1 \) and \( 0 \leq k \leq m \), it holds that \( q_n z_k + (1 - q_n) \sum_{i=0}^{m} b_{n,i} g_i(z_i) \) is semi-contractive for \( y^* \).

Remark 2.1 If \( g(z_0, z_1, \cdots, z_m) > 0 \) for any \( z_i > 0 \), \( 0 \leq i \leq m \), then there are cases that we may restrict our attention only to \( z_i > 0 \), \( 0 \leq i \leq m \) and the unique positive solution \( y^* > 0 \) of \( g(y^*, y^*, \cdots, y^*) = y^* \), whether or not \( g(y, y, \cdots, y) = y \) has other solutions \( y \leq 0 \).

Example 2.1 Examples of semi-contractive function \( g(z_0, z_1, \cdots, z_m) \) for \( y^* \).
(i) \( g(z_0, z_1, \cdots, z_m) = z_m e^{c(1 - z_m)}, \ y^* = 1 \) and \( c \leq 2 \) (see [1]).
(ii) \( g(z_0, z_1, \cdots, z_m) = z_0 \exp(c(1 - \sum_{i=0}^{m} a_i z_i)), \ y^* = 1/(\sum_{i=0}^{m} a_i) \) and \( c \leq 2 \), where \( a_0 > 0, a_i \geq 0, 1 \leq i \leq m \) and \( (\sum_{i=1}^{m} a_i)/a_0 \leq 2/e \).
This is equivalent to \( h(u_0, u_1, \cdots, u_m) = u_0 - c \sum_{i=0}^{m} b_i (e^{u_i} - 1) \) is semi-contractive for \( u^* = 0 \) and \( c \leq 2 \), where \( z_i = y^*_e^{u_i}, \ b_0 = y^* a_0 > 0, b_i = y^* a_i > 0, 1 \leq i \leq m, \sum_{i=0}^{m} b_i = 1 \),
and \( (\sum_{i=1}^{m} b_i)/b_0 \leq 2/e \) (see [8]).
(iii) \( g(z_0, z_1, \cdots, z_m) = c(1 - e^{c_m}), \ y^* = 0 \) and \( c \leq 1 \) (see [3]).
(iv) \( g(z_0, z_1, \cdots, z_m) = \frac{e^{z_m}}{z_1 + \cdots + z_m}, \ x^* = ((c - 1)/b)^{1/p} \) and \( c \leq \frac{p}{p - 2} \), where \( p > 2 \) and \( b > 0 \) (see [1]).

We consider the following difference equation
\[
y_{n+1} = q_n y_{n-k} + (1 - q_n) g_n(y_n, y_{n-1}, \cdots, y_{n-m}), \quad n = 0, 1, \cdots,
\]
(2.3)
where we assume (2.1) and

\[
\begin{cases}
0 \leq q_n < 1, & \lim_{n \to \infty} q_n = q < 1, \quad k \in \{0, 1, \ldots, m\}, \text{ and} \\
\lim_{n \to \infty} g_n(z_0, z_1, \ldots, z_m) = g(z_0, z_1, \ldots, z_m) \text{ for any } z_0, z_1, \ldots, z_m \in (-\infty, +\infty).
\end{cases}
\] (2.4)

**Theorem 2.1** If \(g(z_0, z_1, \ldots, z_m)\) is semi-contractive for \(y^*\), then \(y^*\) of (2.3) is globally asymptotically stable for any \(0 \leq q < 1\).

**Colloary 2.3** Assume that there exists a constant \(0 \leq q_0 < 1\) and some \(0 \leq k \leq m\) such that \(q_0 z_k + (1 - q_0)g(z_0, z_1, \ldots, z_m)\), is semi-contractive for \(y^*\). Then, for any \(q_0 \leq q_n < 1\) and \(g_n(z_0, z_1, \ldots, z_m)\) which satisfy (2.4), the solution \(y^*\) of (2.3) is globally asymptotically stable.

**Remark 2.2** (i) The corresponding continuous case (2.3) is the following differential equation

\[
\begin{cases}
y'(t) = -p(t)\{y(t) - \frac{1}{1-q_n}g_n(y(n), y(n-1), \ldots, y(n-m))\}, \quad n \leq t < n+1, \quad n = 0, 1, 2, \ldots, \\
p(t) > 0, \quad q_n = e^{-\int_{n}^{n+1}p(t)dt} < 1.
\end{cases}
\]

(ii) In Theorem 2.1, a semi-contractivity condition is a delays and \(q_n\)-independent condition for the solution \(y^*\) of (2.3) to be globally asymptotically stable.

By Theorem 2.1 and Example 2.1, we obtain the following result:

**Example 2.2** Examples of delays and \(q\)-independent stability conditions.

(i) Ricker model \(y_{n+1} = qy_n + (1-q)y_{n-m}e^{c(1-y_{n-m})}\), \(n = 0, 1, 2, \ldots\). The positive equilibrium \(y^* = 1\) is globally asymptotically stable, if \(c \leq 2\) (see [1]).

(ii) Ricker model with delayed-density dependence \(y_{n+1} = qy_n + (1-q)y_n \exp\{c(1-\sum_{i=0}^{m}a_iy_{n-i})\}\). The positive equilibrium \(y^* = 1/(\sum_{i=0}^{m}a_i)\) is globally asymptotically stable, if \(c \leq 2\), where \(a_0 > 0, \quad a_i \geq 0, \quad 1 \leq i \leq m\) and \((\sum_{i=1}^{m}a_i)/a_0 \leq 2/e\) (see [8]).

(iii) Wazewska-Czyzewska and Lasota model \(y_{n+1} = qy_n + (1-q)c \sum_{i=0}^{m}b_ie^{-\gamma y_{n-i}}, \quad n = 0, 1, 2, \ldots\), where \(\gamma > 0, \quad b_i \geq 0, \quad 0 \leq i \leq m\), and \(\sum_{i=0}^{m}b_i = 1\).

The positive equilibrium \(y^*\) is the positive solution of the equation \(y^* = ce^{-\gamma y^*}\). Put \(x_n = y^* - y_n\). Then, this equation is equivalent to

\[
x_{n+1} = qx_n - (1-q){\gamma y^*} \sum_{i=0}^{m}b_i(e^{\gamma y^*} - 1), \quad \text{where} \quad b_i \geq 0, \quad 0 \leq i \leq m, \quad \sum_{i=0}^{m}b_i = 1.
\] (2.5)

Thus, the positive equilibrium \(y^*\) is globally asymptotically stable, if \(c \leq e/\gamma\) which is equivalent that the zero solution of (2.5) is globally asymptotically stable if \(\gamma y^* \leq 1\) (see [3]).

(iv) Bobwhite quail population model \(y_{n+1} = qy_n + (1-q)\frac{qy_{n-m}}{1+by_{n-m}}, \quad n = 0, 1, 2, \ldots\), where \(c > 1, \quad b > 0\). The positive equilibrium \(y^* = ((c-1)/b)^{1/p}\) is globally asymptotically stable, if \(c \leq \frac{p}{p-2}\) for \(p > 2\) (see [1]).

We have the following counter example:

**Example 2.3** Examples of \(q\)-dependent and delay-dependent stability conditions.

(i) A model in hematopoiesis \(y_{n+1} = qy_n + (1-q)e^{2(1-y_{n-m})}, \quad n = 0, 1, 2, \ldots\).

The equilibrium \(y^* = 1\) is globally asymptotically stable if \(q \in [1/3, 1]\), and 2-cycle if \(q \in [0, 1/3]\) (see [2]).

(ii) A delayed model in hematopoiesis \(y_{n+1} = qy_n + (1-q)e^{2(1-y_{n-m-1})}, \quad n = 0, 1, 2, \ldots\).

The characteristic equation takes the form \(\lambda^3 - q\lambda^2 = -2(1-q)\). Then for \(q = q_2 = \frac{3-\sqrt{3}}{2}\)
the roots are $-1 < \lambda_1 < 0$, $|\lambda_2| = |\lambda_3| = 1$. For $q_2 < q < 1$, the equilibrium $y^* = 1$ is locally attractive but it becomes unstable for $q = q_2$, and Hopf bifurcation occurs (see [2]).

(iii) Ricker’s equation with delayed-density dependence $y_{n+1} = y_n \exp\{c_n(1-\sum_{i=0}^{m} b_{n,i}y_{n-i})\}$, $n = 0, 1, \cdots$, which is equivalent to $x_{n+1} = x_n - c_n \sum_{i=0}^{m} b_{n,i}(e^{x_{n-i}} - 1)$, $n = 0, 1, \cdots$, where $c_n, b_{n,i} > 0$, $\sum_{i=0}^{m} b_{n,i} = 1$ and $y_n = e^{x_n}$.

The positive equilibrium $y^* = 1$ is globally asymptotically stable if $\lim\sup_{n \to \infty} \sum_{i=n}^{n+m} r_i < \frac{3}{2} + \frac{1}{2(m+1)}$ (see [7]).

(iv) A model of the growth of bobwhite quail populations $y_{n+1} = qy_n + (1-q)c \sum_{i=0}^{m} b_{i}e^{-\gamma y_n}$, $n = 0, 1, \cdots$, where $c, \gamma > 0$.

If $c \leq 1$, then for any $0 < q < 1$, $\lim_{n \to \infty} y_n = 0$. If $c > 1$, then the positive equilibrium $y^* = (c-1)^{1/p}$ of the model exists. Moreover, if $p \leq \frac{2c}{(c-1)(1-q)}$ for $m = 0$, or $p < \frac{c}{(c-1)(1-q)} \frac{3m+4}{2(m+1)^2}$ for $m \geq 1$, then the positive equilibrium $y^*$ is globally asymptotically stable (see [4]).

3 Delays-independent stability conditions for (1.1)

After setting

$$r_1 = a_0, \quad r_2 = \sum_{i=1}^{m} a_i, \quad r = r_1 + r_2, \quad \varphi(x) = qx - r_1 f(x), \quad \hat{z}(q) = (-1 + \sqrt{1+4q})/(2q),$$

we have the following result.

Theorem 3.1 Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$, and suppose that

$$r_1 < q, \quad r \leq q + (1-q)\ln(q/r_1) \quad \text{and} \quad (q/r_1)^{q}e^{-q}(r_1 - r_2) + (1-q) \geq 0,$$

or

$$r_1 \leq q, \quad r > q + (1-q)\ln(q/r_1), \quad qr_2 \leq r_1, \quad r - r_2(q/r_1)^{q}e^{-q} - (1-q)(L - 1) \geq 0 \quad \text{and} \quad \ell = \ln \frac{r-q-(1-q)\ln(q/r_1)}{r_2} \leq 0,$$

or

$$r_1 > q, \quad r \leq 1 + q, \quad r - r_2(q/r_1)^{q}e^{-q} - (1-q)(\ln(q/r_1) - 1) \geq 0,$$

and $\frac{\ell}{q(r_1)}r^{-q} \leq \frac{q}{1-\ell(q)}$.

Then, the zero solution of (1.1) is globally asymptotically stable.

Numerical result 3.1 Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$.

(i) The last inequality in (3.4) can be eliminated from (3.4).

(ii) Under the condition $\frac{p^2}{q} \leq \frac{2}{q}$ and $r \leq 1 + q$, the third inequality of (3.4) is satisfied, and hence the zero solution of (1.1) is globally asymptotically stable.

Example 3.1 Wazewska-Czyzewska and Lasota model (see [9]).

$$y_{n+1} = qy_n + (1-q)c \sum_{i=0}^{m} b_ie^{-\gamma y_{n-i}}, \quad \text{where} \quad c, \gamma > 0, \quad b_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{m} b_i = 1.$$ (3.5)

(3.5) is equivalent to (2.5). For equation (3.5), the positive equilibrium of (3.5), say $y^*$, is globally asymptotically stable, if $\gamma y^* \leq 1$ (see [3] and Example 2.2 iii)). For the case $\gamma y^* > 1$, by using
the generalized Yorke condition, [6, Theorem 8] extended these to $\gamma y^* \leq (1 + q^{m+1})/(1 - q^{m+1})$ with some restricted conditions "$V_k(q) < 0$, $W_k(q) < 0$". Note that the last condition contains the restriction $(q + q^2 + \cdots + q^m)q^m \leq 1$ for $0 < q < 1$. On the other hand, by applying Theorem 3.1 and Numerical result 3.1 to (2.5) for $a_i = (1 - q)\gamma y^* b_i$, $0 \leq i \leq m$, we obtain another sufficient condition, for example, $\sum_{i=1}^{m} b_i \leq \frac{2}{e} b_0$ and $\gamma y^* \leq (1 + q)/(1 - q)$ for the solution $y^*$ of (3.5) to be globally asymptotically stable. Note that $e^x - 1 < x/(1 - x)$ for $0 < x < 1$ and $\frac{1 + x^{m+1}}{1 - x^m} < \frac{1 + x^m}{1 - x}$ for $0 < q < 1$. Thus, compared with [6, Proof of Theorem 2] (and [1]-[9] and references therein), one can see that our results offer new stability conditions to (3.5).

4 Semi-contractivity with a sign condition

For $0 \leq q < 1$, consider the following nonautonomous equation

$$x_{n+1} = qx_n - \sum_{j=0}^{m} a_{n,j} f_j(x_{n-j}), \quad n = 0, 1, \ldots,$$

(4.1)

where $0 < q \leq 1$, $a_{n,j} \geq 0$, $0 \leq j \leq m$, $n = 0, 1, \ldots$, and $\sum_{j=0}^{m} a_{n,j} > 0$, and we assume that there is a function $f(x)$ such that (1.2) holds.

For (4.1) and any $0 \leq l_n \leq m$, we can derive the following equation.

$$x_{n+1} = \left\{ q^{l_{n+1}} x_{n-l_n} + (1 - q) \sum_{k=0}^{l_n} q^k \sum_{j=0}^{m-k} a_{n-k,j} f_j(x_{n-k-j}) \right\}$$

$$- \sum_{k=1}^{l_n} q^k \sum_{j=m-k+1}^{m} a_{n-k,j} f_j(x_{n-k-j}), \quad n = 2m, 2m + 1, \ldots$$

(4.2)

Similar to the proofs of [5, Lemmas 2.3 and 2.4], we have the following two lemmas for (4.1).

Lemma 4.1 Let $\{x_n\}_{n=0}^{\infty}$ be the solution of (4.1). If there exists an integer $n \geq m$ such that $x_{n+1} \geq 0$ and $x_{n+1} > x_n$, then there exists an integer $g_n \in [n-m, n]$ such that

$$x_{g_n} = \min_{0 \leq j \leq m} x_{n-j} < 0.$$  

(4.3)

If there exists an integer $n \geq m$ such that $x_{n+1} \leq 0$ and $x_{n+1} < x_n$, then there exists an integer $\bar{g}_n \in [n-m, n]$ such that

$$x_{\bar{g}_n} = \max_{0 \leq j \leq m} x_{n-j} > 0.$$  

(4.4)

After setting

$$\bar{r}_1 = \sup_{n \geq m} \sum_{k=0}^{m} q^k \sum_{j=0}^{m-k} a_{n-k,j}, \quad \bar{r}_2 = \sup_{n \geq m} \sum_{k=1}^{m} q^k \sum_{j=m-k+1}^{m} a_{n-k,j},$$

(4.5)

and

$$\bar{r} = \bar{r}_1 + \bar{r}_2, \quad \bar{\varphi}(x) = \bar{q} x - \bar{r}_1 f(x), \quad \bar{q} = q^{m+1}, \quad \bar{\xi} = (-1 + \sqrt{1 + 4\bar{q}})/(2\bar{q})$$

(4.6)

we are able to prove the following results.
If there exists an integer \( n \geq m \) such that \( x_{n+1} \geq 0 \) and \( x_{n+1} > x_n \), then by (4.3) and (4.2) with \( l_n = n - g_n \), we have that

\[
x_{n+1} \leq \varphi(x_{g_n}) - r_2 f(L_n), \quad L_n = \min_{0 \leq j \leq 2m} x_{n-j}.
\] (4.7)

If there exists an integer \( n \geq m \) such that \( x_{n+1} \leq 0 \) and \( x_{n+1} < x_n \), then by (4.4) and (4.2) with \( l_n = n - g_n \), we have that

\[
x_{n+1} \geq \varphi(x_{g_n}) - r_2 f(R_n), \quad R_n = \max_{0 \leq j \leq 2m} x_{n-j}.
\] (4.8)

**Lemma 4.2** Suppose that the solution \( x_n \) of (4.1) is oscillatory about 0. If for some real number \( L < 0 \), there exists a positive integer \( n_L \geq 2m \) such that \( x_n \geq L \) for \( n \geq n_L \), then for any integer \( n \geq n_L + 2m \),

\[
x_{n+1} \leq R_L \text{ for } n \geq n_L + 2m, \quad \text{and } x_{n+1} \geq S_L \text{ for } n \geq n_L + 4m,
\] (4.9)

where \( R_L = \max_{L \leq x < 0} \varphi(x) - r_2 f(L) > 0 \) and \( S_L = \min_{0 \leq x \leq L} \varphi(x) - r_2 f(R_L) < 0 \). Moreover, if \( S_L > L \) for any \( L < 0 \), then \( \lim_{n \to \infty} x_n = 0 \).

Assume that \( g(z_0, z_1, \cdots, z_m) \) is continuous for \( (z_0, z_1, \cdots, z_m) \in \mathbb{R}^{m+1} \) and \( g(y^*, y^*, \cdots, y^*) = y^* \) has a unique solution \( y^* \).

**Definition 4.1** The function \( g(z_0, z_1, \cdots, z_m) \) is said to be semi-contractive with a sign condition \( z_0 \) for \( y^* \), if

(i) for any constants \( \tilde{z} < y^* \) and \( z_i \geq \tilde{z}, 0 \leq i \leq m \) with \( z_0 \leq y^* \), there exists a constant \( y^* < \tilde{z} < +\infty \) such that \( g(z_0, z_1, \cdots, z_m) \leq \tilde{z} \) and for any \( \tilde{z} \leq z_i \leq \tilde{z}, 0 \leq i \leq m \) with \( z_0 \geq y^* \), there exists a constant \( \tilde{z} \geq z \) such that \( \tilde{z} \leq g(z_0, z_1, \cdots, z_m) \),

or

(ii) for any constants \( \tilde{z} > y^* \) and \( z_i \leq \tilde{z}, 0 \leq i \leq m \) with \( z_0 \geq y^* \), there exists a constant \( y^* > \tilde{z} > -\infty \) such that \( g(z_0, z_1, \cdots, z_m) \geq \tilde{z} \) and for any \( \tilde{z} \leq z_i \leq \tilde{z}, 0 \leq i \leq m \) with \( z_0 \leq y^* \), there exists a constant \( \tilde{z} \geq z \) such that \( \tilde{z} \geq g(z_0, z_1, \cdots, z_m) \).

Then by (4.7), (4.8) and Lemma 4.2, we can obtain the following result.

**Theorem 4.1** If \( \bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - r_2 f(z_1) \) is semi-contractive with a sign condition \( z_0 \) for \( x^* = 0 \), then the zero solution of (4.1) is globally asymptotically stable.

Note that if \( \bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - r_2 f(z_1) \) is semi-contractive with a sign condition \( z_0 \) for \( x^* = 0 \), then the zero solution \( x^* = 0 \) of (4.1) is uniformly stable and hence \( x^* = 0 \) is globally asymptotically stable.

For the special case \( f(x) = e^x - 1 \), we establish the following sufficient conditions for \( 0 < q < 1 \) which are some extensions of the result in [5] for \( q = 1 \).

**Theorem 4.2** Suppose that \( f(x) = e^x - 1 \) and that one of the following condition is fulfilled:

\[
\begin{align*}
\begin{cases}
\bar{r} \leq 1 & \text{and } \frac{\bar{r} e^{q}}{q} e^{q} \leq \frac{\bar{r}}{1-\bar{q}} \\
\bar{r} \leq 1 + \bar{q} & \text{and } \frac{\bar{r} e^{q}}{q} e^{q} \geq \frac{\bar{r}}{1-\bar{q}}
\end{cases}
\end{align*}
\]

or

\[
\begin{align*}
\begin{cases}
\bar{r} \leq 1 & \text{and } \frac{\bar{r} e^{q}}{q} e^{q} \geq \frac{\bar{r}}{1-\bar{q}} \\
\bar{r} \leq 1 + \bar{q} & \text{and } \frac{\bar{r} e^{q}}{q} e^{q} \leq \frac{\bar{r}}{1-\bar{q}}
\end{cases}
\end{align*}
\] (4.10) (4.11)
with \[
\begin{align*}
G_1(x) &= q\left(\tilde{q}\ln(\tilde{q}/\tilde{r}_1) + \tilde{r} - \tilde{q} - \tilde{r}_2 z^2\right) + \tilde{r} - \tilde{r}(\tilde{q}/\tilde{r}_1)^q e^{\tilde{r}_{2} e^{\tilde{r}_2} - x}, \\
G_3(x) &= (\tilde{r}_1 + (1 + \tilde{q})\tilde{r}_2) - \tilde{q}\tilde{r}_2 e^{\tilde{r}_{2}} - \tilde{r} e^{\tilde{r}_{2} - \tilde{r}_2 z^2} - x,
\end{align*}
\] (4.12)

where \(\alpha\) and \(\delta\) are the lowest solutions of \(G_1(x) = 0\) and \(G_3(x) = 0\), respectively, and \(\tilde{r}\) is a positive solution of \(\tilde{q}z^2 + z - 1 = 0\). Then, the solution \(x^* = 0\) of (4.1) is globally asymptotically stable.

As an immediate consequence we have the following corollary.

**Corollary 4.1** Assume that \(f(x) = e^x - 1\) and that \(p \leq 1 + \tilde{q}\) and \(\tilde{r}_1 \geq \tilde{q}\tilde{r}_2\).

If
\[
(i) \frac{\tilde{r}}{\tilde{q}}(\tilde{q}/\tilde{r}_1)^q e^{\tilde{r}_{2} - \tilde{q}} \leq \frac{e^{\tilde{r}}}{1 - \tilde{r}}, \quad \text{or} \quad (ii) \frac{\tilde{r}}{\tilde{q}}(\tilde{q}/\tilde{r}_1)^q e^{\tilde{r}_{2} - \tilde{q}} > \frac{e^{\tilde{r}}}{1 - \tilde{r}} \quad \text{and} \quad G_1(\alpha) > 0,
\]
(4.14)

then, the zero solution of (4.1) is globally asymptotically stable.

**Example 4.1** Consider a model \(x_{n+1} = qx_n - \sum_{i=0}^{m} a_i (e^{-x_{n-i}} - 1), \quad n = 0, 1, 2, \ldots\), where \(a_i \geq 0, \quad 0 \leq i \leq m, \quad \text{and} \quad \sum_{i=0}^{m} a_i > 0\). This equation is equivalent to (2.5), if \(\sum_{i=0}^{m} a_i = (1-q)\gamma y^*\) and \(0 < q < 1\). By Corollary 4.1, the zero solution \(x^* = 0\) is globally asymptotically stable for \(p \leq 1 + \tilde{q}\), if for the setting (4.5) and \(\tilde{r}_1 = \tilde{q}(1+2)(1-\tilde{r})e^{1-\tilde{r}}\), it holds that \(\frac{\tilde{r}_1}{\tilde{r}} \leq \frac{1+2}{\tilde{r}_1} - 1\). Since \(e^x - 1 < x/(1-x)\) for \(0 < x < 1\) and we do not need the restriction \((q + q^2 + \cdots + q^m)q^m \leq 1\) for \(0 < q < 1\) in [6, Theorem 2], our results improve some of [6, Theorem 8] (see [5]).

**References**


